# MA 138 – Calculus 2 with Life Science Applications Rational Functions and Partial Fractions (Section 7.3)

#### Alberto Corso

(alberto.corso@uky.edu)

Department of Mathematics University of Kentucky

Wednesday, January 25, 2017

http://www.ms.uky.edu/~ma138

Lecture 6

(1) 
$$\int \frac{5}{(3x+2)^4} dx \qquad \text{ set } \boxed{u = 3x+2} \qquad \text{so + 0 at } \frac{du}{dx} = 3$$

$$\boxed{dx = \frac{1}{3} du}$$

hence if we substitute back
$$= \int \frac{5}{u^4} \cdot \frac{1}{3} du = \int \frac{5}{3} u^{-4} du = \frac{5}{3} \cdot \frac{1}{-3} u^{-3} + C$$

$$= -\frac{5}{9} \cdot \frac{1}{u^3} + C = \left[ -\frac{5}{9} \cdot \frac{1}{(3x+2)^3} + C \right]$$

(2) 
$$\int \frac{2x-2}{(x^2-2x+5)^3} dx \qquad \text{set} \qquad \frac{u=x^2-2x+5}{du} \qquad \text{so}$$

$$\frac{du}{dx} = 2x-2 \qquad (2x-2)dx = du$$
hence
$$= \int \frac{du}{u^3} = \int u^{-3} du = -\frac{1}{2}u + C = -\frac{1}{2u^2} + C$$

$$= \left[-\frac{1}{2(x^2-2x+5)^2} + C\right]$$

## Example 1

Evaluate the following indefinite integrals

http://www.ms.uky.edu/~ma138

Lecture 6

## Section 7.3: Rational Functions and Partial Fractions

 $\blacksquare$  A rational function f is the quotient of two polynomials. That is,

$$f(x) = \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomials.

- To integrate such a function, we write f(x) as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form

$$\frac{A}{(ax+b)^n}$$
 or  $\frac{Bx+C}{(ax^2+bx+c)^n}$ 

where A, B, C, a, b, and c are constants and n is a positive integer.

■ In this form, the quadratic polynomial  $ax^2 + bx + c$  can no longer be factored into a product of two linear functions with real coefficients.

http://www.ms.uky.edu/~ma138

Lecture

## **Proper Rational Functions**

■ The rational function f(x) = P(x)/Q(x) is said to be **proper** if the degree of the polynomial in the numerator, P(x), is strictly less than the degree of the polynomial in the denominator, Q(x),

$$f(x) = \frac{P(x)}{Q(x)}$$
 proper  $\iff$  deg  $P(x) <$  deg  $Q(x)$ .

■ Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x + 2} \qquad f_3(x) = \frac{2x - 3}{x^2 + x}$$
 is proper? Only  $f_3(x)$  is proper.

■ The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write f(x) as a sum of a polynomial and a **proper** rational function.

http://www.ms.uky.edu/~ma138

Lecture 6

## Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4$$
 by  $B(x) = x - 3$ .

(Complete the above table and check your work!)

http://www.ms.uky.edu/~ma138

#### Lecture 6

## Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

#### Long Division Algorithm

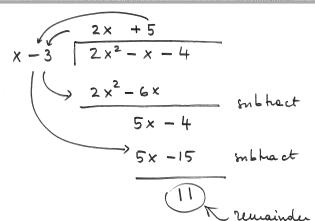
If A(x) and B(x) are polynomials, with  $B(x) \neq 0$ , then there exist unique polynomials Q(x) and R(x), where R(x) is either 0 or of degree strictly less than the degree of B(x), such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials A(x) and B(x) are called the **dividend** and **divisor**, respectively; Q(x) is the **quotient** and R(x) is the **remainder**.

http://www.ms.uky.edu/~ma138

Lecture 6



This means that

$$2x^{2}-x-4 = (x-3)\cdot(2x+5) + 11$$

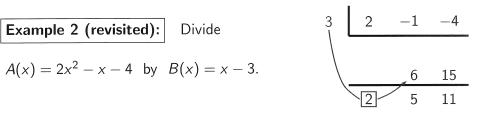
$$cleck: (2x^{2}+5x-6x-15)+11$$

$$= 2x^{2}-x-4$$

- Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form x - c, where c is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial A(x) by writing only its coefficients.

Moreover, instead of B(x) = x - c, we simply write 'c.' Writing c instead of -c allows us to add instead of subtract!

## Example 2 (revisited): Divide

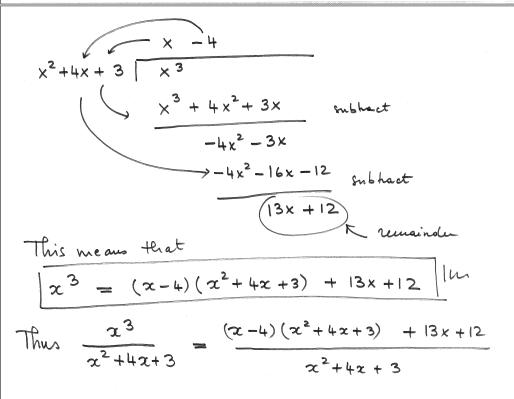


We obtain Q(x) = 2x + 5 and R(x) = 11. That is,

$$2x^2 - x - 4 = (2x + 5)(x - 3) + 11.$$

http://www.ms.uky.edu/~ma138

Lecture 6



## **Example 3** (Online Homework # 3)

Use the Long Division Algorithm to write f(x) as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

http://www.ms.uky.edu/~ma138

Now split the faction on the right hand side as the sum of 2 factions:  $\frac{x^{3}}{x^{2}+4x+3} = \frac{(x-4)(x^{2}+4x+3)}{x^{2}+4x+3} + \frac{13x+12}{x^{2}+4x+3}$ rational fraction

## Partial Fraction Decomposition (linear factors)

#### Case of Distinct Linear Factors

Q(x) is a product of m distinct linear factors. Q(x) is thus of the form

$$Q(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_m)$$

where  $\alpha_1, \alpha_2, \dots, \alpha_m$  are the *m* distinct roots of Q(x).

The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left[ \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \dots + \frac{A_m}{x - \alpha_m} \right]$$

We will see in the next examples how the constants  $A_1, A_2, \ldots, A_m$  are determined.

http://www.ms.uky.edu/~ma138

Lecture 6

We have shown before that
$$\frac{x^3}{x^2+4x+3} = x-4 + \frac{13x+12}{x^2+4x+3}$$

Thus :

$$\int \frac{x^3}{x^2 + 4x + 3} dx = \int (x - 4) dx + \int \frac{13x + 12}{x^2 + 4x + 3} dx$$

$$\int \frac{eary}{x^2 + 4x + 3} dx = \int \frac{13x + 12}{x^2 + 4x + 3} dx$$

$$= \frac{1}{2}x^2 - 4x + C \qquad \text{this}$$

Notice: 
$$\frac{13x+12}{x^2+4x+3} = \frac{13x+12}{(2+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1}$$
want
for some constants

## Example 3 (cont.d)

Evaluate the indefinite integral:  $\int \frac{x^3}{x^2 + 4x + 3} dx.$ 

**Note:** from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4) dx + \int \frac{13x+12}{(x+3)(x+1)} dx.$$

http://www.ms.uky.edu/~ma138

Lecture 6

This mean that 
$$\frac{13 \times +12}{2^2 + 4z + 3} = \frac{27}{2} \cdot \frac{1}{2+3} - \frac{1}{2} \cdot \frac{1}{2+1}$$

Thus

$$\int \frac{13x+12}{x^2+4x+3} dx = \frac{27}{2} \int \frac{1}{x+3} dx - \frac{1}{2} \int \frac{1}{z+1} dz$$
$$= \frac{27}{2} \ln|x+3| - \frac{1}{2} \ln|z+1| + C$$

Thus:

$$\int \frac{x^3}{x^2+4x+3} = \frac{\frac{1}{2}x^2-4x}{2^2 + \frac{27}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C}$$

## **Example 4** (Online Homework # 8)

Find the integral:  $\int_{2}^{3} \frac{2}{x^2 - 1} dx.$ 

## (Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

A(x+1) + B(x+3) = 13x + 12 (\*)

Set x = -1 in (\*). We obtain

$$0 + B \cdot (-1 + 3) = 13(-1) + 12$$

$$B \cdot (2) = -1$$

$$B = -1/2$$

Set x = -3 in (\*). We obtain

$$A \cdot 0 + B \cdot (-1 + 3) = 13(-1) + 12$$
  $A \cdot (-3 + 1) + 0 = 13(-3) + 12$ 

 $A\cdot (-2)=-27$ 

$$A = 27/2$$

http://www.ms.uky.edu/~ma138

Lecture 6

Counda the fraction  $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$ 

We want to decomposit as:

$$\frac{2}{x^{2}-1} = \frac{A}{x-1} + \frac{B}{x+1} = give common dens uni volon$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

Thus 2 = A(x+1) + B(x-1)

evaluate at 
$$\boxed{z=1}$$
:  $2=2\cdot A+B\cdot O$  ...  $\boxed{A=1}$ 

evaluate at 
$$[x=-1]$$
:  $2 = A \cdot (0) + B(-2)$  :  $B=-1$ 

Thus: 
$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

http://www.ms.uky.edu/~ma138

Thus: 
$$\int \frac{2}{x^2-1} dx = \int \frac{1}{z-1} dz - \int \frac{1}{z+1} dz$$

$$= \ln |z-1| - \ln |z+1| + C$$

$$= \ln \left| \frac{x-1}{z+1} \right| + C$$

Finally:
$$\int_{2}^{5} \frac{2}{x^{2}-1} dx = \lim_{|x-1| \ge 2} \frac{1}{2} = \lim_{|x-1| \ge 2}$$

**Example 5** (Online Homework # 6)

Evaluate the indefinite integral:  $\int \frac{1}{x(x+1)} dx$ .

http://www.ms.uky.edu/~ma138

We need to decompose the faction  $\frac{1}{x(x+1)}$  as  $\frac{A}{x} + \frac{B}{x+1}$ . Thus:

$$\frac{1}{\alpha(z+1)} = \frac{\Delta}{\alpha} + \frac{B}{z+1} = \frac{\Delta(z+1) + B \cdot z}{\alpha(z+1)}$$

Thus 
$$1 = A(x+1) + Bx$$

• evaluate at  $x=0$ 

$$1 = A \cdot 1 + B \cdot 0$$

$$A=1$$
• evaluate at  $x=-1$ 
• evaluate at  $x=-1$ 

 $l = A \cdot 0 + B(-1)$ 

No matter which method we choose, we obtain  $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ 

Thus
$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

$$= \ln |x| - \ln |x+1| + C$$

$$= \ln \left| \frac{x}{x+1} \right| + C$$