MA 138 – Calculus 2 with Life Science Applications
Integration by Parts
(Section 7.2)

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Section 7.2: Integration by Parts

We saw that **integration by parts is the product rule in integral form.**

We also recall the following **general formula**:

Rule for Integration by Parts

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx;$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 1

A particle that moves along a straight line has velocity

$$v(t) = t^2 e^{-2t}$$

meters per second after t seconds.
How many meters will it travel during the first t seconds?

Using the notion of cumulative change

$$\underbrace{s(t) - s(0)}_{\text{distance traveled}} = \int_0^t \frac{ds}{du}\cdot du = \int_0^t v(u)\cdot du$$

$$= \int_0^t u^2 \cdot e^{-2u}\cdot du \quad = \text{set} \quad w = -2u \quad \frac{dw}{du} = -2$$

$$\boxed{u = -\tfrac{1}{2}w} \qquad \boxed{du = -\tfrac{1}{2}dw}$$

Substitute and change the integration limits:

$$= \int_0^{-2t} \left(-\tfrac{1}{2}w\right)^2 \cdot e^w \cdot \left(-\tfrac{1}{2}dw\right) = -\int_0^{-2t} \tfrac{1}{8} w^2 e^w\,dw =$$

$$= \int_{-2t}^{0} \tfrac{1}{8} w^2 e^w\,dw \quad = \text{if you prefer relabel} \quad w \longleftrightarrow x$$

$$\therefore \quad s(t) - s(0) = \int_{-2t}^{0} \underbrace{\tfrac{1}{8}x^2}_{f} \underbrace{e^x}_{g'} dx = \text{integration by parts}$$

$$= \tfrac{1}{8}x^2 \cdot e^x \Big]_{-2t}^{0} - \int_{-2t}^{0} \tfrac{1}{8}\, 2x \cdot e^x \, dx$$

$$= \left(0 - \tfrac{1}{8}(-2t)^2 e^{-2t}\right) - \int_{-2t}^{0} \tfrac{1}{4}\, x\, e^x \, dx$$

integrate by parts again

$$= -\tfrac{1}{2}t^2 e^{-2t} - \left\{ \left[\tfrac{1}{4} x e^x\right]_{-2t}^{0} - \int_{-2t}^{0} \tfrac{1}{4} e^x \, dx \right\}$$

$$= -\tfrac{1}{2}t^2 e^{-2t} - \left\{ \left(0 - \tfrac{1}{4}(-2t) e^{-2t}\right) - \left[\tfrac{1}{4} e^x\right]_{-2t}^{0} \right\}$$

$$= -\tfrac{1}{2}t^2 e^{-2t} - \tfrac{1}{2} t e^{-2t} + \left[\tfrac{1}{4}e^0 - \tfrac{1}{4}e^{-2t}\right]$$

$$\boxed{= \tfrac{1}{4} - \left(\tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{4}\right) e^{-2t}}$$

Example 2 (Online Homework # 10)

Suppose that $f(1) = 4$, $f(4) = 6$, $f'(1) = -5$, $f'(4) = -5$, and f'' is continuous. Find the value of

$$\int_1^4 x f''(x)\, dx.$$

$$\int_1^4 x f''(x)\, dx = ?$$

$f(1) = 4$

$f(4) = 6$

$f'(1) = -5$

$f'(4) = -5$

f'' continuous

integrate by parts:

$$\int_1^4 x \underbrace{(f')'}_{f''} dx = \Big[x f'\Big]_1^4 - \int_1^4 \underbrace{1 \cdot f' \, dx}_{\to \text{F.T.C.}} =$$

$$= \Big[x f'(x)\Big]_1^4 - \Big[f(x)\Big]_1^4 =$$

$$= \Big[4 \cdot f'(4) - 1 \cdot f'(1)\Big] - \Big[f(4) - f(1)\Big]$$

$$= 4(-5) - (-5) - (6 - 4) = \boxed{-17}$$

Example 3 (Problem # 8, Section 7.2, page 342)

Evaluate the indefinite integral: $\displaystyle\int 3xe^{-x/2}\, dx.$

$$\int 3x\, e^{-x/2}\, dx = \text{set } \boxed{u = -\frac{x}{2}} \qquad \frac{du}{dx} = -\frac{1}{2}$$

$$\therefore \boxed{x = -2u}, \quad \boxed{-2\, du = dx}; \quad \text{substitute}$$

$$= \int -6u\, e^{u} (-2\, du) = \int \underbrace{12u}_{f}\, \underbrace{e^{u}}_{g'}\, du \underset{\text{by parts}}{=}$$

$$= \underbrace{12u}_{f} \cdot \underbrace{e^{u}}_{g} - \int \underbrace{12}_{f'}\, \underbrace{e^{u}}_{g}\, du = 12u\, e^{u} - 12 e^{u} + C$$

$$= \left(12\left(-\frac{x}{2}\right) - 12\right) e^{-x/2} + C = \boxed{-6(x+2)e^{-x/2} + C}$$

Example 4 (Online Homework # 7)

Find the integral: $\displaystyle\int_0^1 x^2 \sqrt[4]{e^x}\, dx.$

$$\int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^1 x^2 (e^x)^{1/4}\, dx = \int_0^1 x^2 e^{x/4} \cdot dx$$

Use the substitution $\boxed{u = \frac{x}{4}}$ so $\boxed{x = 4u}$

and $\frac{du}{dx} = 1/4$ so $\boxed{4\, du = dx}$. Change also the limits of integration

$$\therefore \int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^{1/4} (4u)^2 \cdot e^u \cdot (4\, du) = \boxed{\int_0^{1/4} 64\, u^2 \cdot e^u\, du}$$

$$\int \underbrace{64 u^2}_{f} \cdot \underbrace{e^u}_{g'} \cdot du \underset{\text{by parts}}{=} \underbrace{64u^2}_{f} \cdot \underbrace{e^u}_{g} - \int \underbrace{128u}_{f'} \cdot \underbrace{e^u}_{g}\, du =$$

$$\underset{\text{by parts again}}{=} 64 u^2 e^u - \left[128u\, e^u - \int 128 \cdot e^u\, du\right]$$

In conclusion:

$$\int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^{1/4} 64\, u^2 \cdot e^u\, du = \Big[64u^2 e^u - 128u e^u + 128 e^u\Big]_0^{1/4}$$

$$= \Big[64(u^2 - 2u + 2)e^u\Big]_0^{1/4}$$

$$= 64\left(\frac{1}{16} - \frac{1}{2} + 2\right)e^{1/4} - 64(2) \cdot e^0 =$$

$$= 64\left(\frac{1 - 8 + 32}{16}\right) e^{1/4} - 128 = \underline{100 e^{1/4} - 128}$$

$$\approx \underline{0.4025}$$

Example 5 (Problem # 35, Section 7.2, page 343)

Evaluate the indefinite integral: $\displaystyle\int \frac{1}{x} \ln x \, dx.$

(1) Our textbook suggests to compute $\int \frac{1}{x} \ln x \, dx$ using integration by parts.

$$\int \underbrace{\frac{1}{x}}_{g'} \cdot \underbrace{\ln x}_{f} \, dx = \underbrace{(\ln x)}_{g} \cdot \underbrace{\ln(x)}_{f} - \int \underbrace{(\ln x)}_{g} \cdot \underbrace{\frac{1}{x}}_{f'} \, dx$$

move this to the right-hand side

$$\therefore \quad 2\int \frac{1}{x} \cdot \ln x \, dx = (\ln x)^2 + C$$

$$\therefore \quad \boxed{\int \frac{1}{x} \ln x \, dx = \frac{1}{2}(\ln x)^2 + \tilde{C}}$$

(2) Using the substitution $\boxed{u = \ln x}$ so $\boxed{du = \frac{1}{x} dx}$

we also get: $\displaystyle\int \frac{1}{x} \cdot \ln x \, dx = \int u \cdot du = \frac{1}{2} u^2 + \tilde{C} =$

$$= \boxed{\frac{1}{2}(\ln x)^2 + \tilde{C}}$$

Example 6 (Problem # 48, Section 7.2, page 343)

Evaluate the definite integral: $\displaystyle\int_0^1 x^3 \ln(x^2+1) \, dx.$

$\displaystyle\int_0^1 x^3 \cdot \ln(x^2+1) \, dx =$ use first the substitution $\boxed{u = x^2+1}$

so that $\frac{du}{dx} = 2x$ or $\boxed{\frac{1}{2} du = x \, dx}$

and observe $\boxed{x^2 = u-1}$

Substitute and change the limits of integration

$$\int_0^1 x^2 \cdot \ln(x^2+1) \cdot x \, dx = \int_1^2 (u-1) \cdot \ln(u) \cdot \frac{1}{2} du =$$

$$= \int_1^2 \underbrace{\left(\frac{1}{2}u - \frac{1}{2}\right)}_{g'} \cdot \underbrace{\ln u}_{f} \, du \underset{\text{by parts}}{=} \left[\left(\frac{1}{4}u^2 - \frac{1}{2}u\right)\ln u\right]_1^2 - \int_1^2 \left(\frac{1}{4}u^2 - \frac{1}{2}u\right)\frac{1}{u} du$$

$$= \left(\frac{1}{4}u^2 - \frac{1}{2}u\right)\ln u\Big]_1^2 - \int_1^2 \left(\frac{1}{4}u - \frac{1}{2}\right) du =$$

$$= \left[\underbrace{\left(\frac{1}{4}\cdot 2^2 - \frac{1}{2}\cdot 2\right)}_{0} \cdot \ln(2) - \left(\frac{1}{4}(1)^2 - \frac{1}{2}(1)\right) \cdot \underbrace{\ln(1)}_{0}\right] - \left[\frac{1}{8}u^2 - \frac{1}{2}u\right]_1^2$$

$$= -\left[\left(\frac{1}{8}\cdot 4 - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right] = \boxed{\frac{1}{8}} \approx 0.125$$

Useful aside: Trigonometric addition formulas

- We also used the double angle formulas

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

to compute $\int \cos^2 x\, dx$ and $\int \sin x \cos x\, dx$.

- Is there a 'simple' way to remember formulas of this kind?
- **Euler's formula** establishes the fundamental relationship between the trigonometric functions and the complex exponential function. It states that, for any real number x,

$$\boxed{e^{ix} = \cos x + i\sin x,}$$

where i is the imaginary unit ($i^2 = -1$).

- For any α and β, using Euler's formula, we have

$$\begin{aligned}\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} \\ &= e^{i\alpha}\cdot e^{i\beta} \\ &= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta + i^2\sin\alpha\sin\beta) \\ &\quad + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

- Thus, by comparing the terms, we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

- Thus, by setting $\alpha = \beta$, we obtain

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\beta.$$