MA 138 – Calculus 2 with Life Science Applications
Nonlinear Systems: Applications
(Section 11.4)

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- In what follows we illustrate some interesting systems of nonlinear differential equations.
- Our analysis is essentially example driven. We typically include the phase plot of the system of DEs, the nullclines (first three examples only), and the graph of some particular solutions.
- Each plot has been generated using the software package Maple. The computer labs on campus should have a Maple installation. You can find the code to generate the plots by consulting the following website

  `http://www.ms.uky.edu/~ma138/Spring15/Maple_S13_non_linear/MA138_S13_non_linear.html`

- For a more comprehensive analysis of each topic, we refer to our textbook (Section 11.4).

Lotka-Volterra Model of Interspecific Competition

Interspecific competition, in ecology, is a form of competition in which individuals of different species compete for the same resource in an ecosystem. The Lotka-Volterra model of interspecific competition incorporates density-dependent effects of competition.

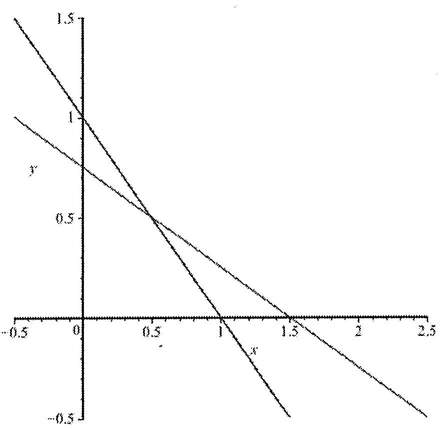
It is an extension of the logistic equation to the case of two species

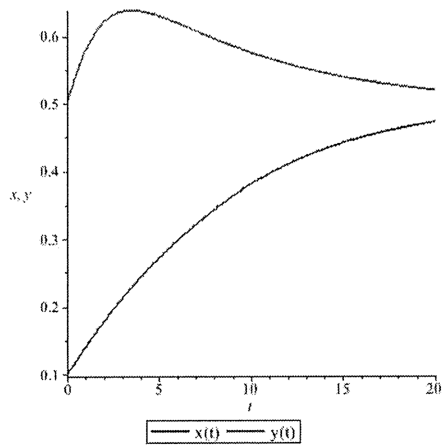
$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1}{K_1} - \alpha_{12}\dfrac{N_2}{K_1}\right) \\ \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2}{K_2} - \alpha_{21}\dfrac{N_1}{K_2}\right) \end{cases}$$

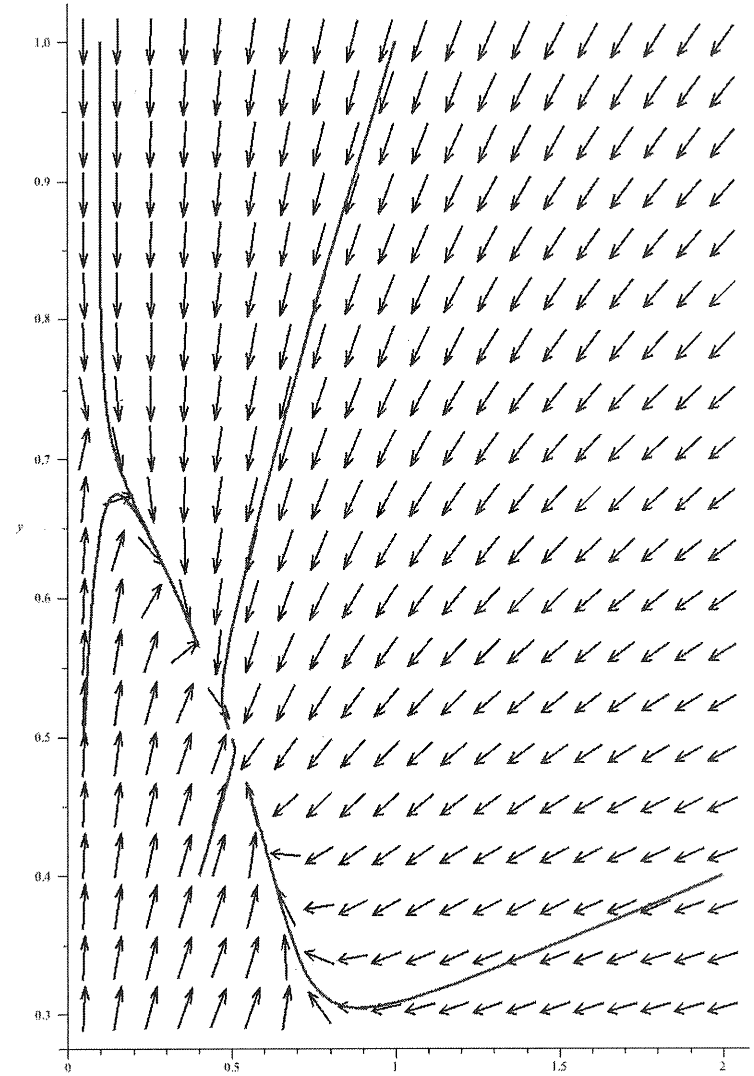
r_1 and r_2 are the **intrinsic growth rates** of species 1 and 2; K_1 and K_2 are the respective **carrying capacities**. In addition, the two species may have inhibitory effects on each other. We measure the effect of species 1 on species 2 by the **competition coefficient** α_{21}; the effect of species 2 on species 1 is measured by the competition coefficient α_{12}.

Example 1 (coexistence is possible)

$$\begin{cases} \dfrac{dx}{dt} &= x(1 - x - y) \\ \dfrac{dy}{dt} &= y(0.75 - y - 0.5x) \end{cases}$$

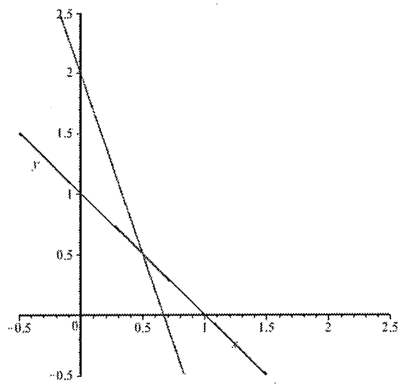


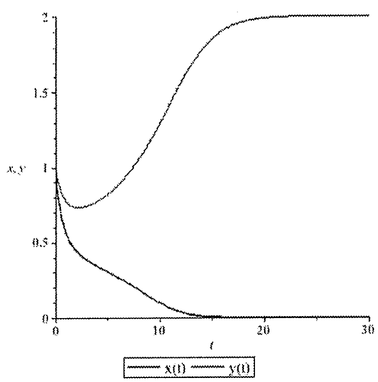


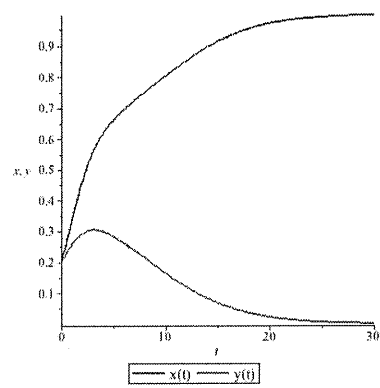


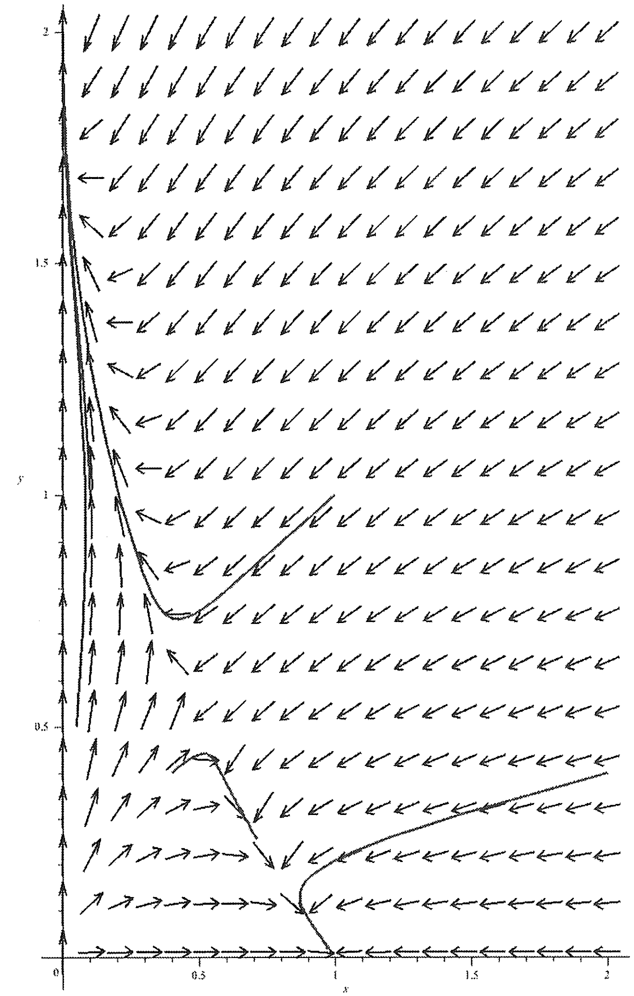
Example 2 (founder control)

$$
\begin{cases}
\dfrac{dx}{dt} &= x(1 - x - y) \\
\dfrac{dy}{dt} &= y(0.5 - 0.25y - 0.75x)
\end{cases}
$$





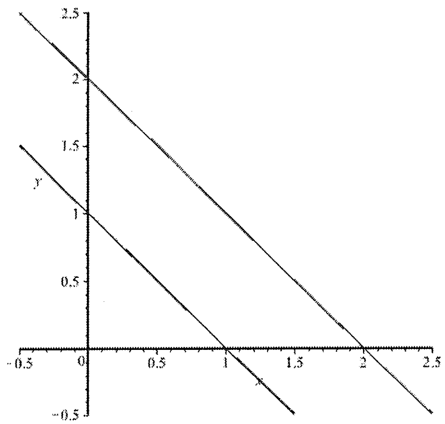


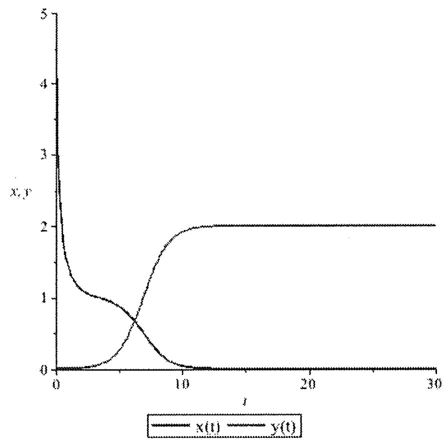


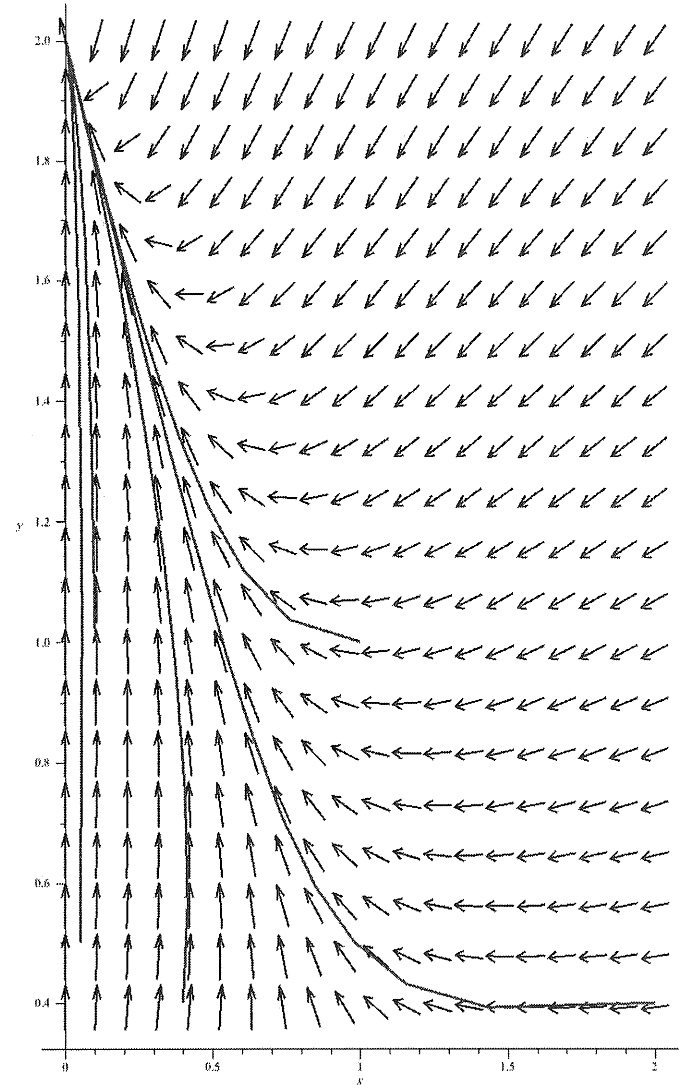
Founder control $\equiv$ No species can invade the other. The competition outcome depends on the initial densities.

Example 3 (competitive exclusion)

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x-y) \\ \dfrac{dy}{dt} &= y(2-y-x) \end{cases}$$







Competitive exclusion $\equiv$ One species, the *strong* one, outcompetes the other, the *weak* one.

The Lotka-Volterra Predator-Prey Model

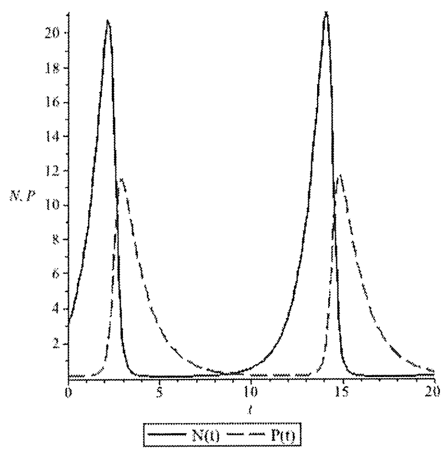
The simplest predator-prey model that exhibits coupled oscillations is the Lotka-Volterra model (Lotka, 1920; Volterra, 1926). We have already discussed this model in several situations.

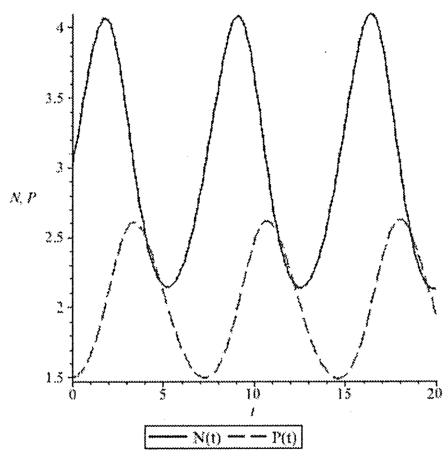
The model they considered is that of two associated species, of which one ($\equiv$the prey), finding sufficient food in its environment, would multiply indefinitely when left to itself, while the other ($\equiv$the predator) would perish for lack of nourishment if left alone; but the second feeds upon the first, and so the two species can coexist together.

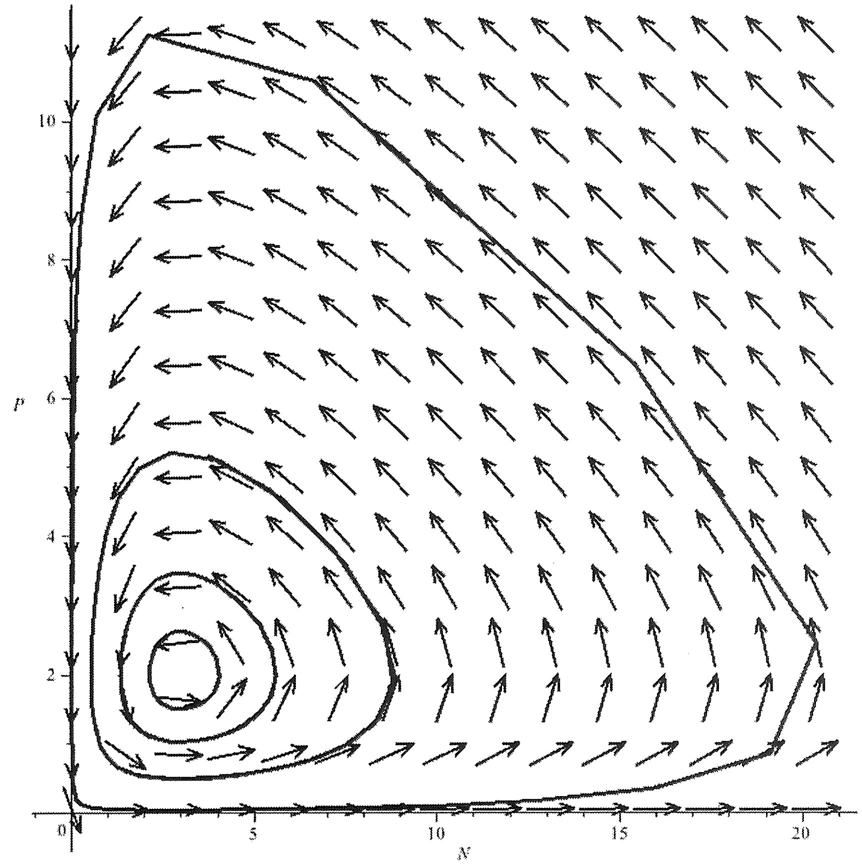
The proportional rate of increase of the eaten species diminishes as the number of individuals of the eating species increases, while augmentation of the eating species increases with the increase of the number of individuals of the eaten species.

Example 4

$$\begin{cases} \dfrac{dN}{dt} = N - 0.5PN \\ \dfrac{dP}{dt} = 0.25PN - 0.75P \end{cases}$$

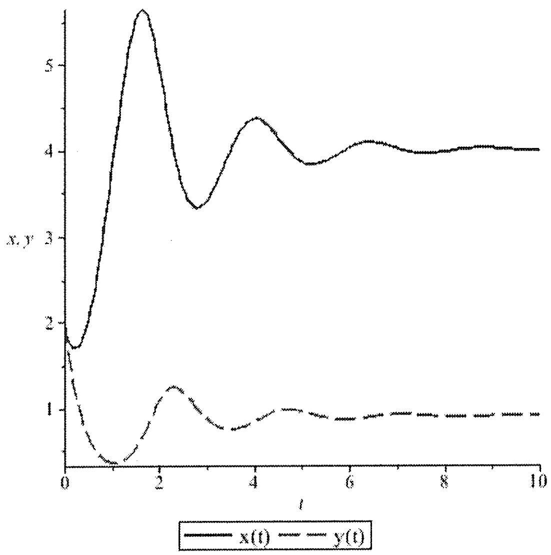


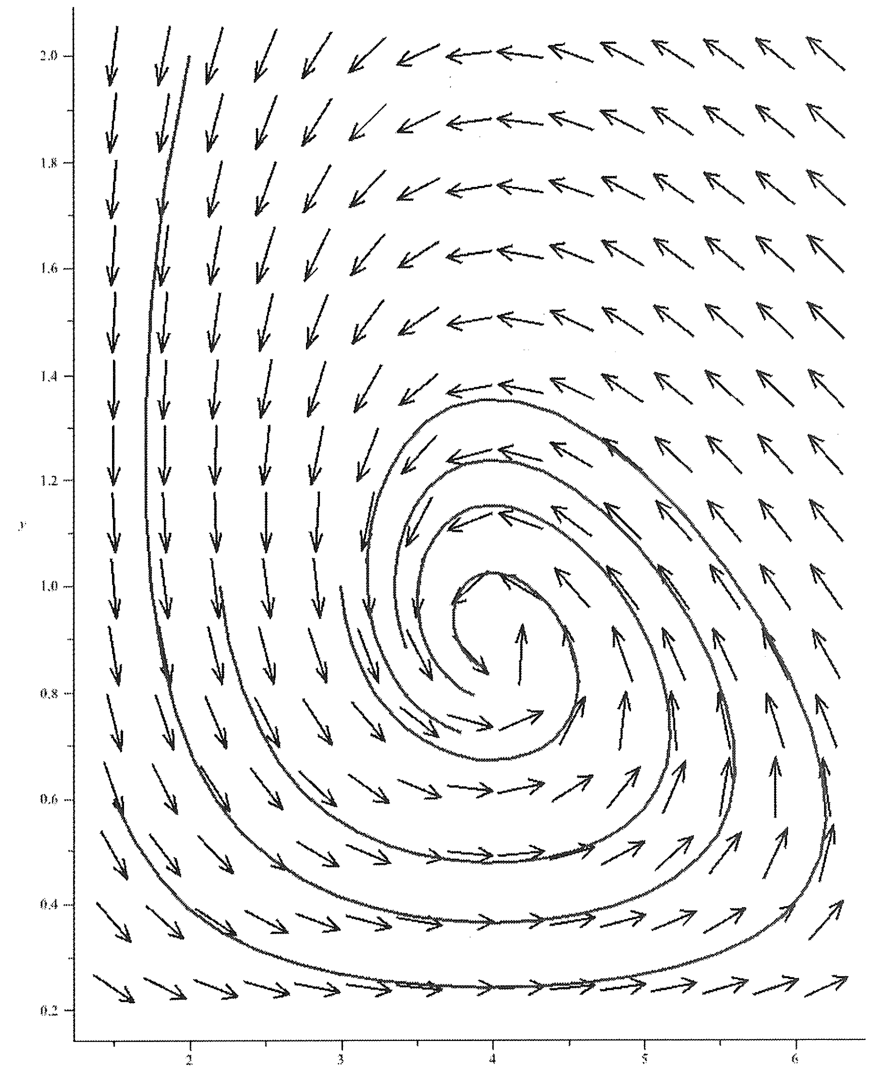




Example 5 (Modified Predator-Prey Model)

$$\begin{cases} \dfrac{dx}{dt} = \underbrace{3x\left(1-\dfrac{x}{10}\right)}_{\text{logistic growth}} - 2xy \\[2ex] \dfrac{dy}{dt} = xy - 4y \end{cases}$$





A Mathematical Model for Neuron Activity

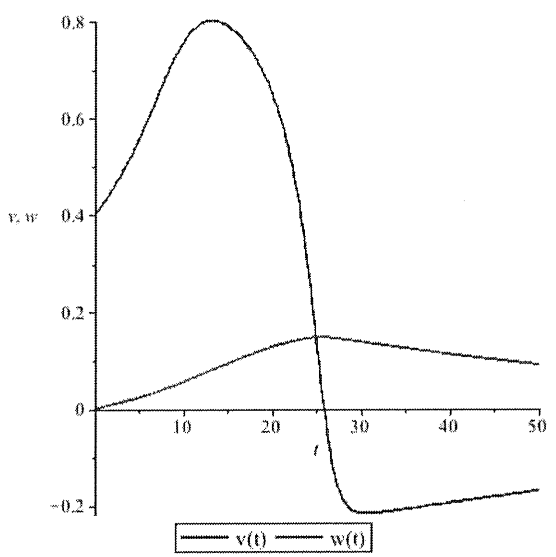
- The nervous system of an organism is a communication network that allows the rapid transmission of information between cells.
- The nervous system consists of nerve cells called **neurons**. A typical neuron has a cell body that contains the cell nucleus and nerve fibers. Nerve fibers that receive information are called **dendrites**, whereas those that transport information are called **axons**; the latter provide links to other neurons via **synapses**.
- Neurons respond to electrical stimuli. When the cell body of an isolated neuron is stimulated with a very mild electrical shock, the neuron shows no response; increasing the intensity of the shock beyond a certain threshold, however, will trigger a response, namely, an impulse that travels along the axon. Increasing the intensity of the electrical shock further does not change the response. The impulse is thus an all-or-nothing response.

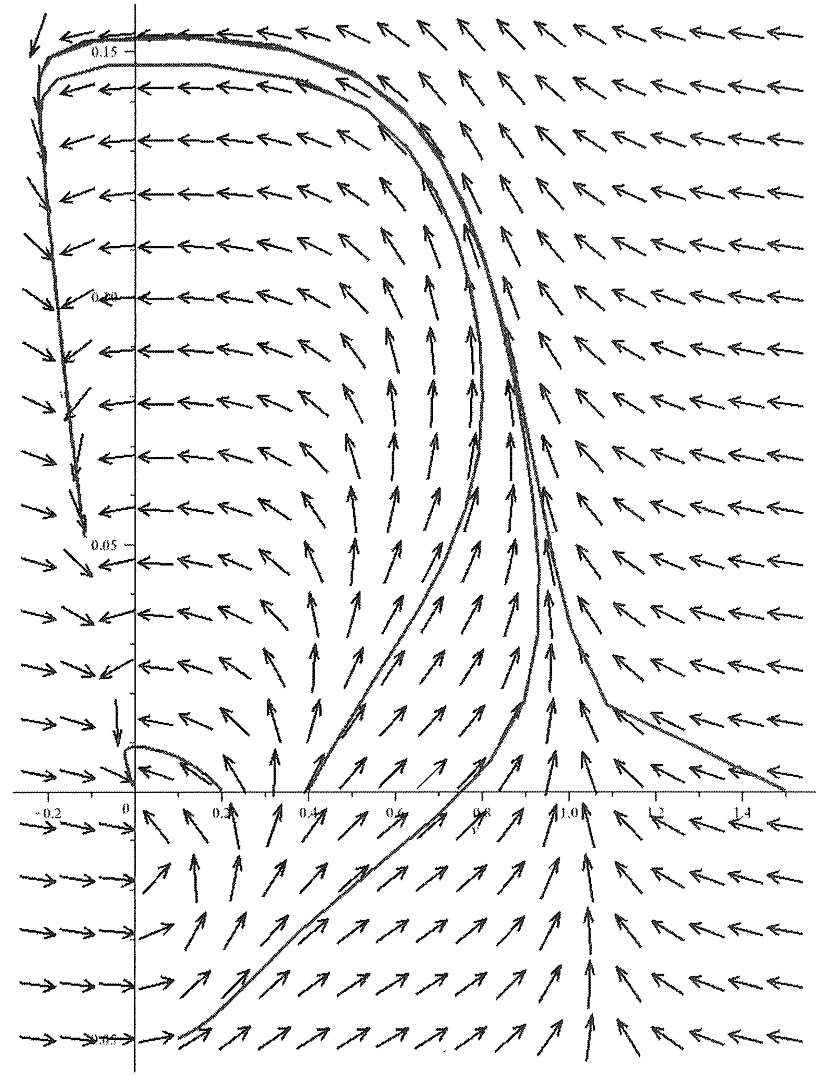
- **A.L. Hodgkin** and **A.F. Huxley** studied the giant axon of a squid experimentally and developed a mathematical model for neuron activity.
- The main players in the functioning of a neuron are sodium (Na^+) and potassium (K^+) ions.
- The model consists of a system of four autonomous differential equations and it is a phenomenological model. That is, the equations are based on fitting curves to experimental data for the various components of the model.
- One equation describes the change of voltage on the cell surface, two equations describe the sodium channel, and one equation describes the potassium channel.
- Their work appeared in a series of papers in 1952. In 1963, Hodgkin and Huxley were awarded the Nobel Prize in Physiology or Medicine for their work on neurons.

- The **Fitzhugh-Nagumo model** is based on the fact that the time scales of the two channels are quite different. The sodium channel works on a much faster time scale than the potassium channel.
- This fact led Fitzhugh and Nagumo to assume that the sodium channel is essentially always in a steady state, an assumption that allowed them to reduce the four equations of the Hodgkin and Huxley model to two.
- The Fitzhugh-Nagumo model is thus an approximation to the Hodgkin and Huxley model, retaining the essential features of the action potential, but much easier to analyze.
- The Fitzhugh-Nagumo model is described by two variables. One variable, denoted by V, describes the potential of the cell surface. The other variable, denoted by w, models the sodium and potassium channels.

Example 6 (Fitzhug-Nagumo Model)

$$\begin{cases} \dfrac{dV}{dt} = -V(V-0.3)(V-1) - w \\ \dfrac{dw}{dt} = 0.01V - 0.004w \end{cases}$$

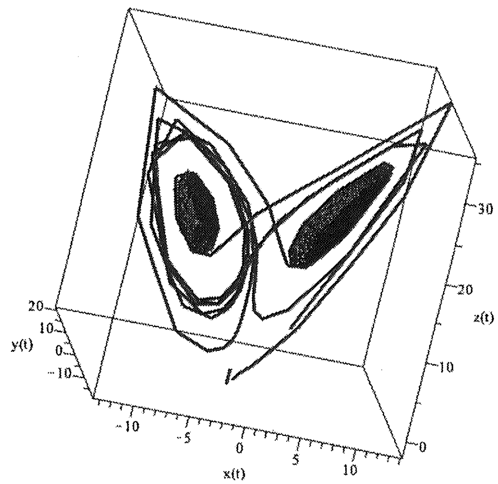




Lorenz Equations

- In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection.
- The model is a system of three ordinary differential equations now known as the **Lorenz equations**.
- This model is notable for having chaotic solutions for certain parameter values and initial conditions.
- In particular, the **Lorenz attractor** is a set of chaotic solutions of the Lorenz system which, when plotted, resemble a butterfly or figure eight.

$$\begin{cases} \dfrac{dx}{dt} &= \sigma(y-x) \\ \dfrac{dy}{dt} &= x(\rho - z) - y \\ \dfrac{dz}{dt} &= yx - \beta z \end{cases}$$



- Here x, y, and z make up the system state, t is time, and σ, ρ, β are the system parameters.
- The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions, and forward osmosis.
- One normally assumes that the parameters σ, ρ, and β are positive.
- Lorenz used the values $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$. The system exhibits **chaotic behavior** for these values.