MA 138 – Calculus 2 with Life Science Applications
Nonlinear Autonomous Systems: Theory
(Section 11.3)

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Analytical Approach

- We consider a system of differential equations of the form

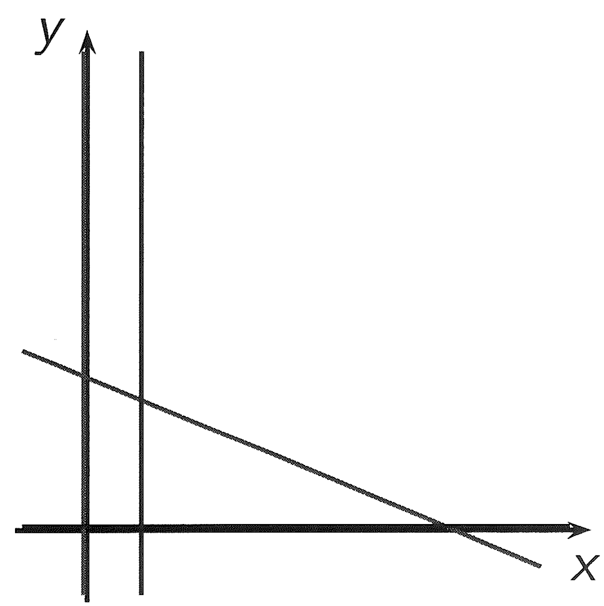
$$\begin{cases} \dfrac{dx_1}{dt} &= f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} &= f_2(x_1, x_2) \end{cases}$$

  where we assume that the functions $f_i(\mathbf{x}) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ do not explicitly depend on t. We also no longer assume that the f_i's are linear.
- Such a system is called autonomous.
- Using vector notation, we can write the system as $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$.
- An equilibrium or critical point, $\widehat{\mathbf{x}}$, of the above nonlinear system satisfies $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{0}$.

Example 1

Identify all the equilibrium points of the following nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = x(5 - x - 6y) \\ \dfrac{dy}{dt} = y(1 - 5x) \end{cases}$$



To find the <u>equilibria</u> of

$$\begin{cases} \dfrac{dx}{dt} = x(5-x-6y) \\ \dfrac{dy}{dt} = y(1-5x) \end{cases}$$

we need to solve the system of equations

$$\begin{cases} x(5-x-6y) = 0 \\ y(1-5x) = 0 \end{cases}$$

From the first equation we get that

$x=0$ or $5-x-6y=0$ ($\Leftrightarrow x = 5-6y$)

Thus we obtain the following 2 systems:

$$\begin{cases} x = 0 \\ y(1-5x) = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 5-6y \\ y(1-5x) = 0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\boxed{(0,0)} \qquad \begin{cases} x = 5-6y \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 5-6y \\ 1-5x = 0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\boxed{(5,0)} \qquad\qquad \begin{cases} x = 1/5 = \underline{0.2} \\ y = \underline{0.8} \end{cases}$$

$$\boxed{(0.2, 0.8)}$$

Thus the system has 3 equilibria

- Suppose that $\widehat{\mathbf{x}}$ is a point equilibrium. Then, as in the case of one nonlinear equation (≡Section 8.2), we look at what happens to a small perturbation of $\widehat{\mathbf{x}}$.
- We perturb $\widehat{\mathbf{x}}$; that is, we look at how $\widehat{\mathbf{x}} + \mathbf{z}$ changes under the dynamics described by our nonlinear system:

$$\frac{d}{dt}(\widehat{\mathbf{x}} + \mathbf{z}) = \frac{d}{dt}\mathbf{z} = \mathbf{f}(\widehat{\mathbf{x}} + \mathbf{z})$$

- The linearization of $\mathbf{f}(\Box)$ about $\mathbf{x} = \widehat{\mathbf{x}}$ is

$$\mathbf{L}(\Box) = \mathbf{f}(\widehat{\mathbf{x}}) + D\mathbf{f}(\widehat{\mathbf{x}})(\Box - \widehat{\mathbf{x}}) = D\mathbf{f}(\widehat{\mathbf{x}})(\Box - \widehat{\mathbf{x}})$$

  where we used the fact that $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{0}$.
- The matrix $D\mathbf{f}(\widehat{\mathbf{x}})$ is the Jacobi matrix of $\mathbf{f}(\mathbf{x})$ evaluated at $\widehat{\mathbf{x}}$.
- If we approximate $\mathbf{f}(\widehat{\mathbf{x}} + \mathbf{z})$ by its linearization $\mathbf{L}(\widehat{\mathbf{x}} + \mathbf{z}) = D\mathbf{f}(\widehat{\mathbf{x}})\mathbf{z}$, then

$$\frac{d\mathbf{z}}{dt} = D\mathbf{f}(\widehat{\mathbf{x}})\,\mathbf{z}$$

  is the linear approximation of the dynamics of the perturbation $\mathbf{z}$.

- We now have a system of linear differential equations that is a good approximation, provided that $\mathbf{z}$ is sufficiently close to $\mathbf{0}$.
- When we linearize a nonlinear system about an equilibrium, the matrix A is the Jacobi matrix evaluated at the equilibrium.
- To classify the equilibrium we can use the same classification scheme as in the linear case. We need to exclude, though, the following cases:

  (i) at least one eigenvalue is equal to 0,

  (ii) the two eigenvalues are purely imaginary, and

  (iii) the two eigenvalues are identical.

- An equilibrium point as described above is often called **hyperbolic** (this is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard!).
- The extension from the linear case to the nonlinear case is possible because of the following result:

Hartman-Grobman Theorem

Hartman-Grobman Theorem

The local phase portrait near a hyperbolic equilibrium point is "topologically equivalent[1]" to the phase portrait of the linearization.

In particular, the stability type of the equilibrium point is faithfully captured by the linearization.

[1] Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other: bending and wrapping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

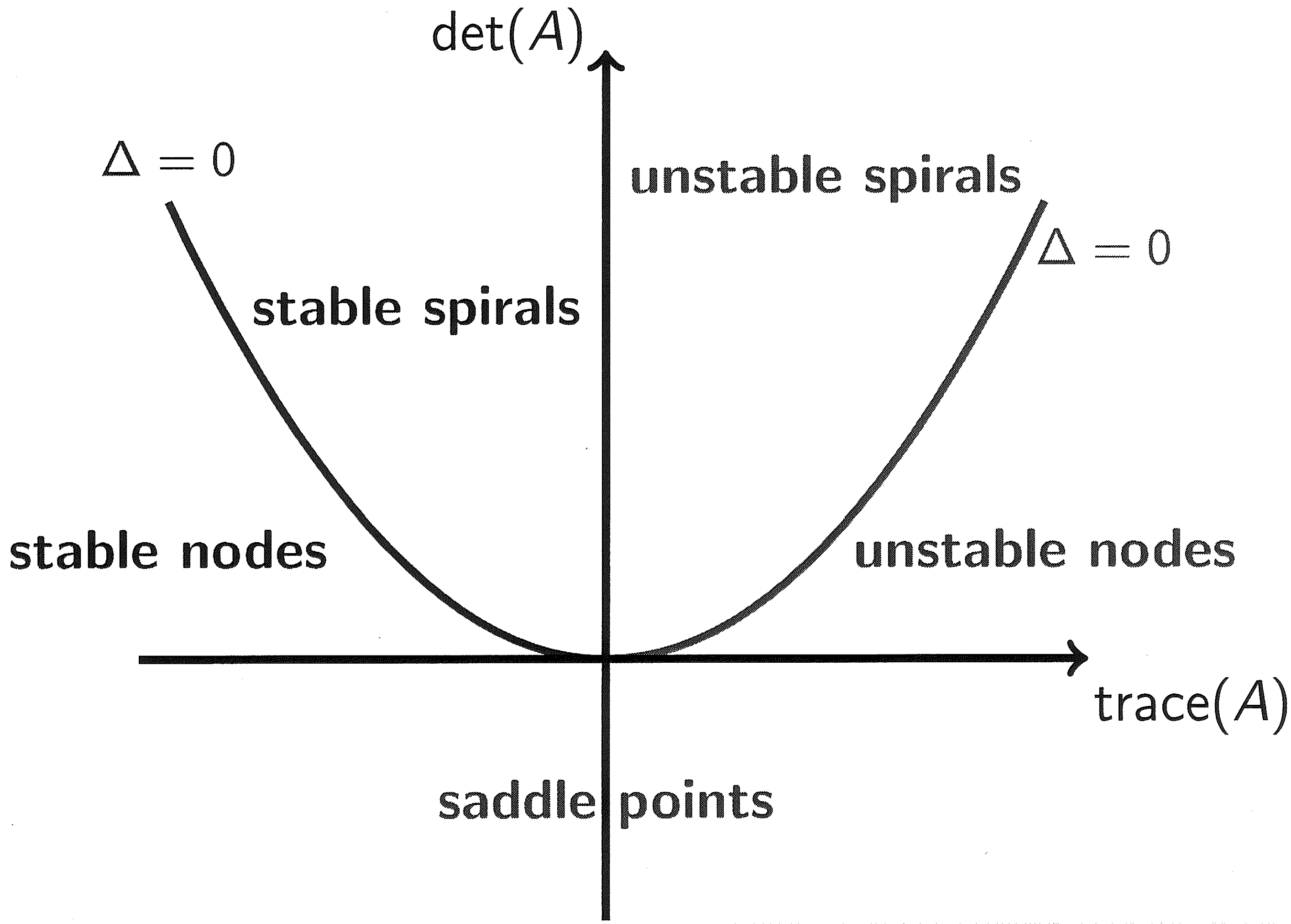
That is... $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ and $\frac{d\mathbf{z}}{dt} = D\mathbf{f}(\widehat{\mathbf{x}})\mathbf{z}$

behave similarly for $\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{z}$ with $\mathbf{z}$ close to $\mathbf{0}$.

More precisely, we find the same classification scheme as in the linear case:

- The equilibrium $\widehat{\mathbf{x}}$ is a **node** if both eigenvalues of $D\mathbf{f}(\widehat{\mathbf{x}})$ are real, distinct, nonzero, and of the same sign; $\widehat{\mathbf{x}}$ is locally stable if the eigenvalues are negative and unstable if the eigenvalues are positive.
- The equilibrium $\widehat{\mathbf{x}}$ is a **saddle point** if both eigenvalues are real and nonzero but have opposite signs. A saddle point is unstable.
- The equilibrium $\widehat{\mathbf{x}}$ is a **spiral** if both eigenvalues are complex conjugates with nonzero real parts. The spiral is locally stable if the real parts of the eigenvalues are negative and unstable if the real parts of the eigenvalues are positive.
- In the exceptional cases, we cannot determine the stability by linearization.

The stability properties of a hyperbolic equilibrium $\widehat{\mathbf{x}}$ can be summarized graphically in terms of the determinant and the trace of the Jacobi matrix $A = D\mathbf{f}(\widehat{\mathbf{x}})$ in the trace-det plane:



Example 1 (cont'd)

Linearize the nonlinear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & x(5 - x - 6y) \\ \dfrac{dy}{dt} & = & y(1 - 5x) \end{cases}$$

at each equilibrium point

- $(\widehat{x}_1, \widehat{y}_1) = (0, 0)$;
- $(\widehat{x}_2, \widehat{y}_2) = (5, 0)$;
- $(\widehat{x}_3, \widehat{y}_3) = (0.2, 0.8)$.

Classify the type of each equilibrium point.

Consider

$$\begin{cases} \dfrac{dx}{dt} = 5x - x^2 - 6xy = f_1(x,y) \\ \dfrac{dy}{dt} = y - 5xy = f_2(x,y) \end{cases}$$

The linearization of $\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ at an equilibrium point $(\hat{x},\hat{y})$ is:

$$\underbrace{\begin{bmatrix} f_1(\hat{x},\hat{y}) \\ f_2(\hat{x},\hat{y}) \end{bmatrix}}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \begin{bmatrix} \frac{\partial f_1}{\partial x}(\hat{x},\hat{y}) & \frac{\partial f_1}{\partial y}(\hat{x},\hat{y}) \\ \frac{\partial f_2}{\partial x}(\hat{x},\hat{y}) & \frac{\partial f_2}{\partial y}(\hat{x},\hat{y}) \end{bmatrix} \cdot \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $(\hat{x},\hat{y})$ is an equilibrium

$(D\underline{f})(x,y)$ = the Jacobi matrix is $\begin{bmatrix} 5-2x-6y & -6x \\ -5y & 1-5x \end{bmatrix}$

At the <u>equilibrium</u> $(0,0)$

$(D\underline{f})(0,0) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ which has eigenvalues $5, 1$

Since the eigenvalues are both positive $(0,0)$ is an <u>unstable equilibrium</u> (source)

At the equilibrium $(5,0)$ the Jacobi matrix becomes

$(D\underline{f})(5,0) = \begin{bmatrix} -5 & -30 \\ 0 & -24 \end{bmatrix}$

Thus at $(5,0)$ the linearization has eigenvalues -5 and -24. These are both negative thus $(5,0)$ is a stable equilibrium (sink)

Finally, at $(0.2, 0.8)$ the Jacobi matrix becomes

$$(Df)(0.2, 0.8) = \begin{bmatrix} -0.2 & -1.2 \\ -4 & 0 \end{bmatrix}$$

Since the determinant is negative, the 2 eigenvalues have opposite signs so the equilibrium is unstable (saddle point).

More explicitly, the eigenvalues are

$$\det \begin{bmatrix} -0.2-\lambda & -1.2 \\ -4 & -\lambda \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad \lambda(0.2+\lambda) - 4.8 = 0$$

$$\lambda^2 + 0.2\lambda - 4.8 = 0$$

$$\lambda_{1,2} = \frac{-0.2 \pm \sqrt{0.2^2 + 4 \cdot 4.8}}{2} = \begin{cases} 2.0932 \\ -2.2932 \end{cases}$$

Example 2 (Problem #10, Exam #4, Spring 2013)

Suppose a *habitat* is divided up into *patches* and each patch can be occupied by at most one individual. If two species A and B live in this habitat, the growth of the population of A is controlled by the internal dynamics of the population growth of A and the interactions between A and B. The situation for B is similar.

Suppose the members of species A are able to outcompete members of species B, that is, the members of A are able to invade patches that are occupied by species B and displace the resident. If p_1 is the fraction of the sites occupied by A and p_2 is the fraction of the sites occupied by B, this situation is described by the differential equations

$$\frac{dp_1}{dt} = c_1 p_1(1-p_1) - m_1 p_1 \qquad \frac{dp_2}{dt} = c_2 p_2(1-p_1-p_2) - m_2 p_2 - c_1 p_1 p_2$$

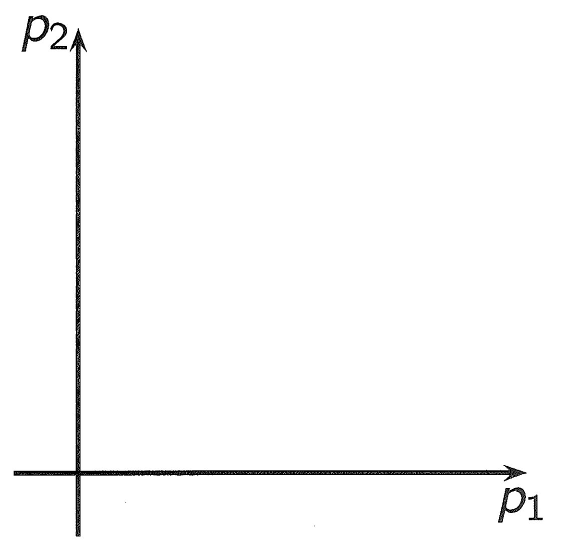
where c_1, c_2, m_1, and m_2 are the *colonization* and *mortality rates* of species A and B, respectively.

Example 2 (cont'd)

Suppose $c_1 = 2$, $c_2 = 10$, $m_1 = 1$, and $m_2 = 2$. Hence, after some algebra, the above system of nonlinear differential equations can be written as

$$\begin{cases} \dfrac{dp_1}{dt} & = & p_1\,(1 - 2p_1) \\ \dfrac{dp_2}{dt} & = & p_2\,(8 - 12p_1 - 10p_2) \end{cases}$$

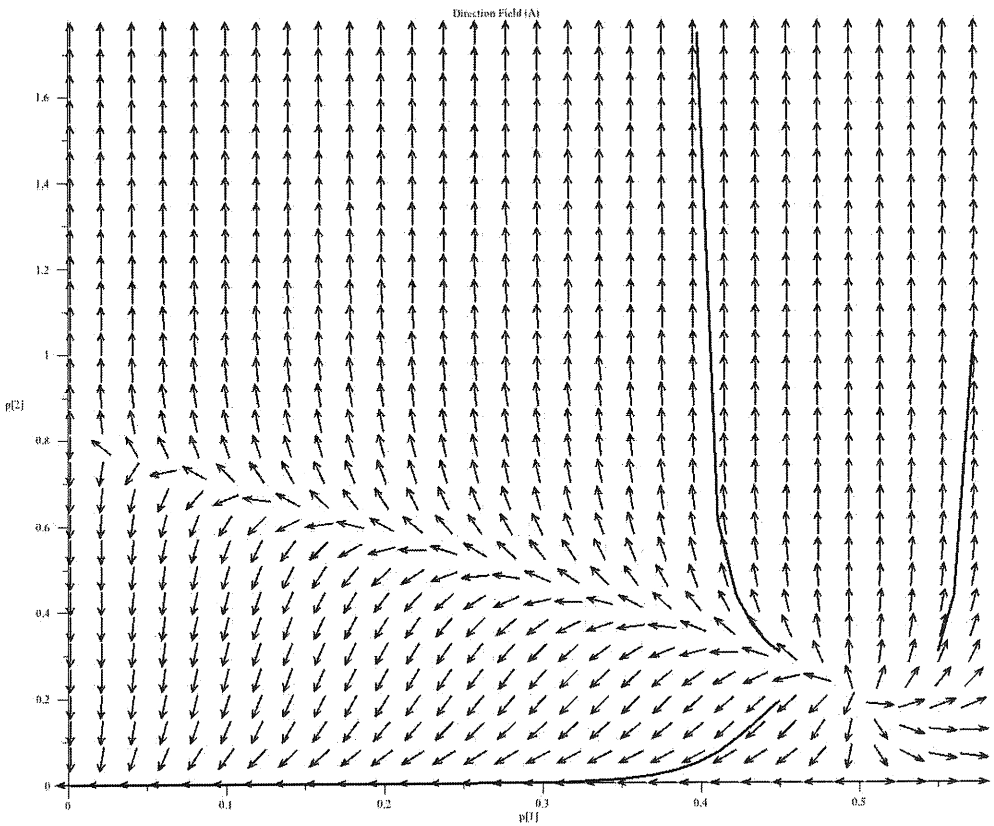
(*a*) There is only one equilibrium point $(\widehat{p}_1, \widehat{p}_2)$ where both species are present. Identify that point and linearize the given system of differential equations at that point. Classify the type of equilibrium.

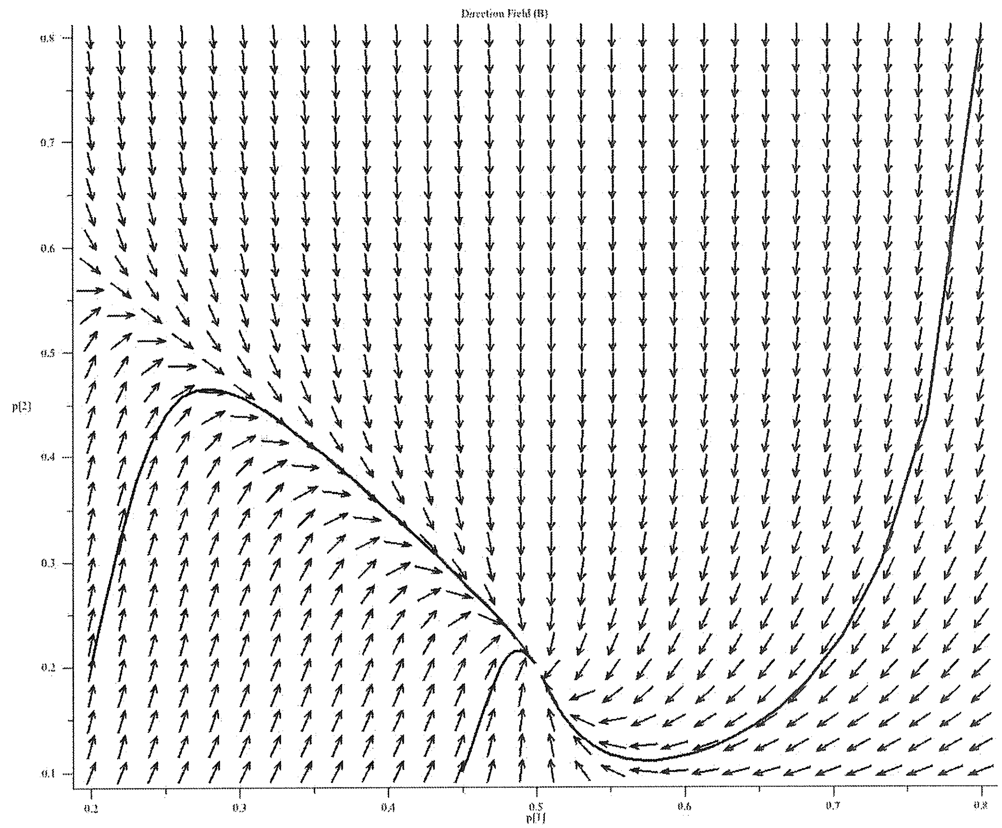


Plot here the nullclines and the nontrivial equilibrium point

Example 2 (cont'd)

(b) Choose the direction field that describes the system of nonlinear differential equations considered in (a).





We the data from our problem we need to analyse

$$\begin{cases} \dfrac{dp_1}{dt} = p_1(1-2p_1) \\ \dfrac{dp_2}{dt} = p_2(8-12p_1-10p_2) \end{cases}$$

To find the equilibria we need to solve

$$\begin{cases} p_1(1-2p_1) = 0 \\ p_2(8-12p_1-10p_2) = 0 \end{cases}$$

Since we are look for the equilibrium when both species are present, $p_1 \neq 0$ and $p_2 \neq 0$. Thus we need to solve

$$\begin{cases} 1-2p_1=0 \\ 8-12p_1-10p_2=0 \end{cases} \iff \begin{cases} p_1 = 1/2 \\ p_2 = \frac{1}{10}(8-12p_1) \end{cases}$$

$$= \frac{1}{10}(8-6) = 1/5$$

$$\boxed{(\hat{p}_1, \hat{p}_2) = (1/2, 1/5)}$$

To analyze the stability we need the Jacobi matrix evaluated at $(1/2, 1/5)$.

$$f_1 = p_1(1-2p_1) = p_1 - 2p_1^2$$

$$f_2 = 8p_2 - 12p_1p_2 - 10p_2^2$$

So that the Jacobi matrix is

$$\begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} 1-4p_1 & 0 \\ -12p_2 & 8-12p_1-20p_2 \end{bmatrix}$$

At the point $(\hat{p}_1, \hat{p}_2) = (1/2, 1/5)$ the matrix becomes:

$$\begin{bmatrix} -1 & 0 \\ -12/5 & 8-\underbrace{12(1/2)-20(1/5)}_{\underbrace{8-6-4}_{-2}} \end{bmatrix}$$

Thus the 2 eigenvalues are both negative so $(1/2, 1/5)$ is a stable equilibrium (sink)

The direction field that describes the situation is "direction field B".

On the Exception to the Hartman-Grobman Thm

The following nonlinear systems of DEs

$$\begin{cases} \dfrac{dx}{dt} = -\mathbf{y}+x(x^2+y^2) \\ \dfrac{dy}{dt} = \mathbf{x}+y(x^2+y^2) \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = -\mathbf{y}-x(x^2+y^2) \\ \dfrac{dy}{dt} = \mathbf{x}-y(x^2+y^2) \end{cases}$$

have rather different phase portraits (see below). However, they have the same linearization at the equilibrium $(0,0)$, with eigenvalues $\pm i$.