

MA 138 – Calculus 2 with Life Science Applications

Linear Maps

(Section 9.3)

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<http://www.ms.uky.edu/~ma138>

Lecture 24

Outline

- We mostly focus on 2×2 matrices, but point out that we can generalize our discussion to arbitrary $n \times n$ matrices.
- Consider a map of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or, in short,} \quad \mathbf{v} \mapsto A\mathbf{v}$$

where A is a 2×2 matrix and \mathbf{v} is a 2×1 (column) vector.

- Since $A\mathbf{v}$ is a 2×1 vector, this map takes a 2×1 vector and maps it into a 2×1 vector. This enables us to apply A repeatedly: We can compute $A(A\mathbf{v}) = A^2\mathbf{v}$, which is again a 2×1 vector, and so on.
- We will **first** look at vectors \mathbf{v} , **then** at maps $\mathbf{v} \mapsto A\mathbf{v}$, and **finally** at iterates of the map A (i.e., $A^2\mathbf{v}$, $A^3\mathbf{v}$, and so on).

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Graphical Representation of (Column) Vectors

We assume that $\mathbf{v} = \begin{bmatrix} x_v \\ y_v \end{bmatrix}$ is a 2×1 matrix.

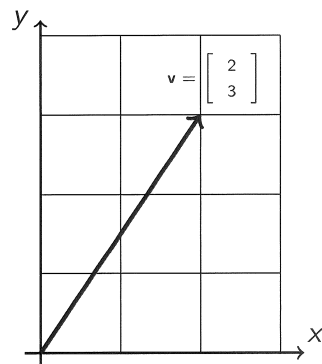
We call \mathbf{v} a column vector or simply a **vector**.

Since a 2×1 matrix has just two components, we can represent a vector in the plane.

For instance, to represent the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

in the x - y plane, we draw an arrow from the origin $(0, 0)$ to the point $(2, 3)$.



Addition of Vectors

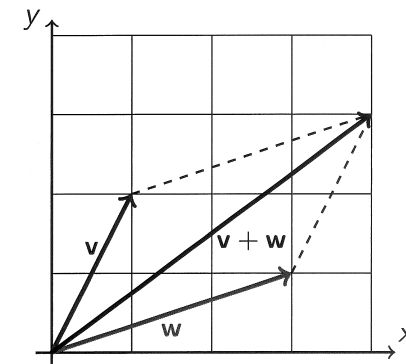
Because vectors are matrices, we can add vectors using matrix addition.

For instance,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

This vector sum has a simple geometric representation. The sum $\mathbf{v} + \mathbf{w}$ is the diagonal in the parallelogram that is formed by the two vectors \mathbf{v} and \mathbf{w} .

The rule for vector addition is therefore referred to as the **parallelogram law**.



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Length of Vectors

The length of the vector $\mathbf{v} = \begin{bmatrix} x_v \\ y_v \end{bmatrix}$, denoted by $\|\mathbf{v}\|$, is the distance from the origin $(0,0)$ to the point (x_v, y_v) .

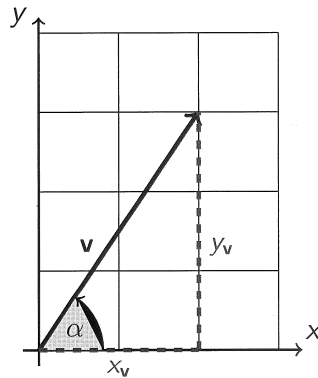
By Pythagoras Theorem we have

$$\text{length of } \mathbf{v} = \|\mathbf{v}\| = \sqrt{x_v^2 + y_v^2}$$

We define the direction of \mathbf{v} as the angle α between the positive x -axis and the vector \mathbf{v} . The angle α is in the interval $[0, 2\pi)$ and satisfies $\tan \alpha = y_v/x_v$.

We thus have two distinct ways of representing vectors in the plane: We can use

- either the endpoint (x_v, y_v)
- or the length and direction $(\|\mathbf{v}\|, \alpha)$.



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Scalar Multiplication of Vectors

Multiplication of a vector by a scalar is carried out componentwise.

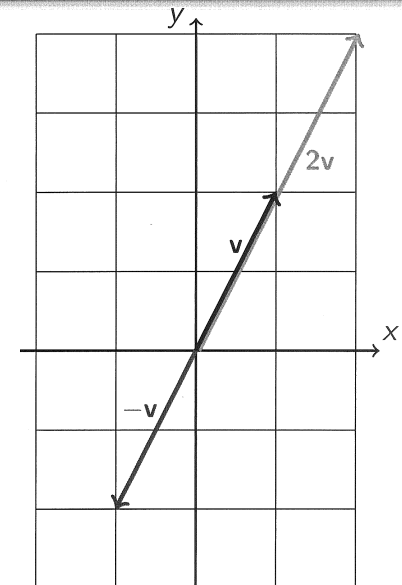
If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 2, we get

$2\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This operation corresponds to

changing the length of the vector by the factor 2.

If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by -1 , then the resulting vector is $-\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, which

has the same length as the original vector, but points in the opposite direction.



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Linear Maps (also called Linear Transformations)

We start with a graphical approach to study maps of the form

$$\mathbf{v} \mapsto A\mathbf{v}$$

where A is a 2×2 matrix and \mathbf{v} is a 2×1 vector.

Since $A\mathbf{v}$ is a 2×1 vector as well, the map A takes the 2×1 vector \mathbf{v} and maps it to the 2×1 vector $A\mathbf{v}$ can be thought of as a map from the plane \mathbb{R}^2 to the plane \mathbb{R}^2 .

We will discuss simple examples of maps from \mathbb{R}^2 into \mathbb{R}^2 defined by $\mathbf{v} \mapsto A\mathbf{v}$, that take the vector \mathbf{v} and rotate, stretch, or contract it.

For an arbitrary matrix A , vectors may be moved in a way that has no simple geometric interpretation.

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Example 1 (Reflections)

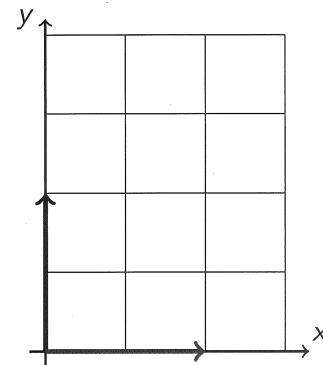
Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



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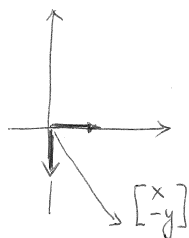
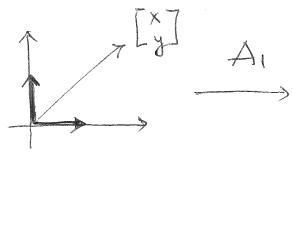
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$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



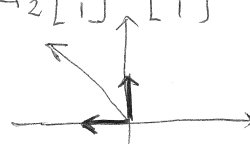
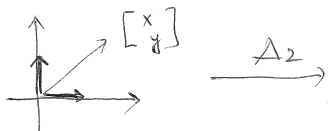
$\therefore A_1$ is a reflection about the x-axis

$$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$A_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



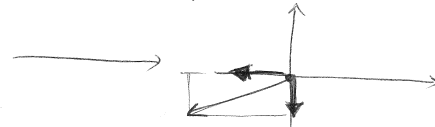
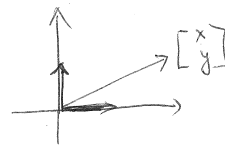
$\therefore A_2$ is a reflection about the y-axis

$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto A_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto A_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$



A_3 is $-\mathbb{I}_2$ that is it sends every vector into its opposite.

Finally, $A_4: \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ -x \end{bmatrix} = -\begin{bmatrix} y \\ x \end{bmatrix}$$

i.e. we reflect a vector about the $y=x$ line and then we send it to its opposite.

Example 2 (Contractions or Expansions)

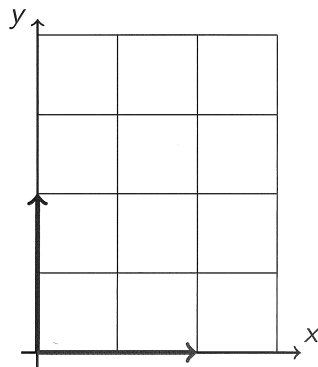
Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$



$$A_1: \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

that is A_1 doubles the x-coordinate of a vector (it is an expansion in the x-direction)

A_2 : it is a contraction by a factor of 2 in the y-direction. i.e.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y/2 \end{bmatrix}$$

A_3 multiplies the x-component of a vector by a factor "a". It all depends if a is positive or negative

It also depends whether $|a| < 1$ or $|a| > 1$.
contraction or expansion

A_4 : the x component of a vector is sent into ax
 the y-component of a vector is sent into bx

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax \\ by \end{bmatrix}$$

Example 3 (Shears)

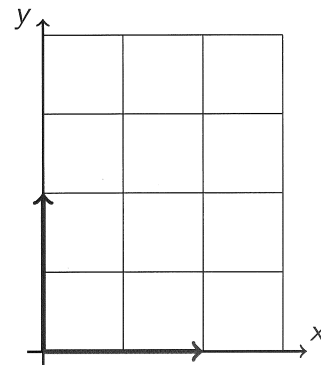
Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

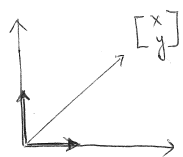


$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

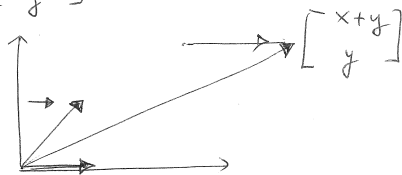
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ y \end{bmatrix}$$



A_1



the y-component of the vector remains the same whereas the x-component of the vector gets moved to the right of y units.

Same as A_1 , but

$$A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix}$$

shift in the x component
 suppose $a > 0$

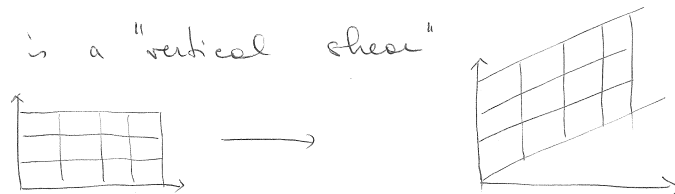
$$A_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-ay \\ y \end{bmatrix}$$

negative shift
 so the components are moved to the left if $a > 0$

$$A_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ bx+y \end{bmatrix}$$

it is now the y-component of the vector that gets changed by adding.

There is a "vertical shear"



Example 4

Sir D'Arcy Wentworth Thompson (May 2, 1860 - June 21, 1948) was a Scottish biologist, mathematician, and classics scholar. He was a pioneer of mathematical biology. Thompson is remembered as the author of the distinctive 1917 book *On Growth and Form*.

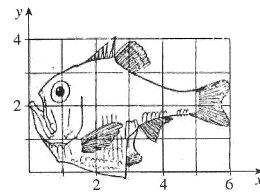
The book led the way for the scientific explanation of morphogenesis, the process by which patterns are formed in plants and animals.

For example, Thompson illustrated the transformation of *Argyropelecus offers* into *Sternoptyx diaphana* by applying a 20° shear mapping (\equiv transvection). What is the form of the matrix that describes this change?

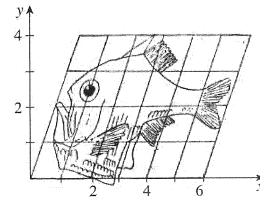
(source: WIKIPEDIA)

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Argyropelecus offers



Sternoptyx diaphana

From the discussion in Example 3

there is a horizontal shear.

the y -components of the vectors are unchanged but we shift by a factor ay the x component of the vectors.

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+ay \\ y \end{bmatrix}$$

$$a = \tan(20^\circ) = 0.36397$$

$$\therefore \begin{bmatrix} 1 & 0.36397 \\ 0 & 1 \end{bmatrix}$$

Rotations

The following matrix rotates a vector in the x - y plane by an angle α :

$$R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

If $\alpha > 0$ the rotation is counterclockwise; if $\alpha < 0$ it is clockwise.

Properties of Rotations:

- $\det(R_\alpha) = \cos^2 \alpha + \sin^2 \alpha = 1$.
- A rotation by an angle α followed by a rotation by an angle β should be equivalent to a single rotation by a total angle $\alpha + \beta$. In fact, using the usual trigonometric identities, we have

$$\begin{aligned} R_\alpha R_\beta &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} = R_{\alpha + \beta} \end{aligned}$$

- The previous identity shows that the product of rotations is commutative: $R_\alpha R_\beta = R_\beta R_\alpha$.

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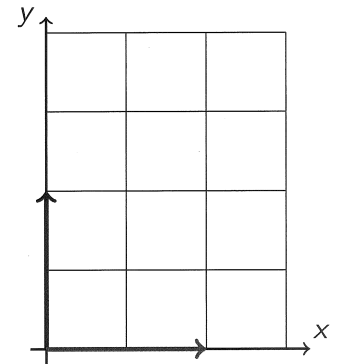
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Example 5 (Rotations)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



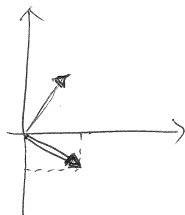
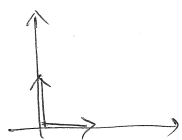
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$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$



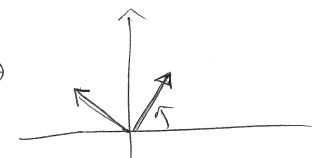
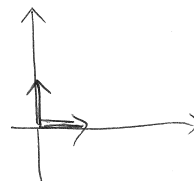
clockwise
rotation of
 $\frac{\pi}{6}$ ($= 30^\circ$)

or counterclockwise rotation of $-\frac{\pi}{6}$ ($= -30^\circ$)

$$B = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$



counterclockwise rotation of $\frac{\pi}{3}$ ($= 60^\circ$)

Properties of Linear Maps

According to the properties of matrix multiplication, the map $\mathbf{v} \mapsto A\mathbf{v}$ satisfies the following conditions:

- $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$, and
- $A(\lambda\mathbf{v}) = \lambda(A\mathbf{v})$, where λ is a scalar.

Because of these two properties, we say that the map $\mathbf{v} \mapsto A\mathbf{v}$ is **linear**.

Example 6 (Problem # 2, Section 9.3, p 486)

Show by direct calculation that $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$ and $A(\lambda\mathbf{v}) = \lambda(A\mathbf{v})$.

$$\begin{aligned} A(\mathbf{v} + \mathbf{w}) &= A\mathbf{v} + A\mathbf{w} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} \right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x+x' \\ y+y' \end{bmatrix} \\ & & &= \begin{bmatrix} a(x+x') + b(y+y') \\ c(x+x') + d(y+y') \end{bmatrix} \\ & & &= \begin{bmatrix} ax+bx' + by+by' \\ cx+cx' + dy+dy' \end{bmatrix} \\ & & &= \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} + \begin{bmatrix} ax'+by' \\ cx'+dy' \end{bmatrix} \\ & & &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A(\lambda\mathbf{v}) &= \lambda(A\mathbf{v}) & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\lambda \begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} a(\lambda x) + b(\lambda y) \\ c(\lambda x) + d(\lambda y) \end{bmatrix} \\ & & &= \begin{bmatrix} \lambda(ax+by) \\ \lambda(cx+dy) \end{bmatrix} = \lambda \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \\ & & &= \lambda \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \end{aligned}$$

Example 7

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find $A\mathbf{u}$ and $A\mathbf{v}$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$\bullet \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\mathbf{u}} \mapsto \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\bullet \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\mathbf{v}} \mapsto \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

That is \mathbf{u} and \mathbf{v} were special vectors

such that $A\mathbf{u} = -\mathbf{u}$ $A\mathbf{v} = 4\mathbf{v}$

Want to find $\boxed{A\mathbf{w} = \lambda\mathbf{w}}$ when possible!

Composition of Linear Maps \equiv Product of Matrices

Consider two linear maps $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$ given by the matrices A_f and A_g

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

That is the coordinates are transformed according to the rules

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \begin{cases} x'' = \alpha x' + \beta y' \\ y'' = \gamma x' + \delta y' \end{cases}$$

If we compose the two maps we obtain the transformation

$$\begin{cases} x'' = \alpha(ax + by) + \beta(cx + dy) = (\alpha a + \beta c)x + (\alpha b + \beta d)y \\ y'' = \gamma(ax + by) + \delta(cx + dy) = (\gamma a + \delta c)x + (\gamma b + \delta d)y \end{cases}$$

whose matrix representation corresponds to the product $A_g A_f$ of the two matrices

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{bmatrix}}_{A_{g \circ f}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f} \begin{bmatrix} x \\ y \end{bmatrix}$$