

MA 138 – Calculus 2 with Life Science Applications

Eigenvectors and Eigenvalues

(Section 9.3)

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Lecture 25

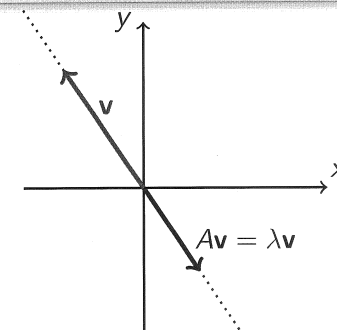
Eigenvalues and Eigenvectors

Definition

Assume that A is a square matrix. A nonzero vector \mathbf{v} that satisfies the equation

$$A\mathbf{v} = \lambda\mathbf{v} \quad (\mathbf{v} \neq \mathbf{0})$$

is an **eigenvector** of the matrix A , and the number λ is an **eigenvalue** of the matrix A .



- The zero vector $\mathbf{0}$ always satisfies the equation $A\mathbf{0} = \lambda\mathbf{0}$ for any choice of λ . Thus $\mathbf{0}$ is not special. That's why we assume $\mathbf{v} \neq \mathbf{0}$.
- The eigenvalue λ can be 0, though.
- **Geometric interpretation**, when the eigenvalue $\lambda \in \mathbb{R}$: If we draw a straight line through the origin in the direction of an eigenvector, then any vector on this straight line will remain on the line after the map A is applied.

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Example 1 (Online Homework # 5)

Determine if \mathbf{v} is an eigenvector of the matrix A :

(a) $A = \begin{bmatrix} -35 & -14 \\ 84 & 35 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 19 & 24 \\ -12 & -17 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$;

(c) $A = \begin{bmatrix} -3 & -10 \\ 5 & 12 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$.

(a) $\begin{bmatrix} -35 & -14 \\ 84 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ yes!

(b) $\begin{bmatrix} 19 & 24 \\ -12 & -17 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 86 \\ -58 \end{bmatrix} = 2 \begin{bmatrix} 43 \\ -29 \end{bmatrix}$ not an eigenvector.

(c) $\begin{bmatrix} -3 & -10 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$ not an eigenvalue.

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Example 2 (Online Homework # 6)

Given that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are eigenvectors

of the matrix $A = \begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix}$,

determine the corresponding eigenvalues λ_1 and λ_2 .

$$\begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

\therefore eigenvalue is $\lambda_1 = 4$

$$\begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

\therefore eigenvalue is $\lambda_2 = 1$

notice $\lambda_1 + \lambda_2 = 1 + 4 = 28 + (-23) = \text{trace}(A)$
 $\lambda_1 \lambda_2 = 1 \cdot 4 = 4 = \det(A)$
 $= 28(-23) - (36)(-18)$

Example 3 (Online Homework # 8)

Determine if λ is an eigenvalue of the matrix A:

(a) $A = \begin{bmatrix} 7 & -10 \\ 0 & -3 \end{bmatrix}$ and $\lambda = -2$;

(b) $A = \begin{bmatrix} -4 & -12 \\ 0 & 8 \end{bmatrix}$ and $\lambda = 3$;

(c) $A = \begin{bmatrix} -92 & 42 \\ -196 & 90 \end{bmatrix}$ and $\lambda = -8$.

(a)

$$\begin{bmatrix} 7 & -10 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \stackrel{?}{=} -2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for some $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We need $\begin{cases} 7v_1 - 10v_2 \stackrel{?}{=} -2v_1 \\ -3v_2 = -2v_2 \end{cases}$

$$\Rightarrow \begin{cases} v_2 = 0 \rightarrow 7v_1 + 2v_1 - 10v_2 = 0 \\ \rightarrow v_1 = 0 \end{cases}$$

impossible

$\lambda = -2$ is not an eigenvalue

(b) Is it possible to find $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that

$$\begin{bmatrix} -4 & -12 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \stackrel{?}{=} 3 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -4v_1 - 12v_2 \stackrel{?}{=} 3v_1 \\ 8v_2 \stackrel{?}{=} 3v_2 \end{cases}$$

$$\Rightarrow v_2 = 0 \Rightarrow v_1 = 0$$

$\therefore \lambda = 3$
is NOT
an eigenvalue

(c)
$$\begin{bmatrix} -92 & 42 \\ -196 & 90 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \stackrel{?}{=} -8 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -92v_1 + 42v_2 = -8v_1 \\ -196v_1 + 90v_2 = -8v_2 \end{cases}$$

$$\Rightarrow \begin{cases} -84v_1 + 42v_2 = 0 \\ -196v_1 + 98v_2 = 0 \end{cases}$$

simplify by 42 the first eq.

and by 98 the second eq.

$$\begin{cases} -2v_1 + v_2 = 0 \\ -2v_1 + v_2 = 0 \end{cases}$$

hence there is only one equation

$$\therefore v_2 = 2v_1$$

Any eigenvector

corresponding to $\lambda = -8$ is of the form $\begin{bmatrix} t \\ 2t \end{bmatrix}$
or $t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $t \in \mathbb{R}$

Finding the Eigenvalues

- We are interested in finding $\mathbf{v} \neq \mathbf{0}$ and λ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- We can rewrite this equation as $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$.
- In order to factor \mathbf{v} , we must multiply $\lambda\mathbf{v}$ by the identity matrix I_2 .
(In the $n \times n$ case we multiply instead by I_n . The procedure and outcome are exactly the same!)
- Multiplication by I_2 yields $A\mathbf{v} - \lambda I_2\mathbf{v} = \mathbf{0}$.
- We can now factor \mathbf{v} , resulting in $(A - \lambda I_2)\mathbf{v} = \mathbf{0}$.
- In Section 9.2, we showed that in order to obtain a nontrivial solution ($\mathbf{v} \neq \mathbf{0}$), the matrix $A - \lambda I_2$ must be singular; that is,

$$\det(A - \lambda I_2) = 0$$

Let us make the previous calculations more explicit (in the 2×2 case):

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_A \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \lambda \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}}$$

\Downarrow

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\Downarrow

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\Downarrow

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\Downarrow

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\Downarrow

$$\underbrace{\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}}_{A - \lambda I_2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation $\det(A - \lambda I_2) = 0$ that determines the eigenvalues of A is a polynomial equation in λ of degree two. This polynomial is referred to as the **characteristic polynomial** of A .

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ define

- $\text{trace}(A) = a + d$,
- $\det(A) = ad - bc$.

The characteristic polynomial has a simple form:

Characteristic Polynomial of the 2×2 Matrix A

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = 0$$

\Downarrow

$$(a - \lambda)(d - \lambda) - bc = 0$$

\Downarrow

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

Corollary

If λ_1 and λ_2 are the solutions of the characteristic polynomial, then they must satisfy

$$\text{trace}(A) = \lambda_1 + \lambda_2 \quad \det(A) = \lambda_1 \lambda_2.$$

Example 4 (Problem #53, Section 9.3, p. 487)

Consider the matrix $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

- Find its eigenvalues λ_1 and λ_2 .
- Find the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 associated with the eigenvalues from part (a).
- Graph the lines through the origin in the direction of the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 , together with the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 and the vectors $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

$$A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$$

- $\det(A - \lambda I_2) = 0 \iff$
 $\det \begin{bmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{bmatrix} = (-4 - \lambda)(1 - \lambda) + 6$
 $= \lambda^2 + 4\lambda - \lambda - 4 + 6 = \boxed{\lambda^2 + 3\lambda + 2 = 0}$
 $\therefore \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$
 $\therefore \boxed{\lambda_1 = -2}, \boxed{\lambda_2 = -1}$
- $\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ find $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} -4v_1 + 2v_2 = -2v_1 \\ -3v_1 + v_2 = -2v_2 \end{cases} \Rightarrow \begin{cases} -2v_1 + 2v_2 = 0 \\ -3v_1 + 3v_2 = 0 \end{cases}$$

Same equation

$$\Rightarrow -v_1 + v_2 = 0 \quad \text{or} \quad v_1 = v_2$$

Hence any eigenvector looks like $\begin{bmatrix} t \\ t \end{bmatrix}$
 for $t \in \mathbb{R}$ or $t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Just pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1$ we need to solve

$$\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \begin{cases} -4w_1 + 2w_2 = -w_1 \\ -3w_1 + w_2 = -w_2 \end{cases}$$

$$\Rightarrow \begin{cases} -3w_1 + 2w_2 = 0 \\ -3w_1 + 2w_2 = 0 \end{cases} \Rightarrow w_1 = \frac{2}{3}w_2$$

Thus if we pick $w_2 = 3t$ then

$$w_1 = \frac{2}{3}(3t) = 2t \quad \text{hence}$$

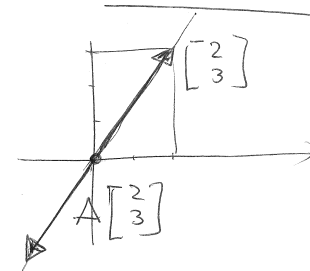
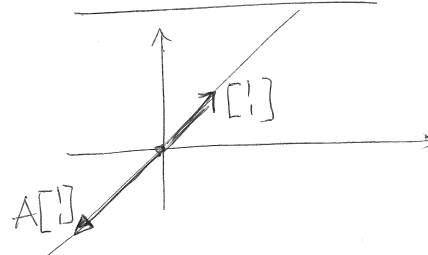
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \end{bmatrix} = t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{for any } t \in \mathbb{R}$$

Thus choose the simplest eigenvector as $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

THUS:

$$\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{and } \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Example 5 (Online Homework #10)Find the eigenvalues and associated unit eigenvectors of the (symmetric)

matrix $A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}$.

$$A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 5-\lambda & -10 \\ -10 & 20-\lambda \end{bmatrix} =$$

$$= (5-\lambda)(20-\lambda) - 100 = 0$$

$$\Leftrightarrow \lambda^2 - 25\lambda + 100 - 100 = 0$$

$$\Leftrightarrow \boxed{\lambda^2 - 25\lambda = 0} \Leftrightarrow \boxed{\lambda_1 = 0 \quad \lambda_2 = 25}$$

(1) $\boxed{\lambda_1 = 0}$ $\begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ hence

$$\begin{cases} 5v_1 - 10v_2 = 0 \\ -10v_1 + 20v_2 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 - 2v_2 = 0 \\ v_1 - 2v_2 = 0 \end{cases}$$

Hence $\boxed{v_1 = 2v_2}$ set $v_2 = t \in \mathbb{R}$ then $v_1 = 2t$. Hence any eigenvector for $\lambda = 0$ looks like $\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.Pick e.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ it has length $\sqrt{4+1} = \sqrt{5}$ If we take $\boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ this has length 1 (unit eigenvector)

(2) $\boxed{\lambda_2 = 25}$ we need $\begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 25 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} 5w_1 - 10w_2 = 25w_1 \\ -10w_1 + 20w_2 = 25w_2 \end{cases} \Leftrightarrow \begin{cases} -20w_1 - 10w_2 = 0 \\ -10w_1 - 5w_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2w_1 + w_2 = 0 \end{cases} \text{ same equation}$$

i.e. $w_2 = -2w_1$ Pick for example $w_1 = t \in \mathbb{R}$ then $w_2 = -2t$ so $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ For $t=1$ we get $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ but it has length $\sqrt{1^2 + (-2)^2} = \sqrt{5}$. Thus divide it by its length and we get an eigenvector of length 1

$$\boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

Example 6 (Online Homework #12)

Let $A = \begin{bmatrix} -4 & 3 \\ 5 & k \end{bmatrix}$.

Find the value of k so that A has 0 as an eigenvalue.

We know that for a 2×2 matrix A with eigenvalues λ_1 and λ_2 then
 $\det(A) = \lambda_1 \lambda_2$ and $\text{trace}(A) = \lambda_1 + \lambda_2$

Since we want one of the 2 eigenvalues to be 0 then $\det(A) = 0$.

$$\det \begin{bmatrix} -4 & 3 \\ 5 & k \end{bmatrix} = 0 \iff -4k - 3 \cdot 5 = 0$$

$$\iff k = -\frac{15}{4}$$

Example 7 (Online Homework #13)

For which value of k does the matrix $A = \begin{bmatrix} -3 & k \\ -8 & -8 \end{bmatrix}$

have one real eigenvalue of multiplicity 2?

$A = \begin{bmatrix} -3 & k \\ -8 & -8 \end{bmatrix}$. To find the eigenvalues of A

$$\det(A - \lambda I_2) = 0 \implies$$

$$\det \begin{bmatrix} -3-\lambda & k \\ -8 & -8-\lambda \end{bmatrix} = (-3-\lambda)(-8-\lambda) + 8 \cdot k = 0$$

$$\implies \lambda^2 + 11\lambda + 24 + 8k = 0$$

$$\lambda_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4(1)(24+8k)}}{2 \cdot 1}$$

To have $\lambda_1 = \lambda_2$ we need the discriminant of the equation to be 0:

$$121 - 96 - 32k = 0$$

$$k = \frac{25}{32}$$

Example 8 (Online Homework #16)

Find a matrix A such that $\mathbf{v}_1 = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ are eigenvectors of A , with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ respectively.

Want $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$(1) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} = 5 \begin{bmatrix} -3 \\ -4 \end{bmatrix} \quad \boxed{\text{AND}}$$

$$(2) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

So we must have from (1) and (2)

$$\begin{cases} -3a - 4b = -15 \\ -3c - 4d = -20 \end{cases} \quad \text{and} \quad \begin{cases} 2a - 5b = -2 \\ 2c - 5d = 5 \end{cases}$$

OR

$$\begin{cases} -3a - 4b = -15 \\ 2a - 5b = -2 \end{cases} \quad \begin{cases} -3c - 4d = -20 \\ 2c - 5d = 5 \end{cases}$$

$$\begin{bmatrix} -3 & -4 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix} \quad \begin{bmatrix} -3 & -4 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -20 \\ 5 \end{bmatrix}$$

same matrix

its inverse is

$$\frac{1}{23} \begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix} \quad \text{so that}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -15 \\ -2 \end{bmatrix} = \begin{bmatrix} 67/23 \\ 36/23 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -20 \\ 5 \end{bmatrix} = \begin{bmatrix} 120/23 \\ 25/23 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 67/23 & 36/23 \\ 120/23 & 25/23 \end{bmatrix}$$

Example 9 (Complex Eigenvalues)

Consider the matrix $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

- Find its eigenvalues.
- Find the eigenvectors associated with the eigenvalues from part (a).

$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ has eigenvalues given by

$$\det[A - \lambda I_2] = 0 \iff \det \begin{bmatrix} \sqrt{3} - \lambda & -1 \\ 1 & \sqrt{3} - \lambda \end{bmatrix} = 0$$

$$\iff (\sqrt{3} - \lambda)^2 + 1 = 0 \iff$$

$$3 - 2\sqrt{3}\lambda + \lambda^2 + 1 = 0 \iff \boxed{\lambda^2 - 2\sqrt{3}\lambda + 4 = 0}$$

Quadratic equation: $\lambda_{1,2} = \frac{2\sqrt{3} \pm \sqrt{12 - 16}}{2}$

$$\lambda_{1,2} = \frac{2\sqrt{3} \pm 2\sqrt{-1}}{2} = \boxed{\sqrt{3} \pm i} \parallel$$

$\lambda_1 = \sqrt{3} + i$ To find the eigenvector we need

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (\sqrt{3} + i) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\iff \begin{cases} \sqrt{3}v_1 - v_2 = (\sqrt{3} + i)v_1 \\ v_1 + \sqrt{3}v_2 = (\sqrt{3} + i)v_2 \end{cases}$$

$$\iff \begin{cases} \sqrt{3}v_1 - v_2 = \sqrt{3}v_1 + iv_1 \\ v_1 + \sqrt{3}v_2 = \sqrt{3}v_2 + iv_2 \end{cases}$$

$$\iff \begin{cases} v_2 = -iv_1 \\ v_1 = iv_2 \end{cases}$$

Notice that they are the same equation !!!

$$\underline{v_1 = iv_2} \implies \begin{pmatrix} iv_1 = i^2 v_2 \\ = -v_2 \end{pmatrix}$$

Hence pick for example the second equation $\boxed{v_1 = iv_2}$ If $v_2 = t \in \mathbb{R}$

then $v_1 = it$ so $\begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$

the simplest eigenvector is $\underline{\underline{\begin{bmatrix} i \\ 1 \end{bmatrix}}}$

For $\boxed{\lambda_2 = \sqrt{3} - i}$ we need

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (\sqrt{3} - i) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \iff$$

$$\begin{cases} \sqrt{3}w_1 - w_2 = (\sqrt{3} - i)w_1 \\ w_1 + \sqrt{3}w_2 = (\sqrt{3} - i)w_2 \end{cases} \iff \begin{cases} w_2 = iw_1 \\ w_1 = -iw_2 \end{cases}$$

Pick eq. : $w_1 = -iw_2$ so $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$

So notice that

$$\lambda_1 = \sqrt{3} + i$$

$$\lambda_2 = \sqrt{3} - i$$

$$\underline{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\underline{w} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

they are complex conjugate of each other!