MA137 – Calculus 1 with Life Science Applications
Course Introduction & Preliminaries and Elementary Functions
(Sections 1.1 & 1.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

August 24, 2016

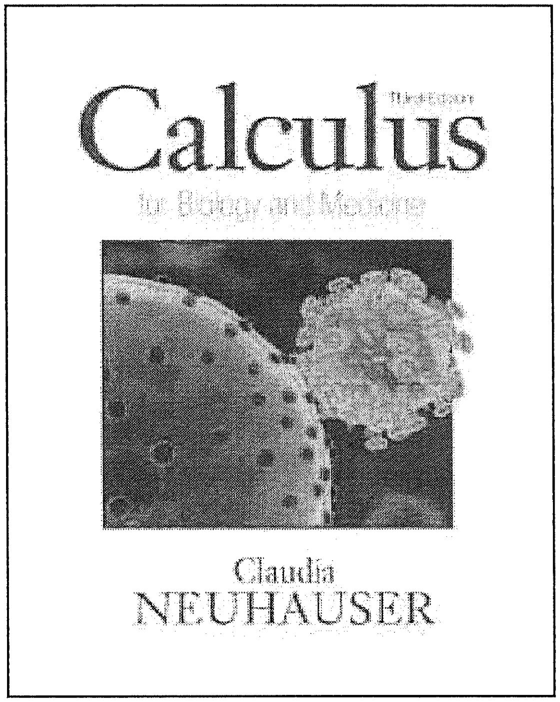
Instructor

- Instructor: Alberto Corso
- Lecture: MWF 10:00-10:50am – CB 118
- Office: POT(≡Patterson Office Tower) 701
- Office Hours: MWF 11 am – 12 noon, and by appointment
- Email: alberto.corso@uky.edu
- Web: http://www.ms.uky.edu/~corso
- Course Web: http://www.ms.uky.edu/~ma137

Teaching Assistants (TAs)

Section	Time/Location	TA information
005	TR 12:00-12:50pm – CB 339	**Stephen Deterding**
006	TR 01:00-01:50pm – CB 339	deterding@uky.edu
003	TR 10:00-10:50am – CB 339	**Kalila Sawyer**
004	TR 11:00-11:50am – CB 339	kalila.sawyer@uky.edu
001	TR 08:00-08:50am – CB 339	**Luis Sordo Vieira**
002	TR 09:00-09:50am – CB 339	sordovieira@uky.edu
007	TR 02:00-02:50pm – CB 339	**Matias von Bell**
008	TR 03:00-03:50pm – CB 339	matias.vonbell@uky.edu

Textbook



Title: Calculus for Biology and Medicine

Author: Claudia Neuhauser

Publisher: Pearson

Edition: Third

ISBN: ISBN 10: 0-321-64468-9
ISBN 13: 978-0-321-64468-8

Course Outline for MA 137

Ch. 1: Preview and review
Ch. 2: Discrete time models, sequences, and difference equations
Ch. 3: Limits and continuity
Ch. 4: Differentiation
Ch. 5: Applications of differentiation
Ch. 6: Integration

If you are planning on taking MA 138, the course outline for MA 138 is:

Ch. 7: Integration techniques and computational methods
Ch. 8: Differential equations
Ch. 9: Linear algebra and analytic geometry
Ch. 10: Multivariable calculus
Ch. 11: Systems of differential equations

Grading

You will be able to obtain a **maximum of 500 points** in this class, divided as follows:

- Three 2-hour exams, 100 points each;
- Final exam, 100 points;
- Homework, 40 points;
- Quizzes, 40 points;
- Final project (Gen Ed requirement), 20 points;

Your final grade for the course will be based on the total points you have earned as follows:

A: 450-500	B: 400-449	C: 350-399	D: 300-349	E: 0-299
$\geq 90\%$	$\geq 80\%$	$\geq 70\%$	$\geq 60\%$	$< 60\%$

Exams (Regular and Alternate)

Regular Exams will be given on Tuesdays from 5:00–7:00 pm

- September 20
- October 18
- November 15

Alternate Exams for Exams 1-3 are given on the same days as the regular exams from 7:30–9:30 pm (September 20, October 18, November 15).

Review Sessions will be held on Monday September 19, October 17 and November 14 from 6:00–8:00 pm.

Homework

- The homework associated to MA137 is mostly done **online**. There are **three exceptions** where there are three **handwritten** homework assignments.
- The online homework (WeBWorK) can be accessed through
  https://webwork.as.uky.edu/webwork2/MA137F16/
- Your username is your **Link Blue user ID** (use capital letters!) and your password is **your 8 digit student ID number**.
- You can try online problems as many times as you like. The system will tell you if your answer is correct or not. You can email the TA a question from each of the problem. TAs will always do their best to respond within 24 hours.
- **Don't wait until the last minute!**

REEF Polling

- If you are taking an introductory Chemistry class you are likely to be required to use REEF Polling by iClicker.
- If you already have a REEF account, add this course by selecting the "+" button on the top-right of your Courses page, selecting the University of Kentucky as your institution, and searching for this course, "MA 137 - Calculus 1 with Life Science Applications."
- REEF polling by iClicker lets you use your laptop, smart phone, tablet, or physical iClicker remote to answer questions in class.
- To create an account, purchase a subscription, and/or register a physical iClicker remote, visit

  http://support.reef-education.com

- If none of your classes uses REEF Polling, there is no need to purchase a subscription.

¿Minoring in Mathematics?

To obtain a **minor in Mathematics**, a student who has completed MA 137/138 Calculus I and II must complete the following:

1. MA 213 – Calculus III (4 credits)
2. MA 322 – Matrix Algebra and Its Applications (3 credits)
3. Six additional credit hours of Mathematics courses (=two courses) numbered greater than 213. Possible courses include: MA 214, MA 261, MA 320, MA 321, **MA 327 (Introduction to game theory)**, MA 330, MA 341, MA 351, MA 361, or any 400 level math course
4. We are also planning to create a new modeling course by Fall 2017 at the upper level in Mathematics. Stay tuned!!
   ¿ **MA 337:** Modeling Nature and the nature of modeling?

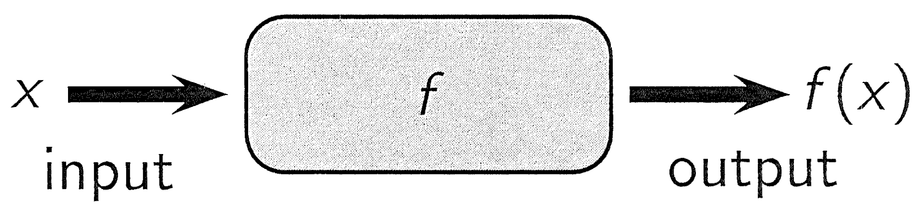
Thus you need 13 additional credit hours in Mathematics classes.

Definition of Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

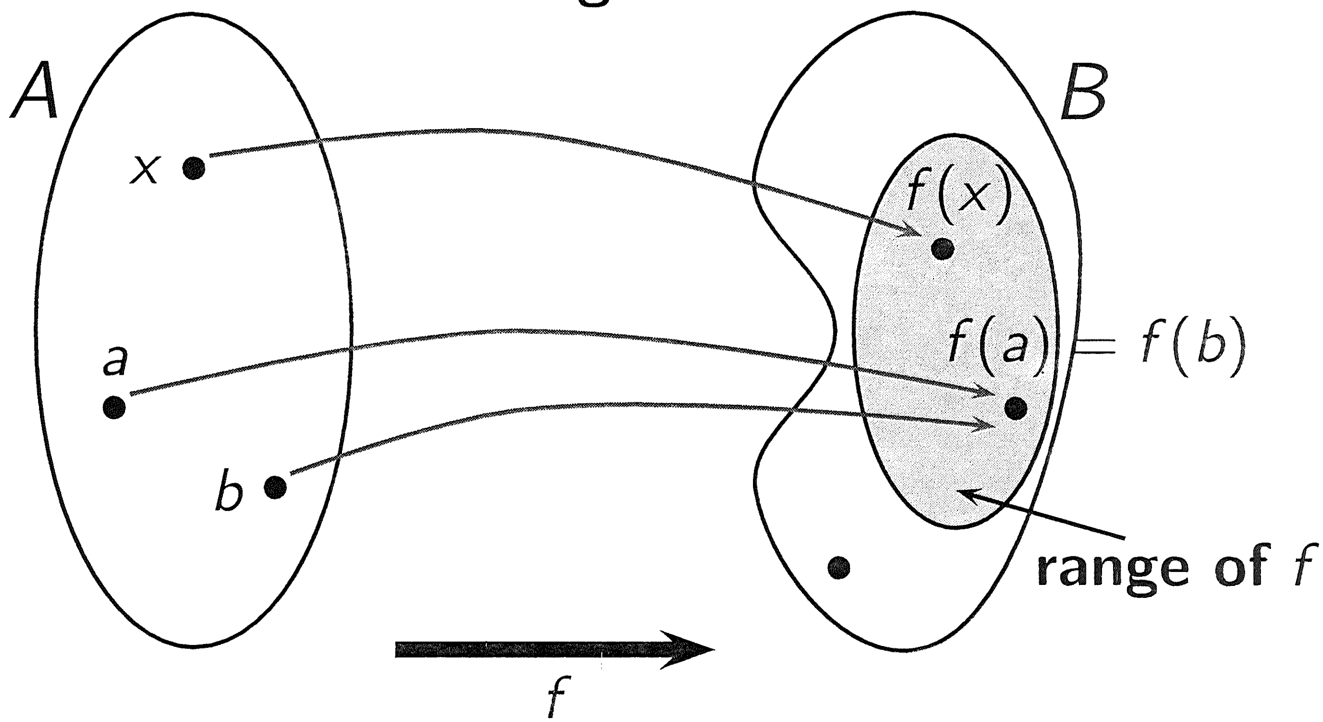
The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of f at** x, or the **image of** x **under** f.

The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \,|\, x \in A\}$.



Machine diagram of f

Arrow diagram of f



Notation: To define a function, we often use the notation

$$f : A \longrightarrow B, \qquad x \mapsto f(x)$$

Evaluating a Function

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**.

The symbol that represents a number in the range of f is called a **dependent variable**.

In the definition of a function the independent variable plays the role of a "placeholder".

For example, the function $f(x) = 2x^2 - 3x + 1$ can be thought of as

$$f(\square) = 2 \cdot \square^2 - 3 \cdot \square + 1.$$

To evaluate f at a number (expression), we substitute the number (expression) for the placeholder.

The Domain of a Function

The domain of a function is the set of all inputs for the function.

The domain may be stated explicitly.

For example, if we write

$$f(x) = 1 - x^2 \qquad -2 \leq x \leq 5$$

then the domain is the set of all real numbers x for which $-2 \leq x \leq 5$.

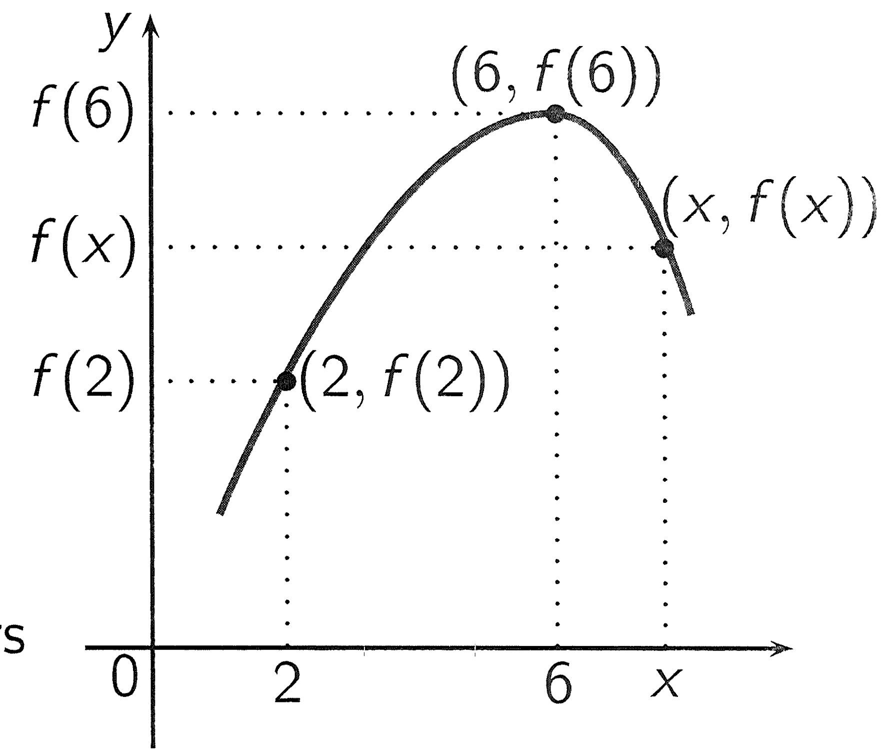
If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain is the set of all real numbers for which the expression is defined.*

Fact: Two functions f and g are equal if and only if

1. f and g are defined on the same domain,
2. $f(x) = g(x)$ for all x in the domain.

Graphs of Functions

The graph of a function is the most important way to visualize a function. It gives a picture of the behavior or 'life history' of the function.
We can read the value of $f(x)$ from the graph as being the height of the graph above the point x.



If f is a function with domain A, then the graph of f is the set of ordered pairs

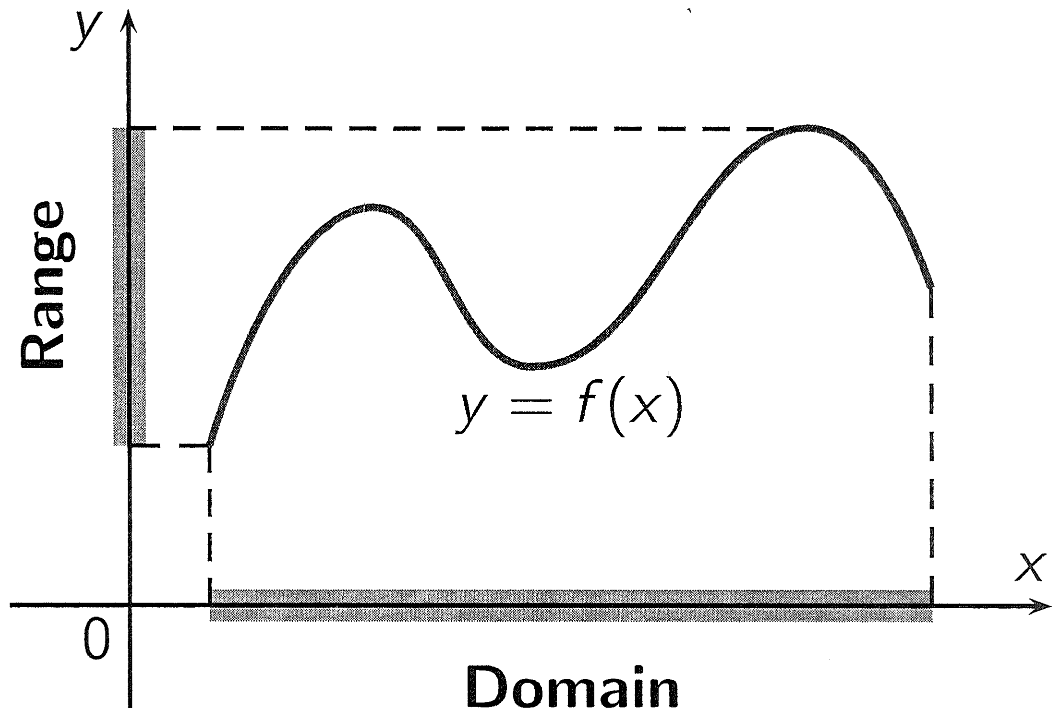
$$\textbf{graph of } f = \{(x, f(x)) \,|\, x \in A\}.$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

Obtaining Information from the Graph of a Function

The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.

In addition, the graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in the picture:

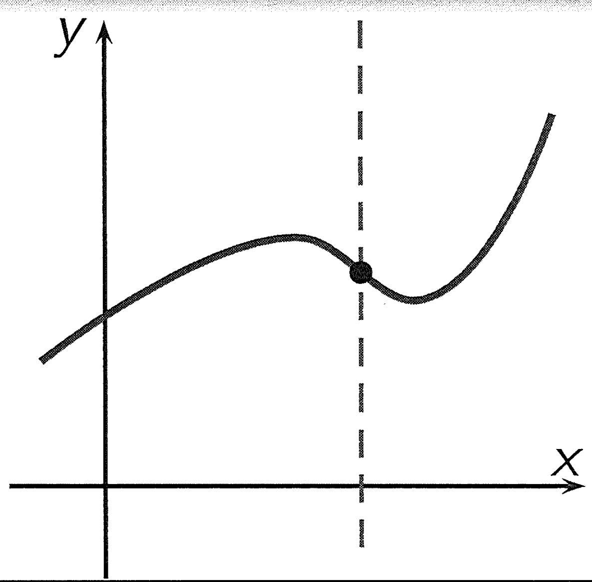


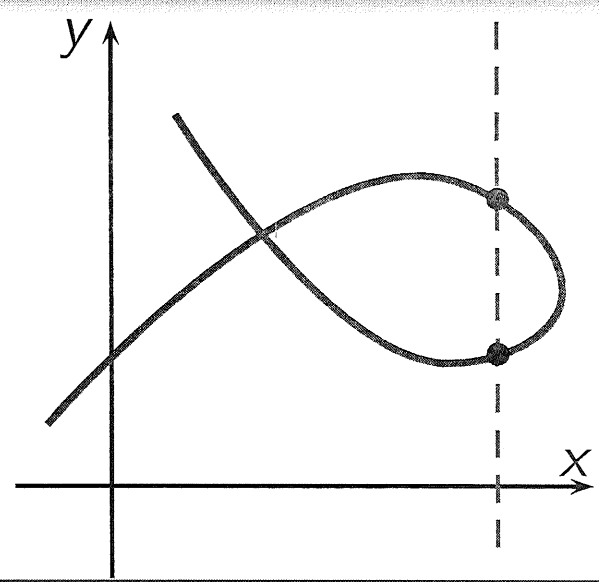
The Vertical Line Test

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



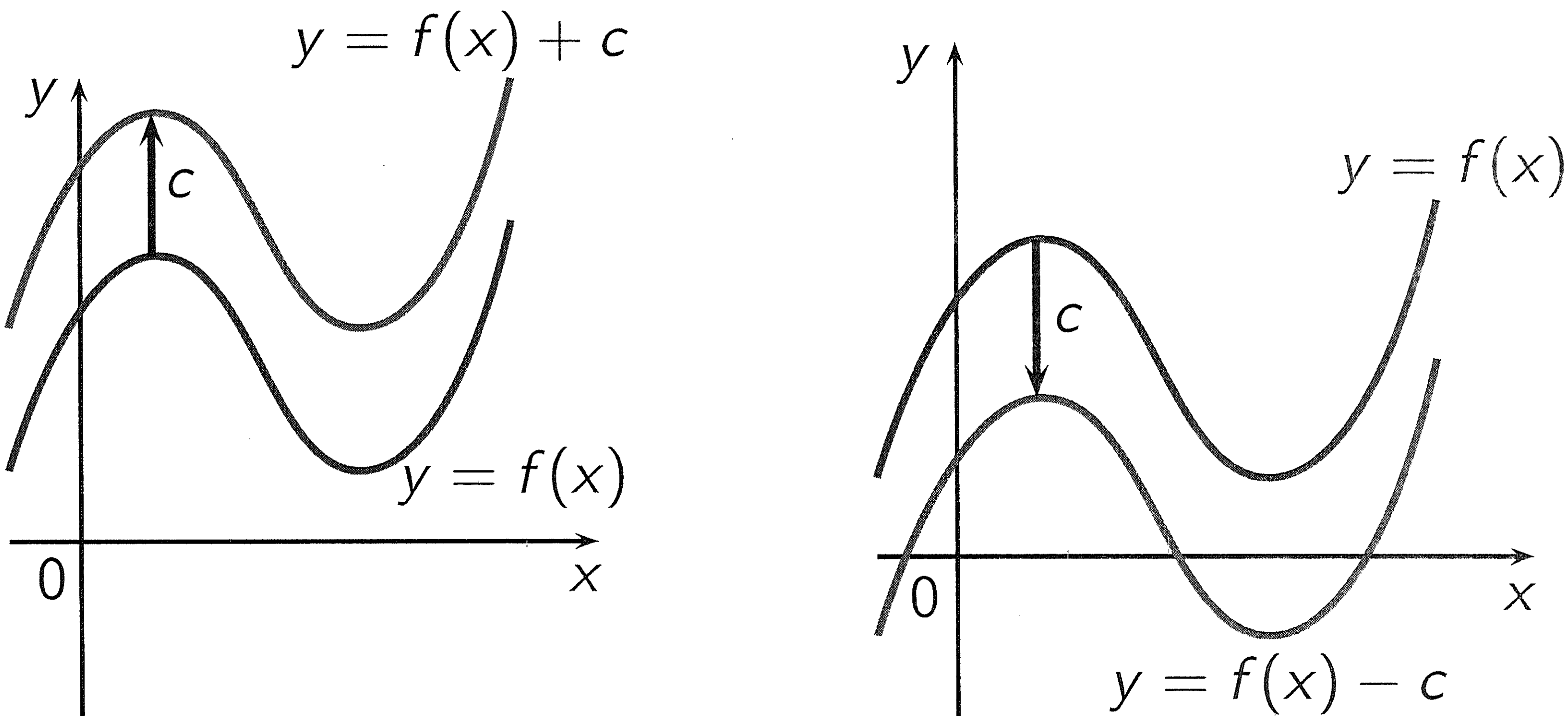


Vertical Shifting

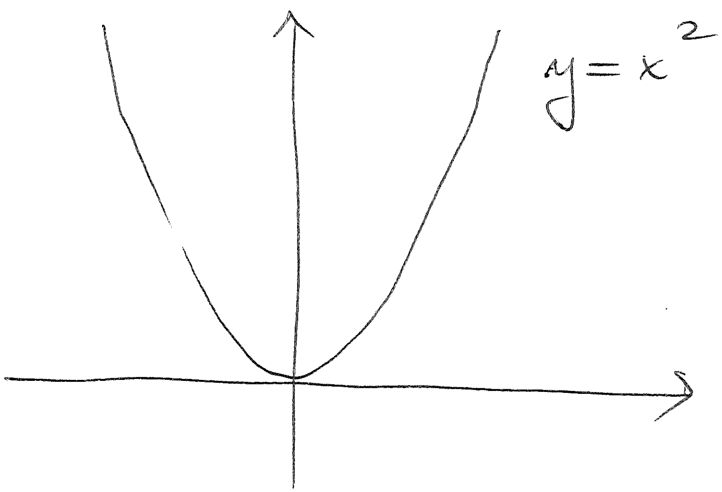
Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

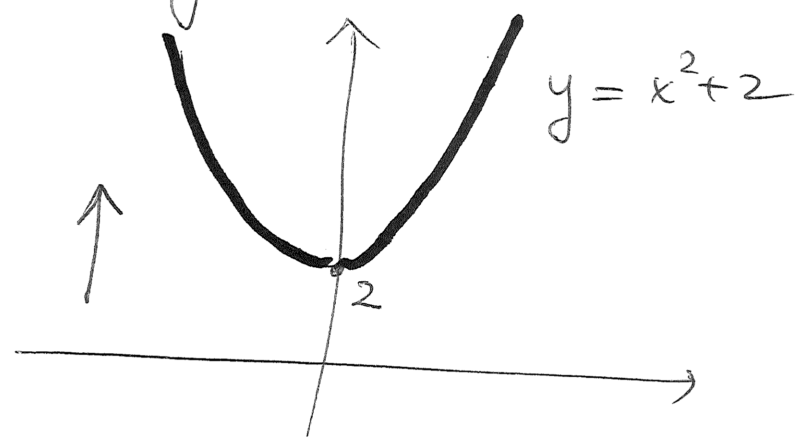
To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



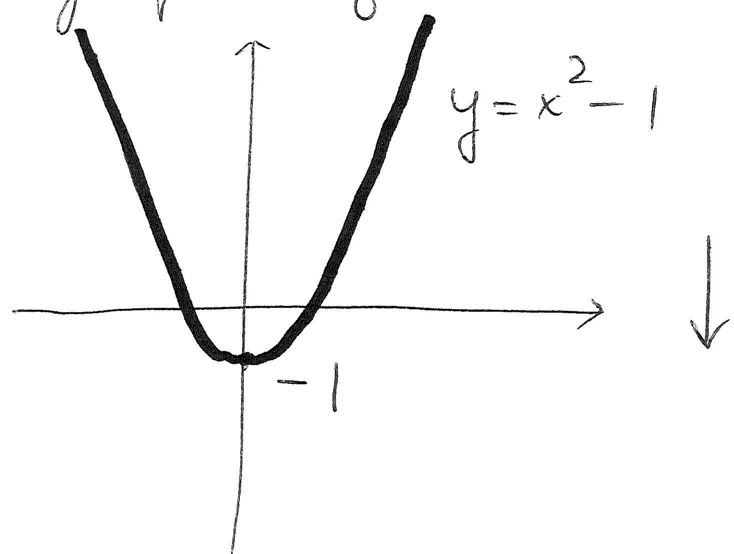
Consider for example the parabola $y = x^2$ whose graph is



Then the graph of $y = x^2 + 2$ is



whereas the graph of $y = x^2 - 1$ is

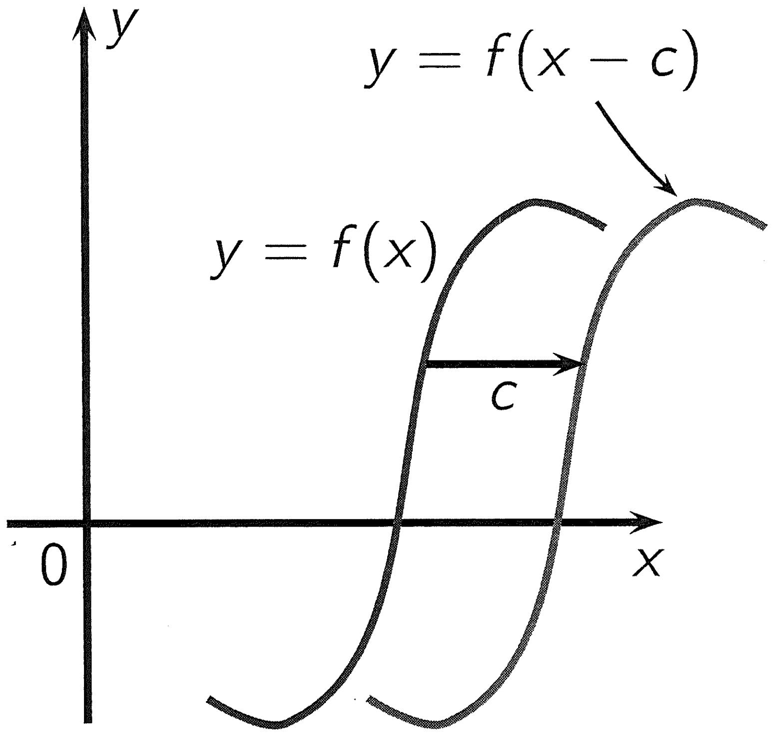


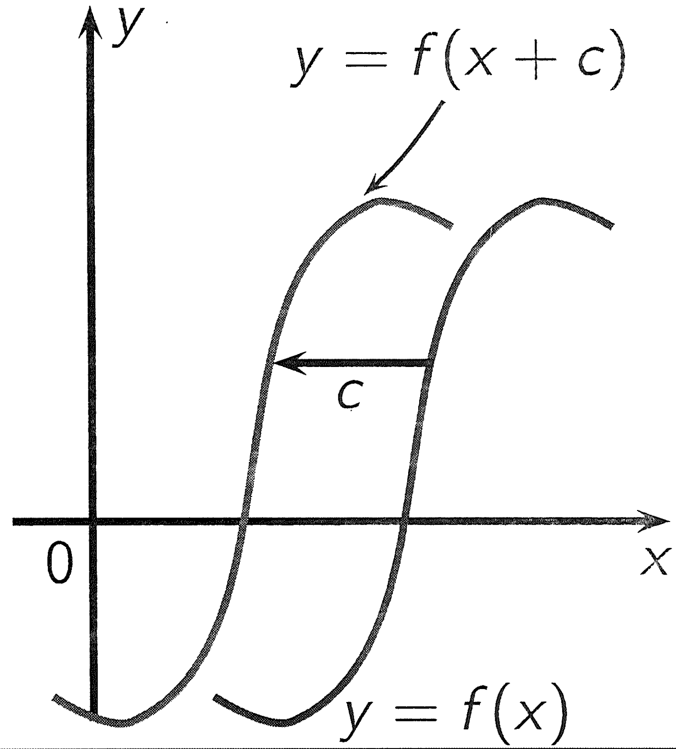
Horizontal Shifting

Suppose $c > 0$.

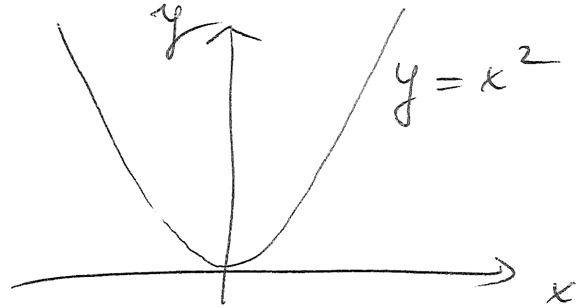
To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.

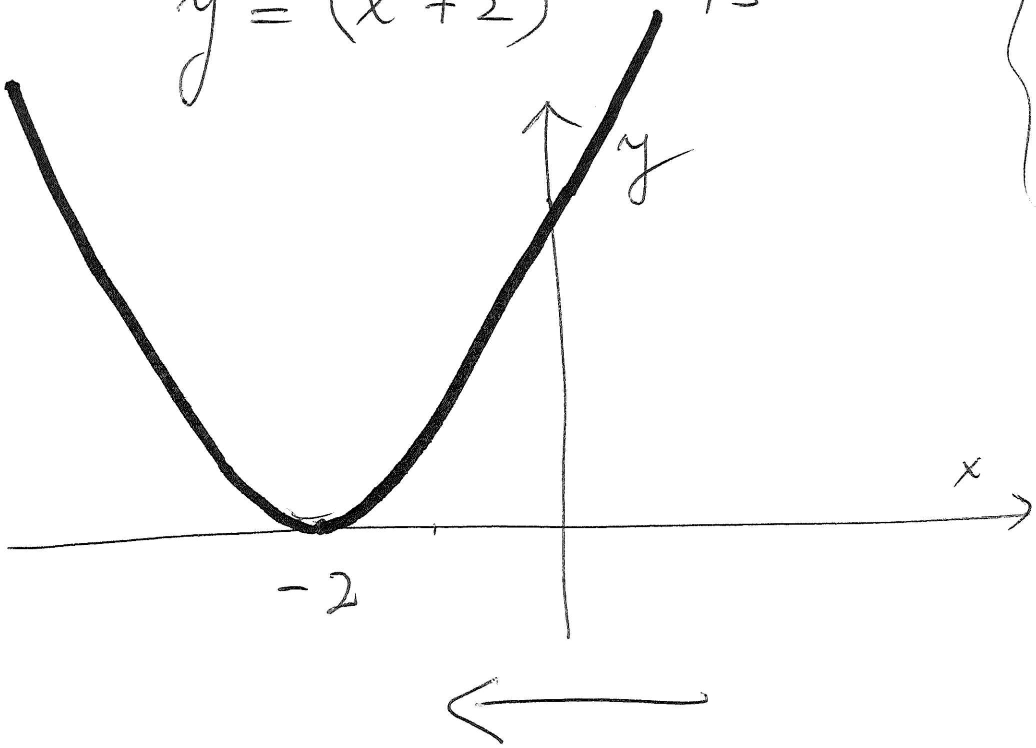




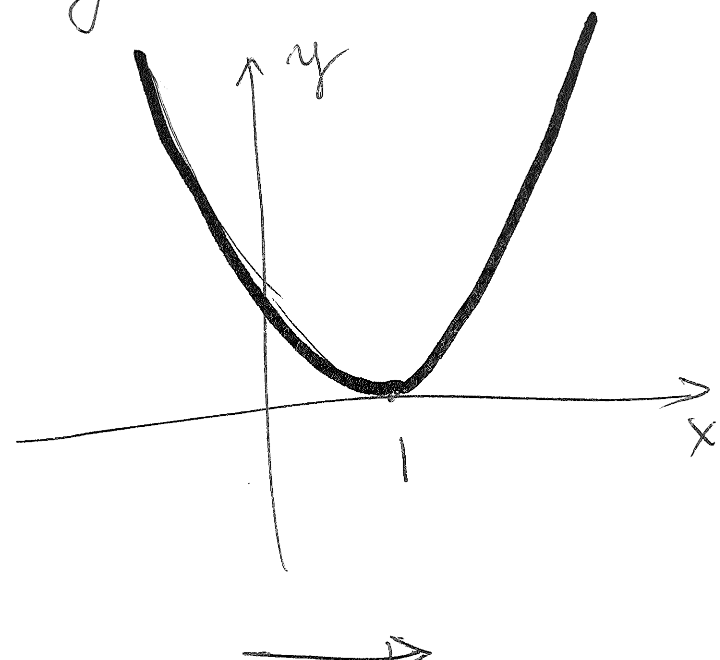
Consider again $y=x^2$



Then the graph of $y=(x+2)^2$ is



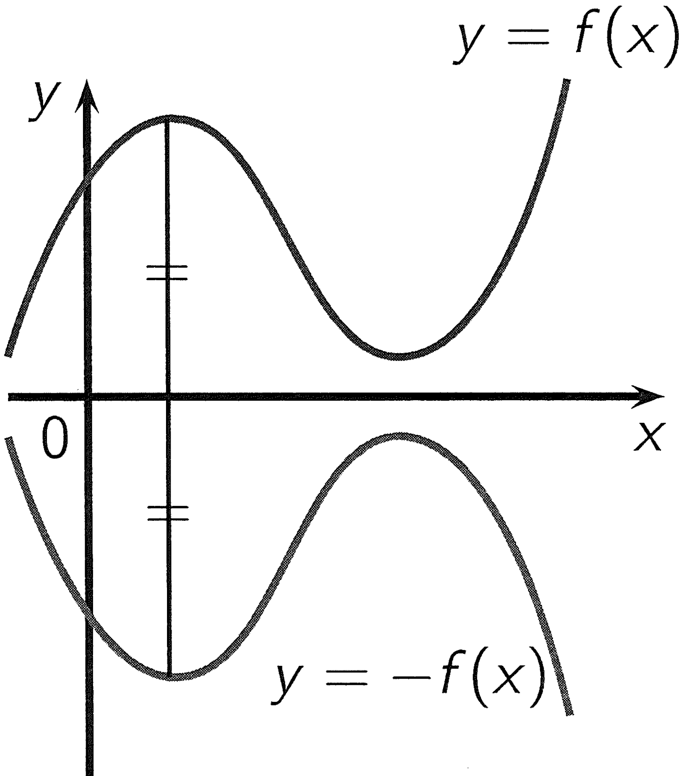
The graph of $y=(x-1)^2$ is

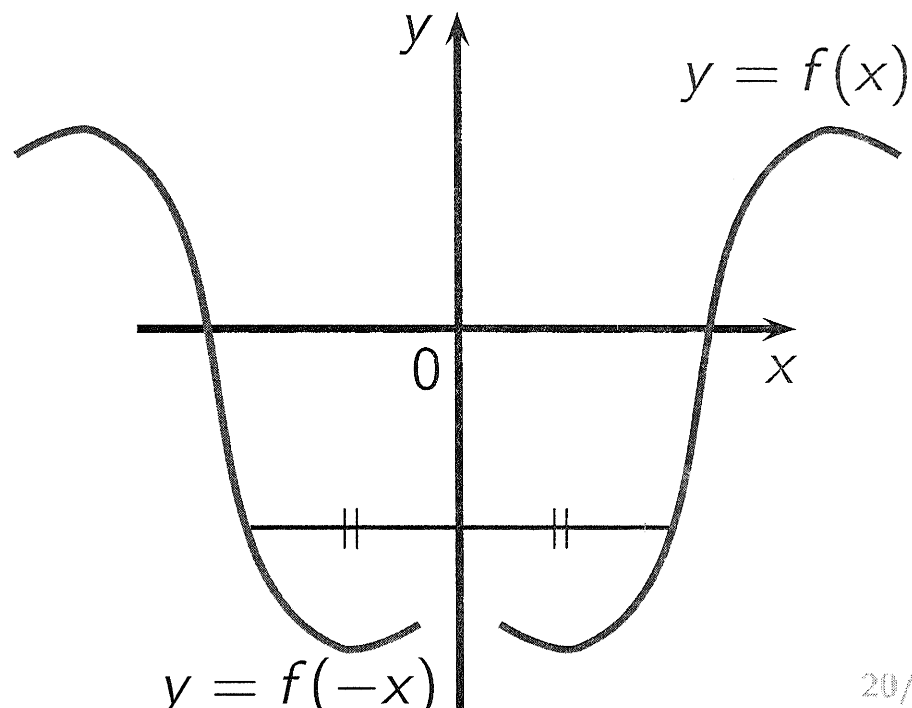


Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

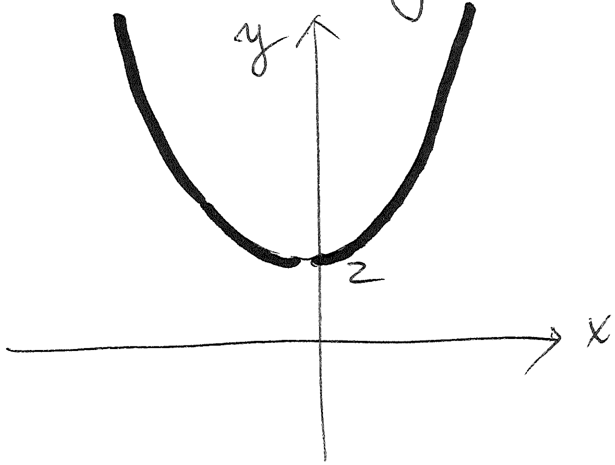
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



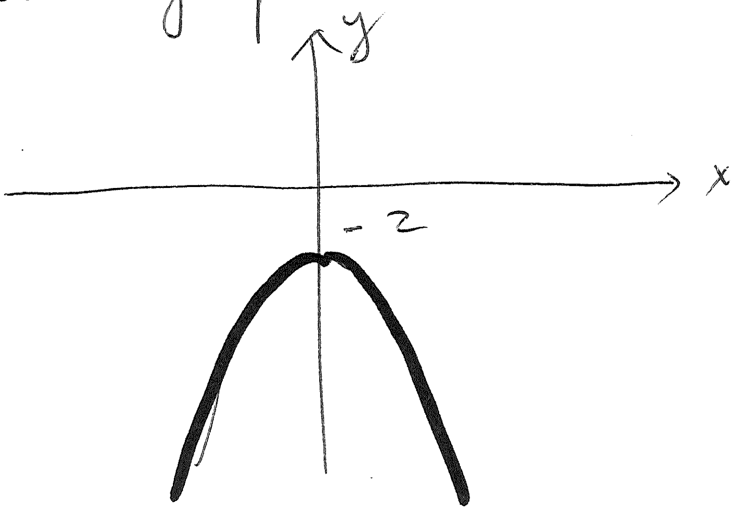


Reflection with respect to the x-axis

Consider $y = x^2 + 2$

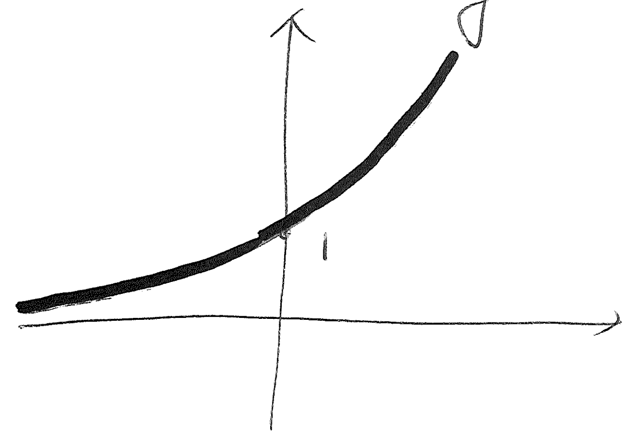


Then $y = -(x^2+2)$ has graph

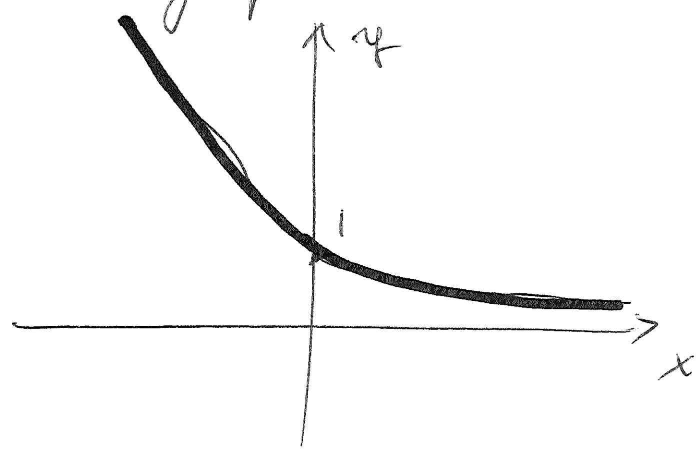


Reflection with respect to the y-axis

Consider $y = e^x$



Then $y = e^{(-x)}$ has graph

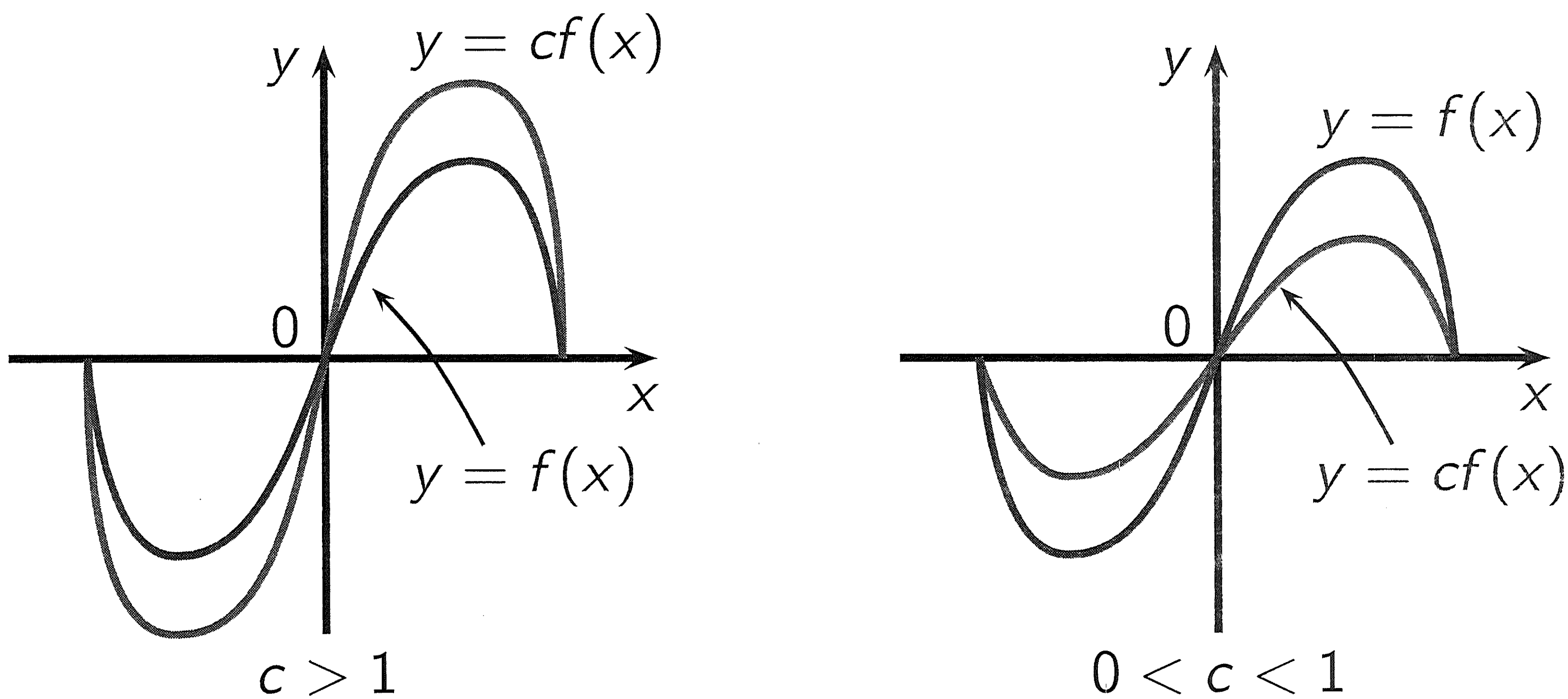


Vertical Stretching and Shrinking

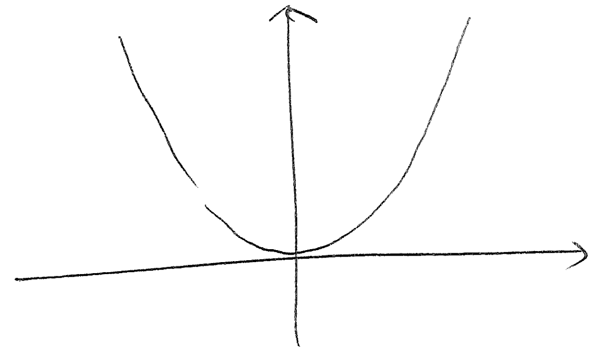
To graph $y = cf(x)$:

If $c > 1$, STRETCH the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, SHRINK the graph of $y = f(x)$ vertically by a factor of c.

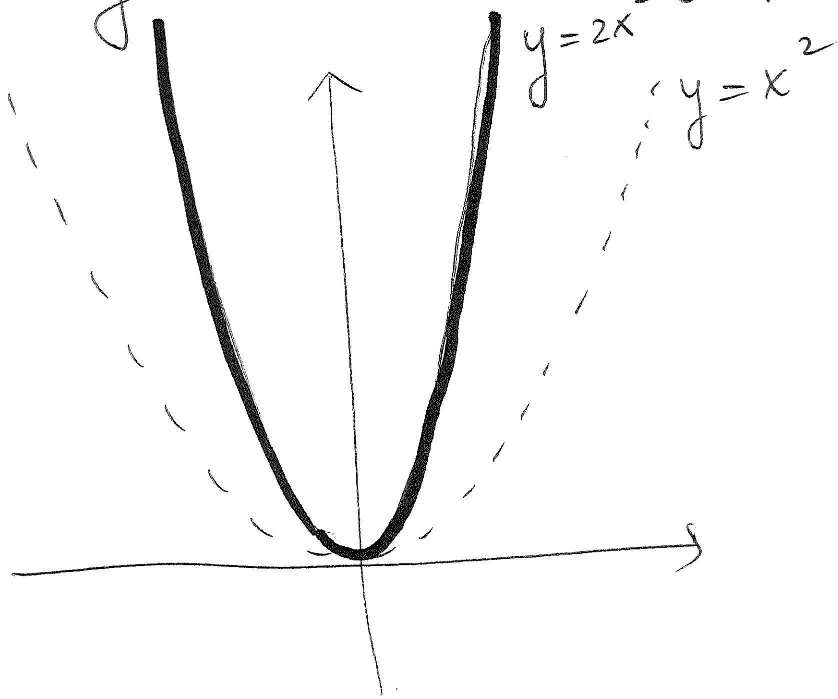


Consider $y = x^2$

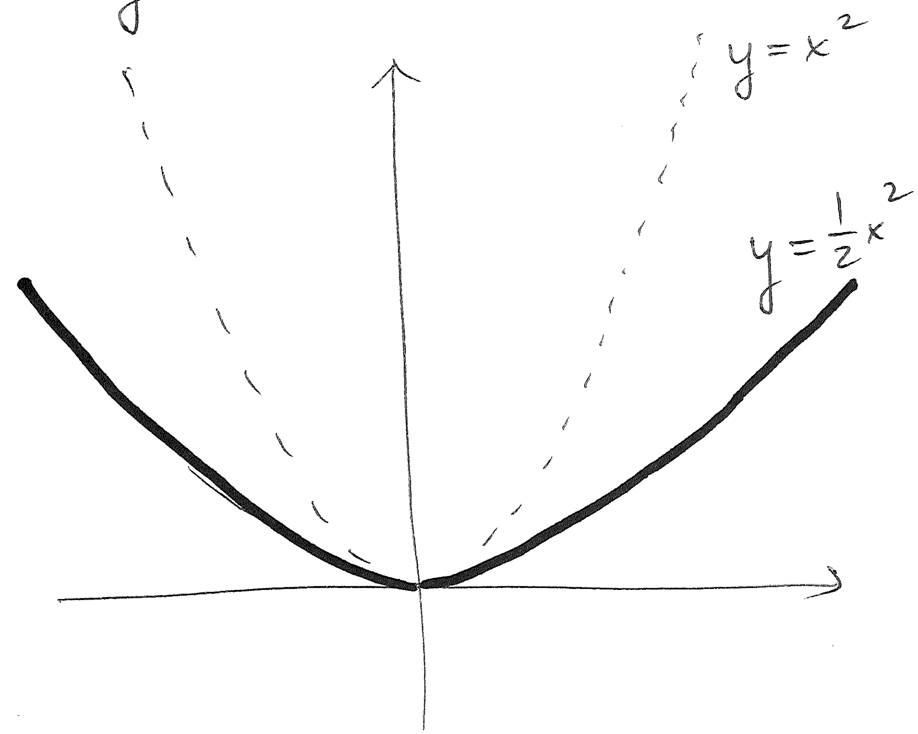


Then we have that:

$y = 2x^2$ has graph



$y = \frac{1}{2}x^2$ has graph

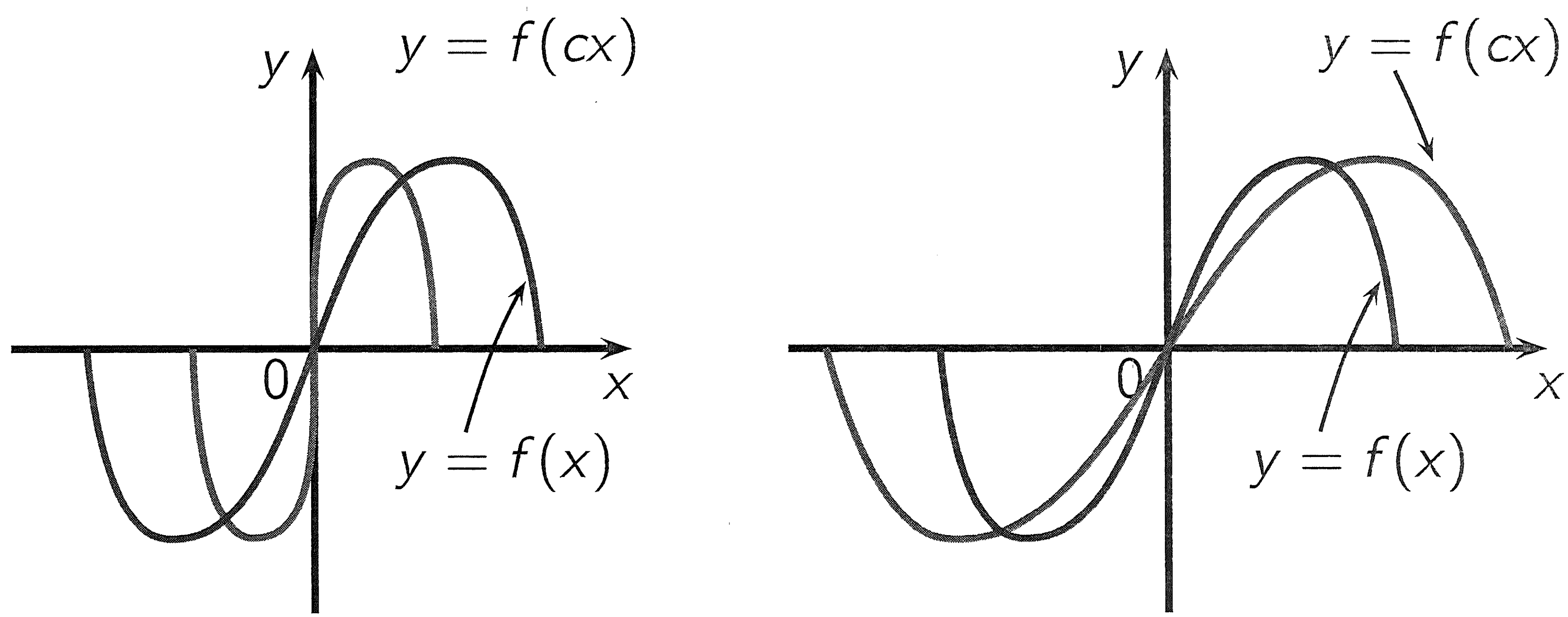


Horizontal Shrinking and Stretching

To graph $y = f(cx)$:

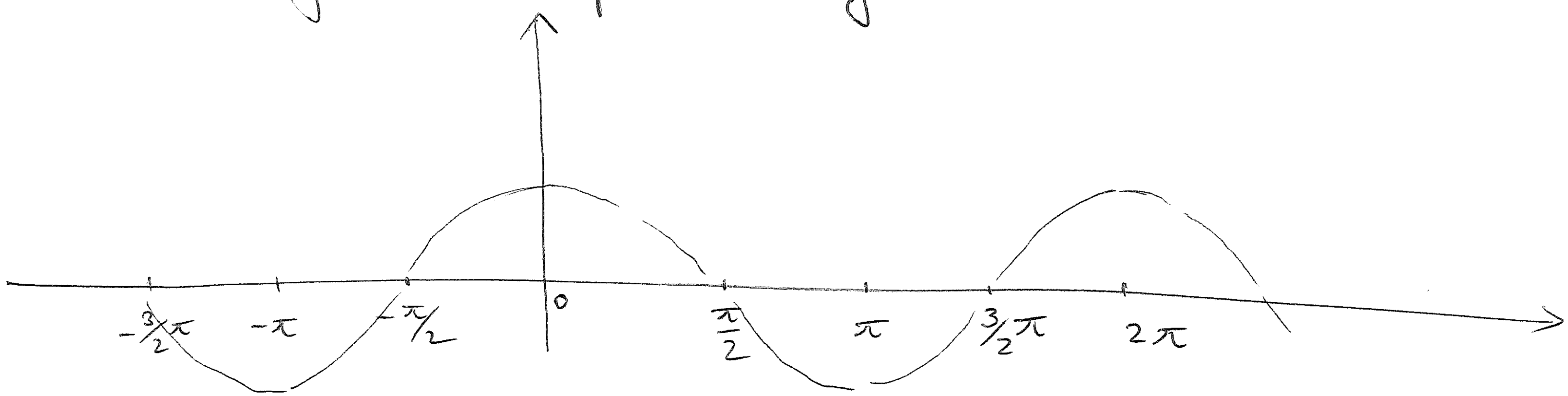
If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.

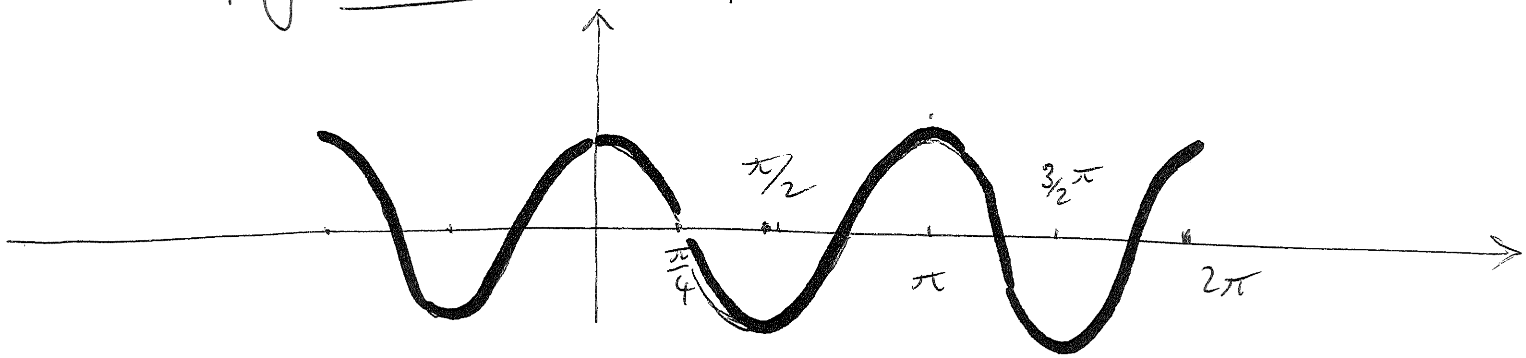


$c > 1$ $\qquad$ $0 < c < 1$

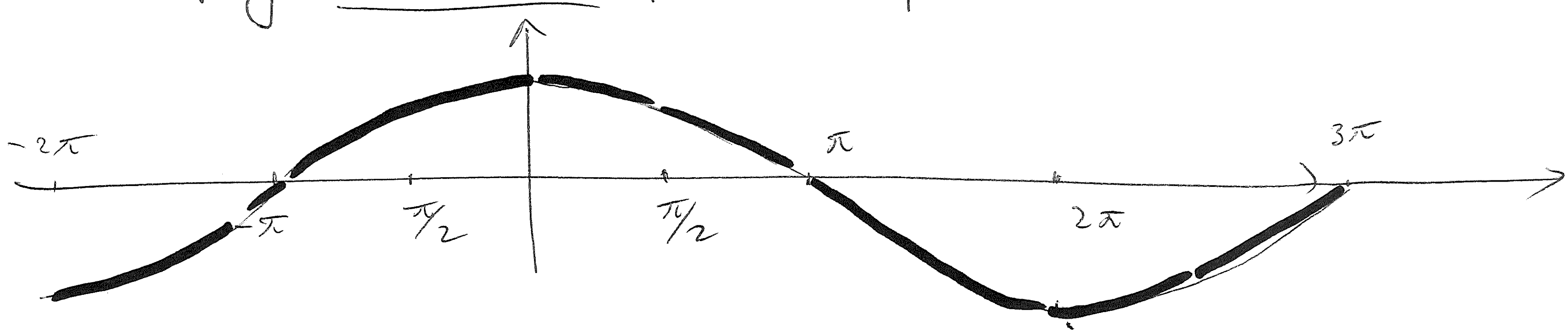
Consider for example $y = \cos(x)$



Then $\boxed{y = \cos(2x)}$ has graph:



Then $\boxed{y = \cos(\frac{1}{2}x)}$ has graph



MA137 – Calculus 1 with Life Science Applications
Preliminaries and Elementary Functions
(Sections 1.1 & 1.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

August 26, 2016

Basic Functions

We introduce the basic functions that we will consider throughout the remainder of the semester.

- **polynomial functions**
  A polynomial function is a function of the form
  $$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$
  where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are (real) constants with $a_n \neq 0$. The coefficient a_n is called the leading coefficient, and n is called the degree of the polynomial function. The largest possible domain of f is $\mathbb{R}$.

  Examples Suppose a, b, c, and m are constants.

  - Constant functions: $f(x)=c$ (graph is a horizontal line);
  - Linear functions: $f(x)=mx+b$ (graph is a straight line);
  - Quadratic functions: $f(x)=ax^2+bx+c$ (graph is a parabola).

- **rational functions**

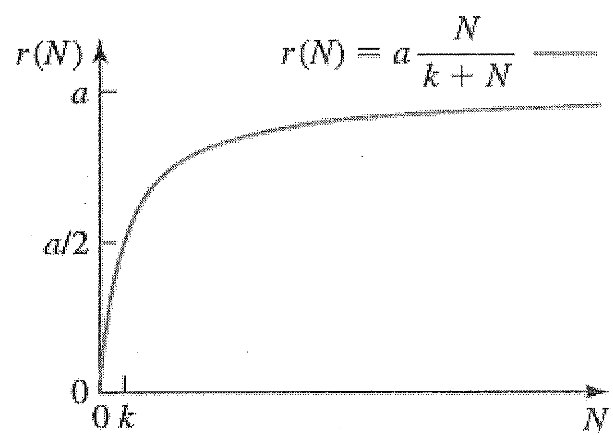
A rational function is the quotient of two polynomial functions $p(x)$ and $q(x)$: $f(x) = \dfrac{p(x)}{q(x)}$ for $q(x) \neq 0$.

Example The **Monod growth function** is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations.

If we denote the concentration of the nutrient by N, then the per capita growth rate $r(N)$ is given by

$$r(N) = \frac{aN}{k+N}, \qquad N \geq 0$$

where a and k are positive constants.



- **power functions**

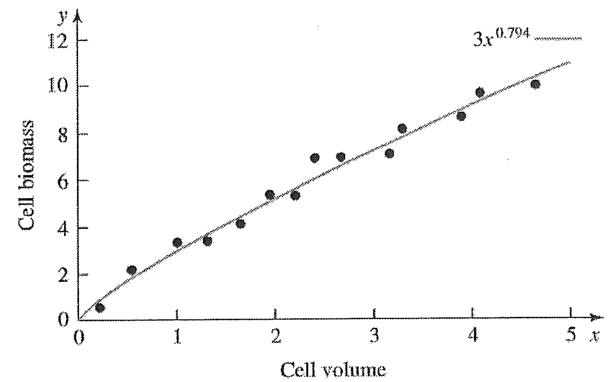
  A power function is of the form $f(x) = x^r$ where r is a real number.

  Example Power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes).

  Finding such relationships is the objective of **allometry**. For example, in a study of 45 species of unicellular algae, a relationship between cell volume and cell biomass was sought. It was found [see, Niklas (1994)] that

$$\text{cell biomass} \propto (\text{cell volume})^{0.794}$$

  Most scaling relations are to be interpreted in a statistical sense; they are obtained by fitting a curve to data points. The data points are typically scattered about the fitted curve given by the scaling relation.



- **exponential and logarithmic functions**
- **trigonometric functions**

Example 1 (Problem #52, Section 1.1, p. 14):

The Celsius scale is devised so that 0°C is the freezing point of water (at 1 atmosphere of pressure) and 100°C is the boiling point of water (at 1 atmosphere of pressure).
If you are more familiar with the Fahrenheit scale, then you know that water freezes at 32°F and boils at 212°F.

(a) Find a linear equation/function that relates temperature measured in degrees Celsius and temperature measured in degrees Fahrenheit.

(b) The normal body temperature in humans ranges from 97.6°F to 99.6°F. Convert this temperature range into degrees Celsius.

$F = aC + b$ a linear relation

* when $C = 0$ then $F = 32$ so that

$32 = a \cdot 0 + b \Longrightarrow \boxed{b = 32}$

* when $C = 100$ then $F = 212$ so that

$212 = a \cdot 100 + 32 \Longrightarrow 100a = 212 - 32$

$\Longrightarrow a = \frac{180}{100} = 9/5 \quad \therefore \boxed{F = \frac{9}{5}C + 32}$

(alternatively, we can solve for C

$F = 9/5\, C + 32 \Longleftrightarrow 5F = 9C + 160$

$\Longleftrightarrow \underline{C} = \underline{\frac{5}{9}F - \frac{160}{9} \cong 5/9\, F - 17.7}$)

(b)

$$\boxed{97.6 \le F \le 99.6}$$

is the range of normal body temperature in humans

Substitute: $97.6 \le \frac{9}{5}C + 32 \le 99.6$

and write it in terms of C alone

$$\Longleftrightarrow \quad 97.6 - 32 \le 9/5\, C \le 99.6 - 32$$

$$\Longleftrightarrow \quad 65.6 \le 9/5\, C \le 67.6$$

$$\Longleftrightarrow \quad 5/9 \cdot 65.6 \le C \le \frac{5}{9}\, 67.6$$

$$\Longleftrightarrow \quad \boxed{36.44 \le C \le 37.55}$$

Example 2:

When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by

$$C(t) = 0.06t - 0.0002t^2,$$

where $0 \leq t \leq 240$ and the concentration is measured in mg/L.
When is the maximum serum concentration reached?
What is that maximum concentration?

Consider $C(t) = 0.06t - 0.0002t^2$

and rewrite it as

$$C(t) = -0.0002t^2 + 0.06t$$

We want to complete the squares:

$$C(t) = -0.0002\left[t^2 - \frac{0.06}{0.0002}t\right] = -0.0002\left[t^2 - 300t\right]$$

$$= -0.0002\left[t^2 - 300t + \left(\frac{300}{2}\right)^2\right] + 4.5$$

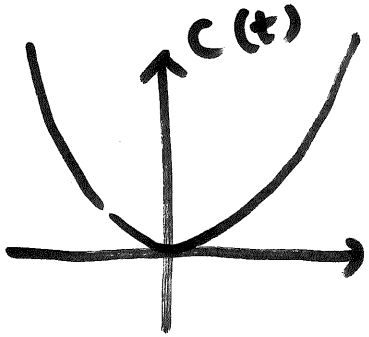
notice that $4.5 = 0.0002\left(\frac{300}{2}\right)^2$

$$\therefore \boxed{C(t) = -0.0002(t-150)^2 + 4.5}$$

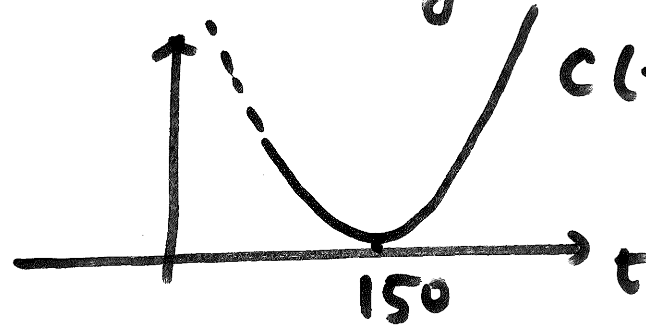
$\therefore$ maximum serum concentration at $t = 150$; $\therefore$ maximum concentration 4.5

The graph of $C(t) = -0.0002\,(t-150)^2 + 4.5$

is obtained as follows via elementary transformat.

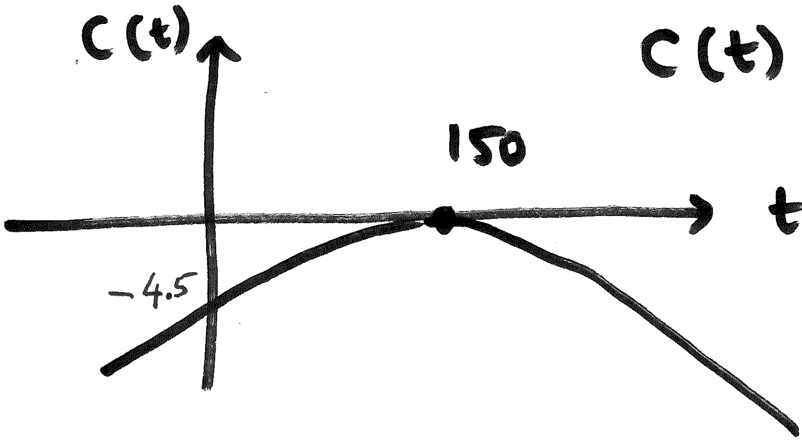


$c(t) = t^2$



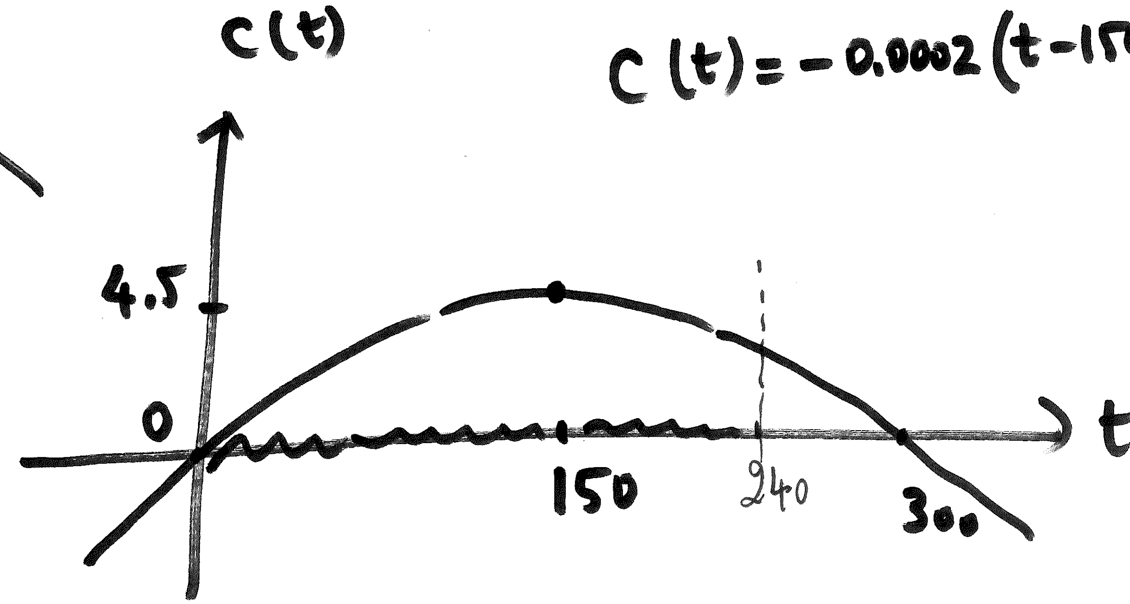
$c(t) = (t-150)^2$

$C(t) = -0.0002\,(t-150)^2$



Finally

$C(t) = -0.0002\,(t-150)^2 + 4.5$



Example 3: (Michaelis-Menten enzymatic reaction)

According to the Michaelis-Menten equation (1913) when a chemical reaction involving a substrate S is catalyzed by an enzime, the rate of reaction $V = V([\mathrm{S}])$ is given by the expression

$$V = \frac{V_{\max}[\mathrm{S}]}{K_m + [\mathrm{S}]},$$

where [S] denotes substrate concentration (for examples in moles per liter), and $V_{\max}$ and K_m are constants.

$V_{\max}$ is the maximal velocity of the reaction and K_m is the Michaelis constant.

K_m is the substrate concentration at which the reaction achieves half of the maximum velocity.

Graph V assuming that $V_{\max} = 3$ and $K_m = 2$. That is,

$$V = \frac{3[\mathrm{S}]}{2 + [\mathrm{S}]}.$$

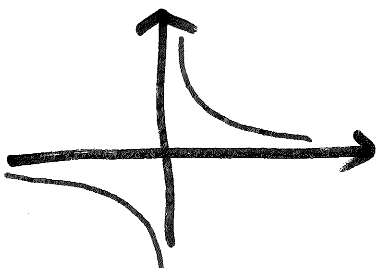
$$V = \frac{V_{max}[S]}{K_m + [S]} \rightsquigarrow V = \frac{3[S]}{2 + [S]}$$ or if you prefer to change variables $$y = \frac{3x}{2+x}$$

Rewrite as: $$\boxed{y = \frac{3x}{2+x}} = \frac{3x + 6 - 6}{x+2} = \frac{3(x+2) - 6}{x+2}$$

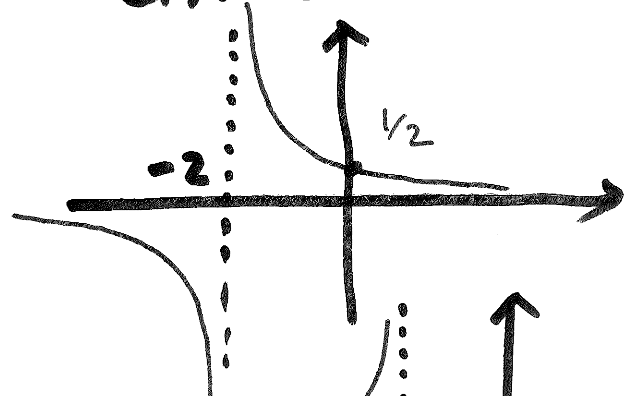
$$= \frac{3(x+2)}{x+2} - \frac{6}{x+2} = \boxed{3 - \frac{6}{x+2}}$$

Now use the elementary transformations:

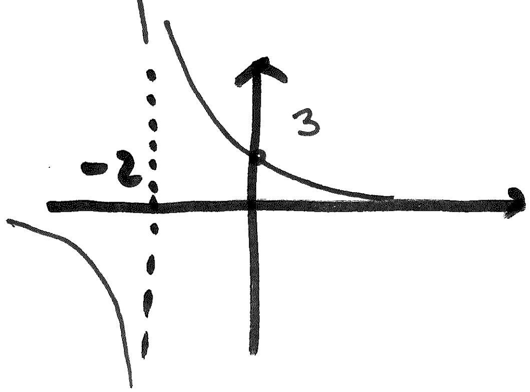
$y = \frac{1}{x}$



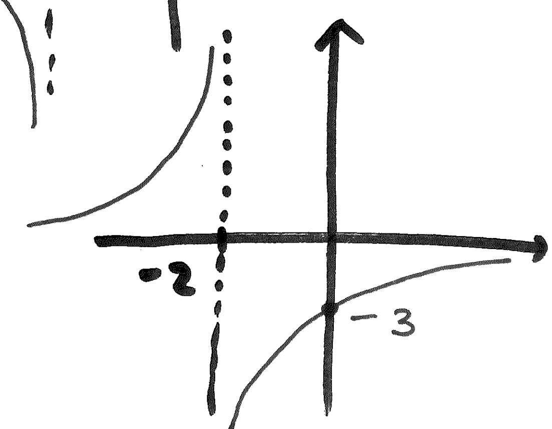
$\rightsquigarrow$ $y = \frac{1}{x+2}$



$y = \frac{6}{x+2}$

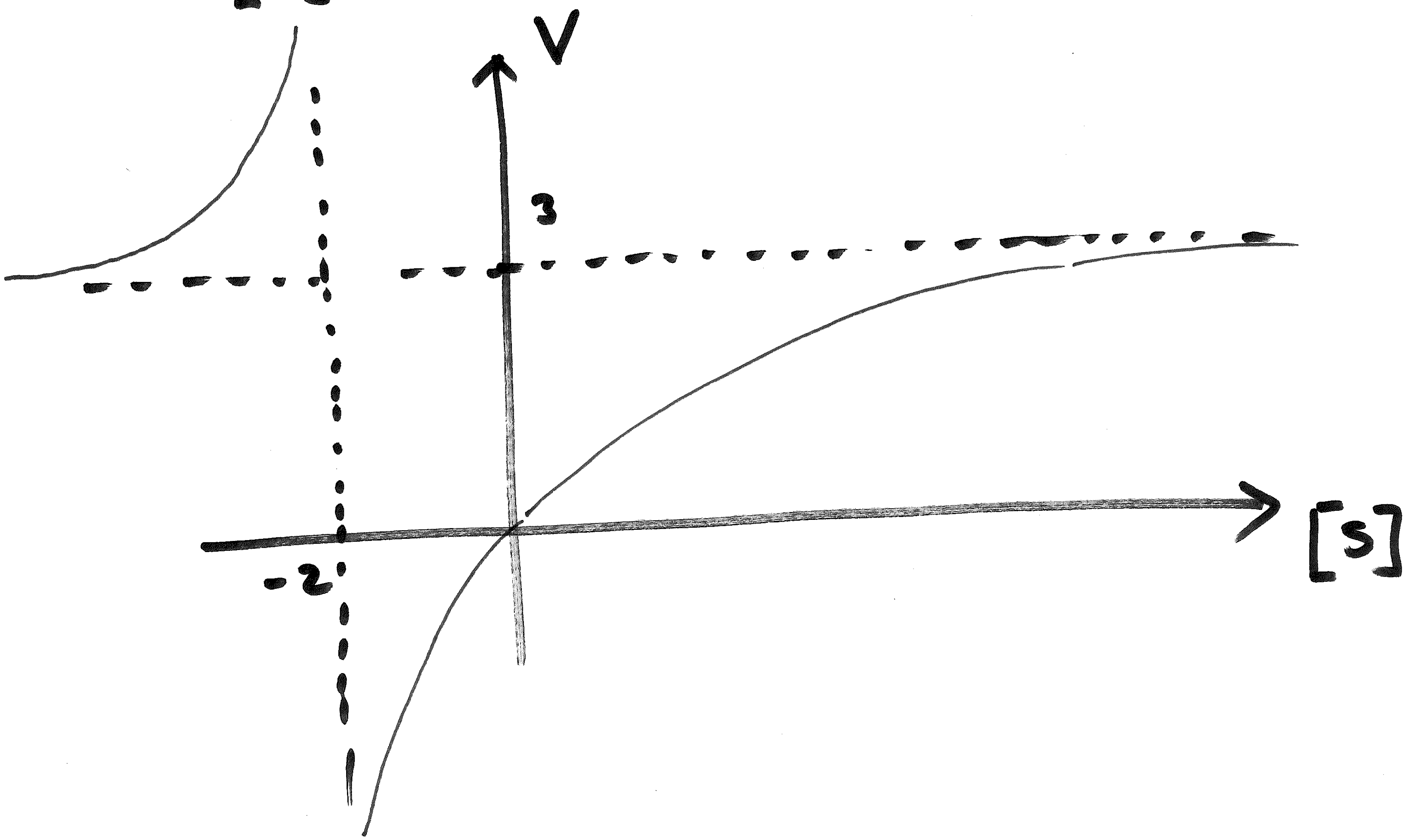


$y = -\frac{6}{x+2}$



Finally and changing back the variables

$$V = \frac{3[S]}{2+[S]} = 3 - \frac{6}{[S]+2}$$



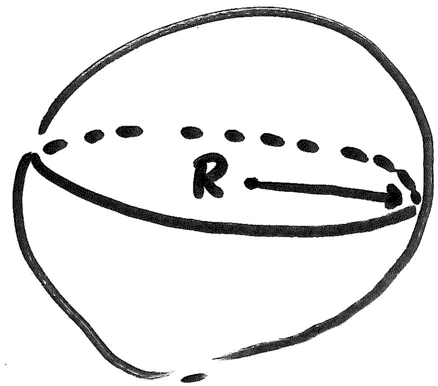
Example 4:

Find the scaling relation between the surface area S and the volume V of a sphere of radius R.

[More precisely, show that $S = (36\pi)^{1/3}\, V^{2/3}$, that is, $S \propto V^{2/3}$.]

Recall that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3$$



The surface area of a sphere of radius R is: $S = 4\pi R^2$

We want to write S as a function of V.

FROM: $V = \frac{4}{3}\pi R^3 \longrightarrow \frac{3}{4\pi} V = R^3$

$$\longrightarrow R = \sqrt[3]{\frac{3}{4\pi} V} = \left(\frac{3}{4\pi} V\right)^{1/3}$$

Substitute in $S = 4\pi R^2$ to get

$$S = 4\pi\left[\left(\frac{3}{4\pi} V\right)^{1/3}\right]^2$$

$$\therefore \quad S = 4\pi \left(\frac{3}{4\pi}\right)^{2/3} \cdot V^{2/3}$$

$$= \left[(4\pi)^3 \left(\frac{3}{4\pi}\right)^2\right]^{1/3} \cdot V^{2/3}$$

$$= \left(64\pi^3 \cdot \frac{9}{16\pi^2}\right)^{1/3} \cdot V^{2/3}$$

$$= (36\pi)^{1/3} \cdot V^{2/3}$$

<u>i.e.</u> $\boxed{S \propto V^{2/3}}$

MA137 – Calculus 1 with Life Science Applications
Operations on Functions
Inverse of a Function and its Graph
(Sections 1.2 & 1.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

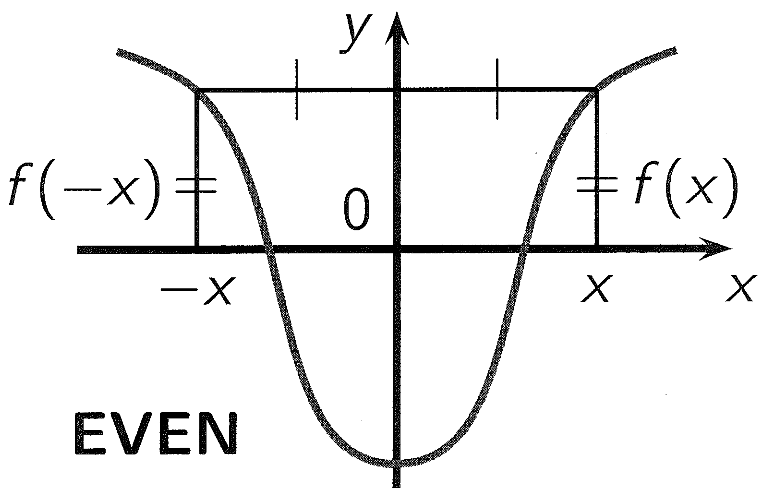
August 29, 2016

Even and Odd Functions

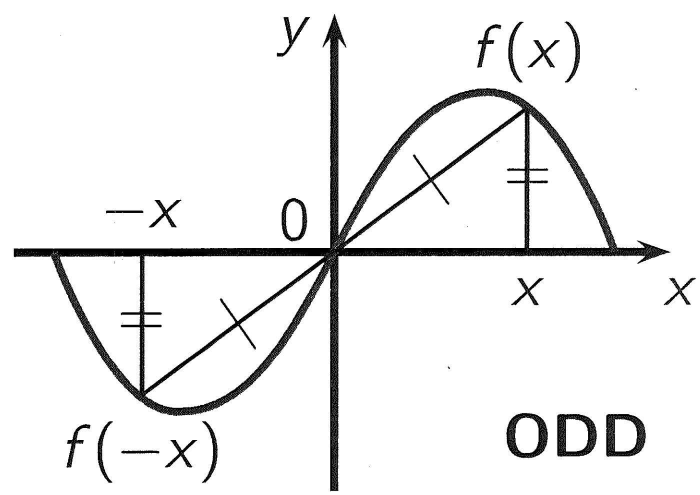
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



Graph symmetric wrt y-axis.



Graph symmetric wrt $(0, 0)$.

Example:

$y = \cos x$ is an **even** function; $y = \sin x$ is an **odd** function.

Example 1:

Determine whether the following functions are even or odd:

$$f(x) = x^3 + 2x^5$$

$$g(x) = x^2 - 3x^4$$

(a) $f(x) = x^3 + 2x^5$

$f(-x) = (-x)^3 + 2(-x)^5 = -x^3 - 2x^5 = -[x^3 + 2x^5]$

$= -f(x)$

thus f is an odd function

(b) $g(x) = x^2 - 3x^4$

$g(-x) = (-x)^2 - 3(-x)^4 = x^2 - 3x^4 = g(x)$

thus g is an even function

NOTE:

In (a) the polynomial only has odd exponents

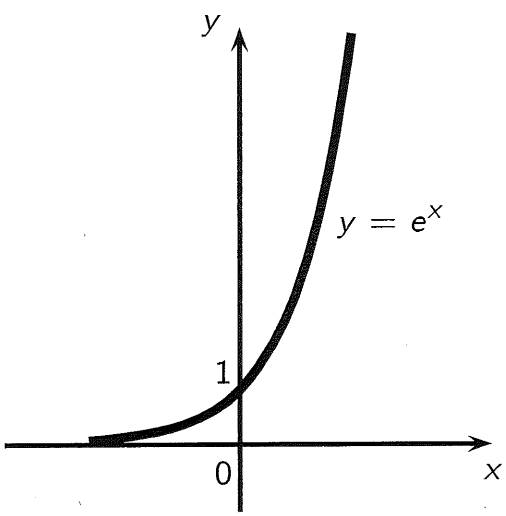
In (b) the polynomial only has even exponents

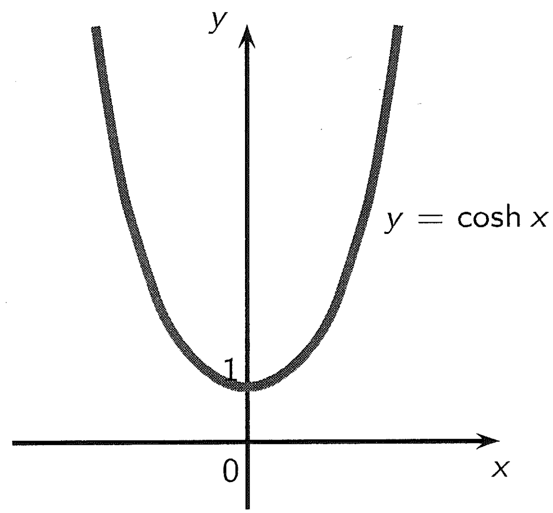
Curious/Amazing Fact!

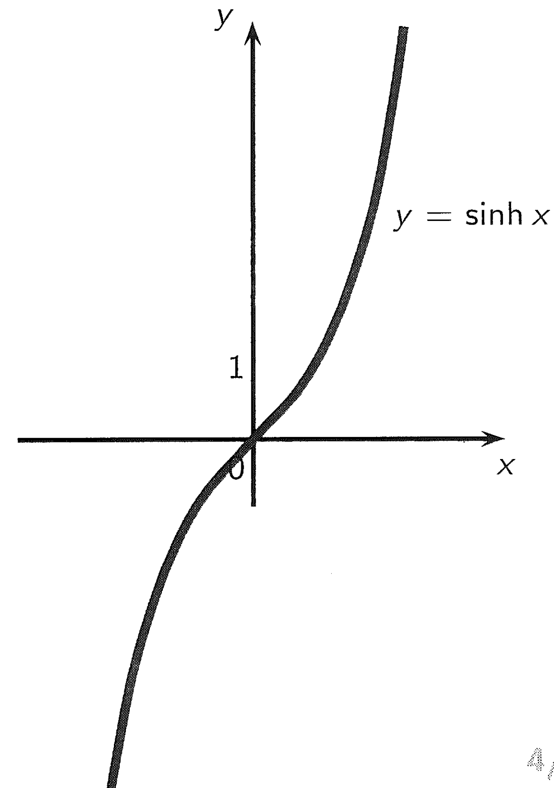
Any function can be uniquely written as an even plus an odd function.

Example:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x}$$







Example 2: (Online Homework HW02, #11)

(Since we talked about trigonometric functions...)

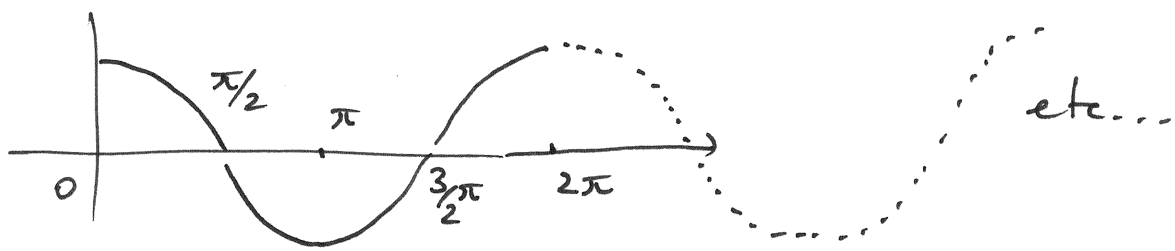
The lungs do not completely empty or completely fill in normal breathing. The volume of the lungs normally varies between 2140 ml and 2700 ml with a breathing rate of 22 breaths/min. This exchange of air is called the *tidal volume*.

One approximation for the volume of air in the lungs uses the cosine function written in the following manner:

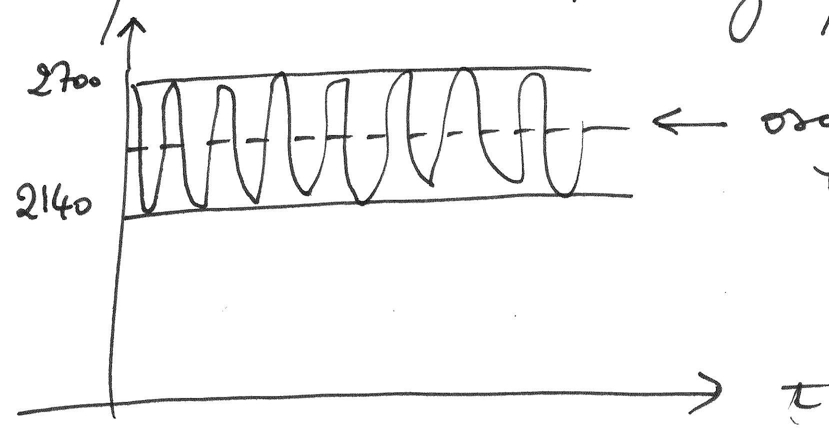
$$V(t) = A + B\cos(\omega t),$$

where A, B, and ω are constants and t is in minutes. Use the data above to create a model, finding the constants $A =$ _______, $B =$ _______, and $\omega =$ _______, that simulates the normal breathing of an individual for one minute.

Fact : $y = \cos(t)$ is a period function, of period 2π, with values between 1 and -1. That is the amplitude of the oscillation is 2 and the oscillation is along the line $y = 0$.



In our case $V(t) = A + B\cos(wt)$ represents the volume of the lungs. This volume ranges between 2140 and 2700 ml. There must be 22 full cycles in a 1 minute period. The graph must look like:



Notice that the amplitute of the oscillations is $2700 - 2140 = 560$ hence B must be half of that: $\boxed{B = 280}$. Moreover the oscillations must be about the line $V_0 = 2140 + 280 = \underline{2420}$

(Observe $2420 = \frac{2140 + 2700}{2} = 2700 - 280$) $\therefore$ $\boxed{A = 2420}$

$V(t) = 2420 + 280 \cos(\underline{\underline{\omega}} t)$

To determine $\underline{\underline{\omega}}$, observe that since we have 22 breaths in one minute, the volume must have the same values at time: $t, \ t + \frac{1}{22}, \ t + \frac{2}{22}, \ t + \frac{3}{22}, \ \ldots$ $\ldots, \ t + \frac{22}{22} = t + 1$. That is

$$\underbrace{V(t) = V\left(t + \tfrac{1}{22}\right)} = V\left(t + \tfrac{2}{22}\right) = \text{etc...}$$

in particular, during the interval $t, \ t + 1/22$ we have one full breath.

This means:

$$V(t) = V(t + 1/22) \iff A + B\cos(\omega t) \overset{\text{MUST}}{=} A + B\cos\left(\omega\left(t + \frac{1}{22}\right)\right)$$

$$\iff \cos(\omega t) = \cos\left(\omega t + \omega \cdot \frac{1}{22}\right)$$

Because $\cos(\cdot)$ is periodic of period 2π

we must have $\omega \cdot \frac{1}{22} = 2\pi$

$$\implies \omega = 2\pi \cdot 22 = 44\pi = 138.23$$

Thus: $\boxed{V(t) = 2420 + 280\cos(138.23\, t)}$ //

Combining functions

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B \,|\, g(x) \neq 0\}$$

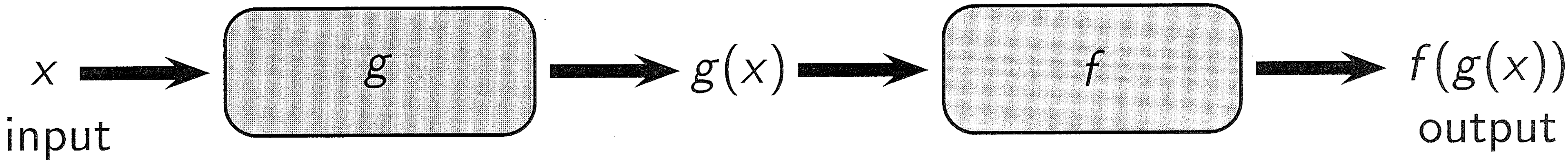
Composition of Functions

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

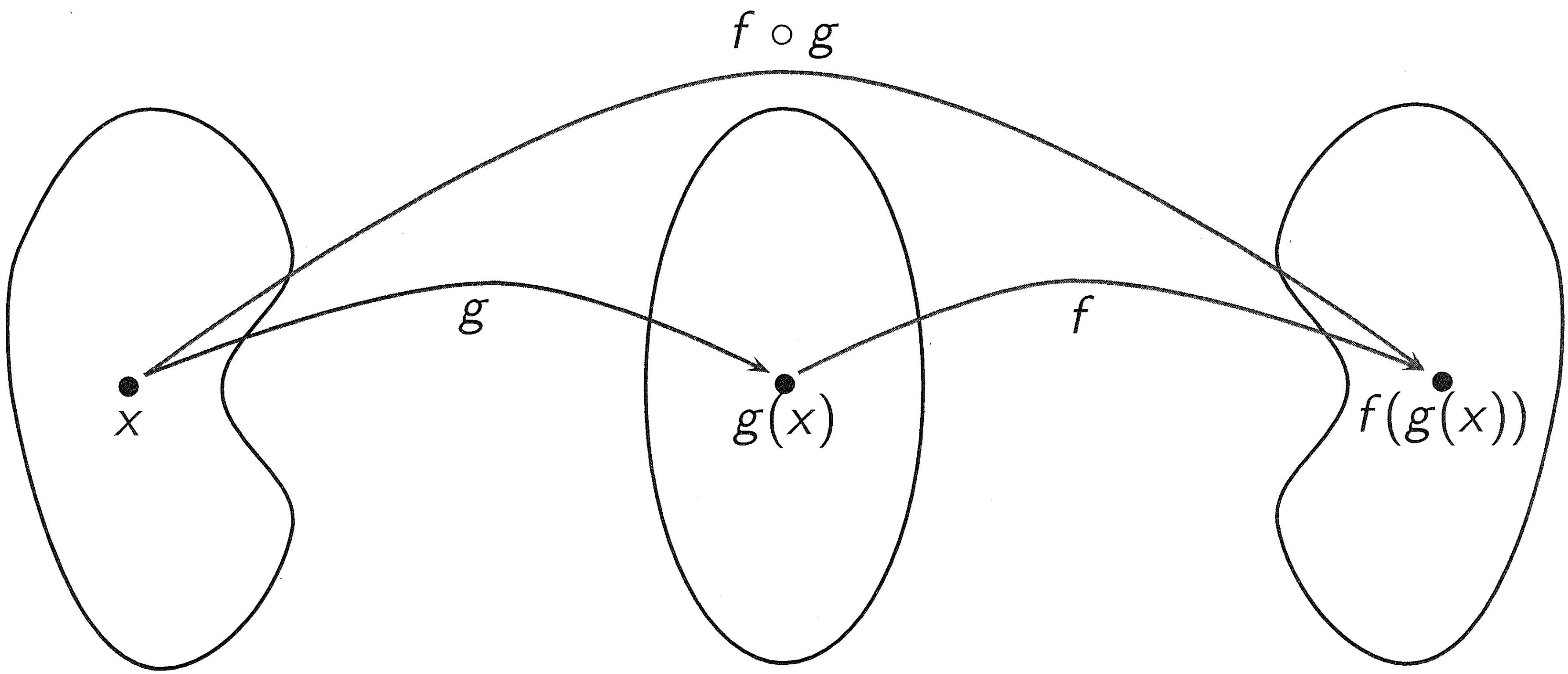
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$\boxed{(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).}$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

Example 3:

Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

$$f(x) = \frac{x}{x+1} \qquad g(x) = 2x-1$$

(*) $$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{2x-1}{(2x-1)+1} = \boxed{\frac{2x-1}{2x}} \checkmark$$

$$= 1 - \frac{1}{2x}$$

(**) $$(g \circ f)(x) = g(f(x)) = 2f(x) - 1 = 2 \cdot \frac{x}{x+1} - 1$$

$$= \frac{2x}{x+1} - 1 = \frac{2x - (x+1)}{x+1} = \boxed{\frac{x-1}{x+1}} \checkmark$$

(***) $$(f \circ f)(x) = f(f(x)) = \frac{f(x)}{f(x)+1} =$$

$$= \frac{x/_{x+1}}{x/_{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{x+(x+1)}{x+1}} = \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{2x+1}$$

$$= \frac{x}{2x+1}$$

Example 4:

Express the function $F(x) = \dfrac{x^2}{x^2 + 4}$ in the form $F(x) = f(g(x))$.

$$F(x) = \frac{x^2}{x^2+4}$$

can be thought of as the following composition

$$x \overset{g}{\longmapsto} x^2 \overset{f}{\longmapsto} \frac{x^2}{x^2+4}$$

thus: $g(x) = x^2$; $f(x) = \dfrac{x}{x+4}$

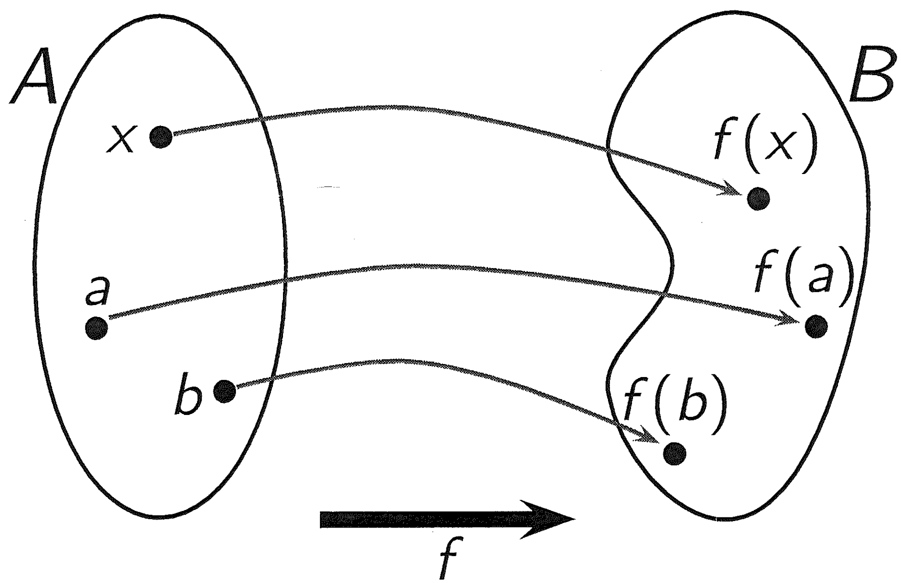
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If} \ \ f(x_1) = f(x_2), \ \text{then} \ \ x_1 = x_2.$$



Horizontal Line Test

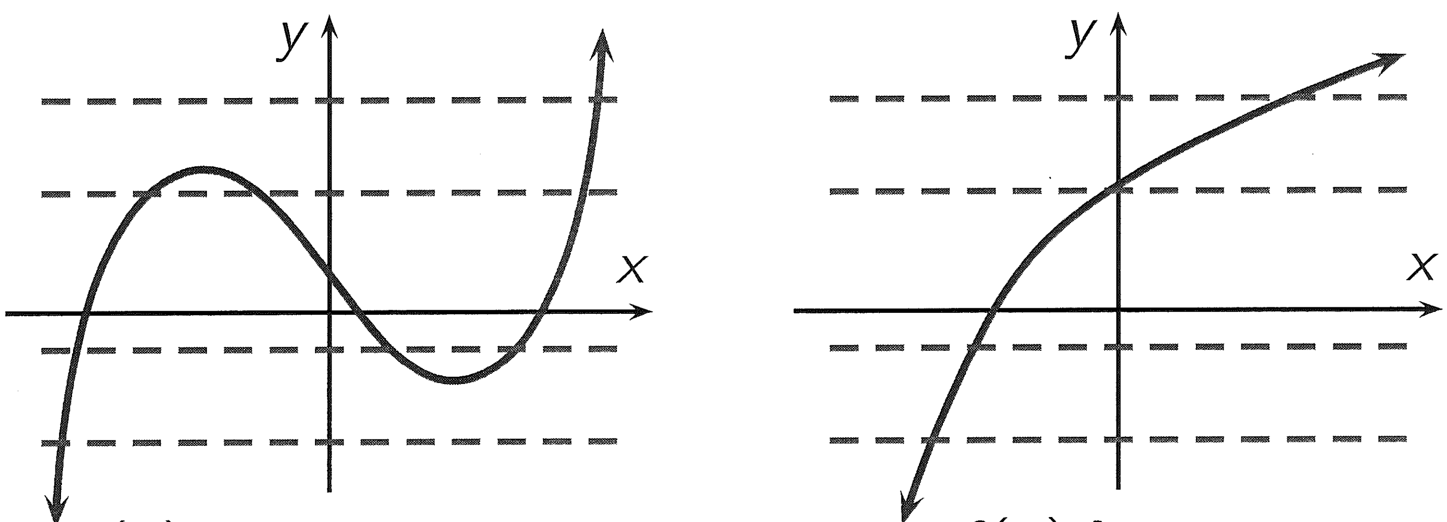
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one

$$\Longleftrightarrow$$

no horizontal line intersects its graph more than once.



$f(x)$ **is not** one-to-one $\qquad$ $f(x)$ **is** one-to-one

The Inverse of a Function

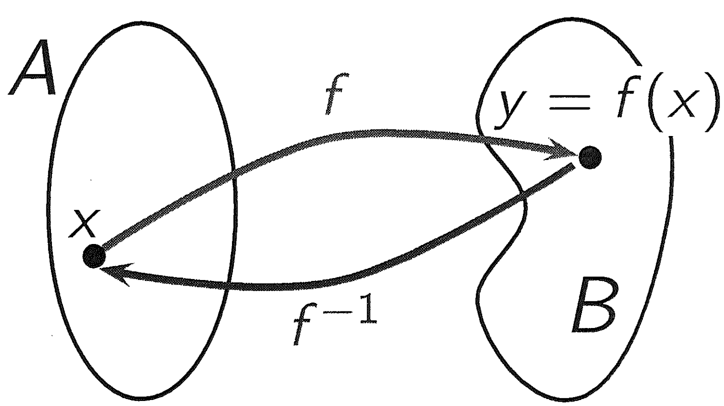
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \iff f(x) = y,}$$

for any $y \in B$.



If f takes x to y,
then f^{-1} takes y back to x.
I.e., f^{-1} undoes what f does.
NOTE:
f^{-1} does NOT mean $\dfrac{1}{f}$.

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B. The inverse function $f^{-1}(y)$ satisfies the following "cancellation" properties:

$$\boxed{f^{-1}(f(x)) = x \quad \text{for every} \quad x \in A}$$

$$\boxed{f(f^{-1}(y)) = y \quad \text{for every} \quad y \in B}$$

Conversely, any function $f^{-1}(y)$ satisfying the above conditions is the inverse of $f(x)$.

Remark:

Typically we write functions in terms of x.
To do this, we need to interchange x and y in $x = f^{-1}(y)$.

Example 5:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$$f(x) = x^5 \qquad g(x) = x^{1/5}$$

$$(f \circ g)(x) = f(g(x)) = [g(x)]^5 = [x^{1/5}]^5 = x$$

$$(g \circ f)(x) = g(f(x)) = [f(x)]^{1/5} = [x^5]^{1/5} = x$$

thus: $(f \circ g)(x) = x$ ✓

and

$(g \circ f)(x) = x$ ✓

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 6: (Online Homework HW02, # 12)

Find the inverse of $y = \dfrac{2-3x}{8-7x}$.

1. $y = \dfrac{2-3x}{8-7x}$

2. Solve for x in terms of y

$y(8-7x) = 2-3x \longrightarrow 8y - 7xy = 2 - 3x$

$3x - 7xy = 2 - 8y \longrightarrow x(3-7y) = 2-8y$

$\longrightarrow x = \dfrac{2-8y}{3-7y}$

3. Interchange x and y

$\therefore \boxed{y = \dfrac{2-8x}{3-7x}}$

Example 7: (Exam 1, Spring 15, # 4)

One of the main quantities that epidemiologists try to measure for infectious diseases is the so-called basic reproduction number, R_0. Biologically, this is the expected number of new infections that an infected individual will produce when introduced into a completely susceptible population.

We can try to modify this by introducing vaccination to control the probability of an outbreak of the disease. We want to know the fraction of the population that we have to vaccinate to achieve a target outbreak probability. If v is the vaccination fraction, then the outbreak probability as a function of v is

$$P = 1 - \frac{1}{R_0(1-v)}.$$

Find the inverse of this function to obtain v, the vaccination coverage needed, as a function of P, the given target outbreak probability.

$$P = 1 - \frac{1}{R_0(1-v)} \longrightarrow \frac{1}{R_0(1-v)} = 1 - P$$

$$\longrightarrow \frac{1}{R_0(1-P)} = 1 - v \longrightarrow \boxed{v = 1 - \frac{1}{R_0(1-P)}}$$

that is we wrote the vaccination coverage needed v in terms of the target outbreak probability P.

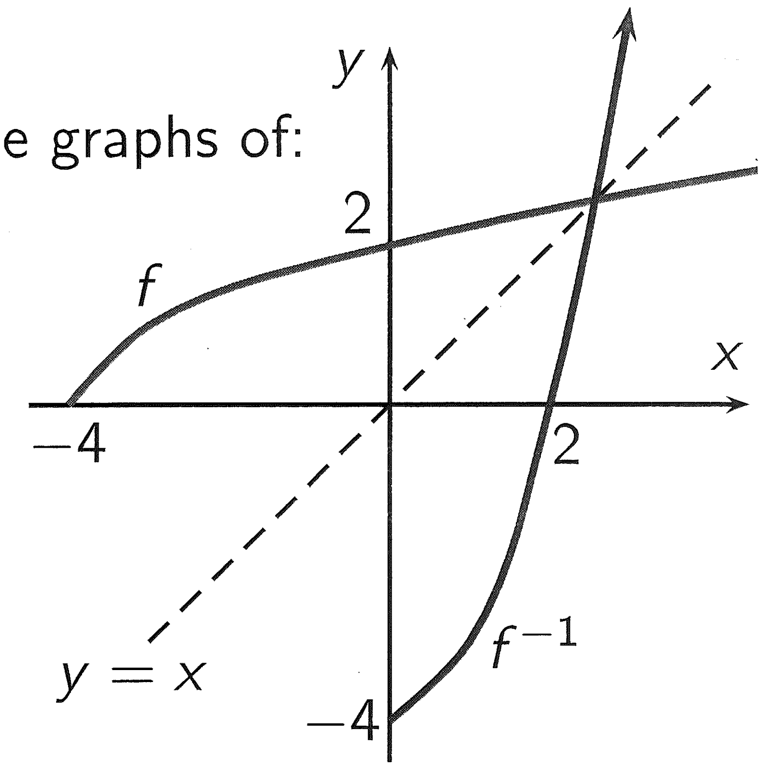
(no need to exchange $v \longleftrightarrow P$)

Graph of the Inverse Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$
and
$$f^{-1}(x) = x^2 - 4, \ x \geq 0.$$



Example 8:

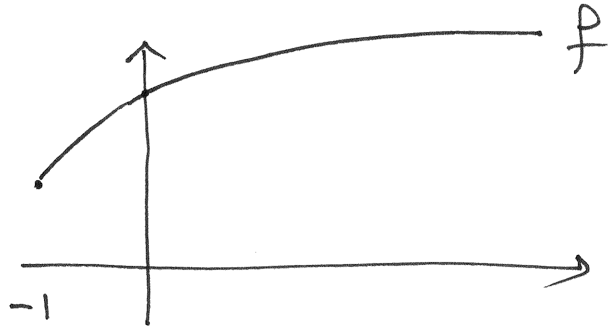
Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.

Find the domain and range of f and f^{-1}.

Graph f and f^{-1} on the same cartesian plane.

$f(x) = 1 + \sqrt{1+x}$

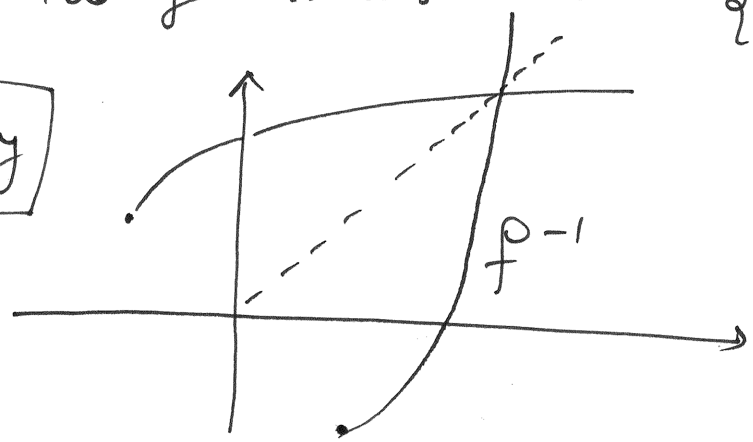
has domain $\{x \in \mathbb{R} \mid x \geq -1\}$ and range $\{y \in \mathbb{R} \mid y \geq 1\}$



the domain of the inverse f^{-1} must be $\{x \in \mathbb{R} \mid x \geq 1\}$

and the range must be $\{y \in \mathbb{R} \mid y \geq -1\}$

graphically



algebraically

$y = 1 + \sqrt{1+x}$

$\longrightarrow y - 1 = \sqrt{1+x}$

$\longrightarrow (y-1)^2 = 1 + x$

$\longrightarrow x = (y-1)^2 - 1 \longrightarrow \boxed{y = (x-1)^2 - 1}$ $\quad x \geq 1$

MA137 – Calculus 1 with Life Science Applications
Exponential and Logarithmic Functions
(Sections 1.1 and 1.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

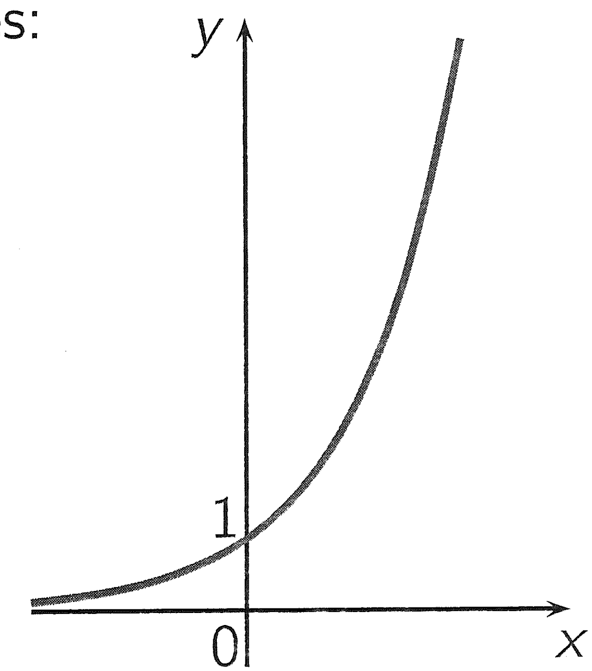
August 31, 2016

Exponential Functions

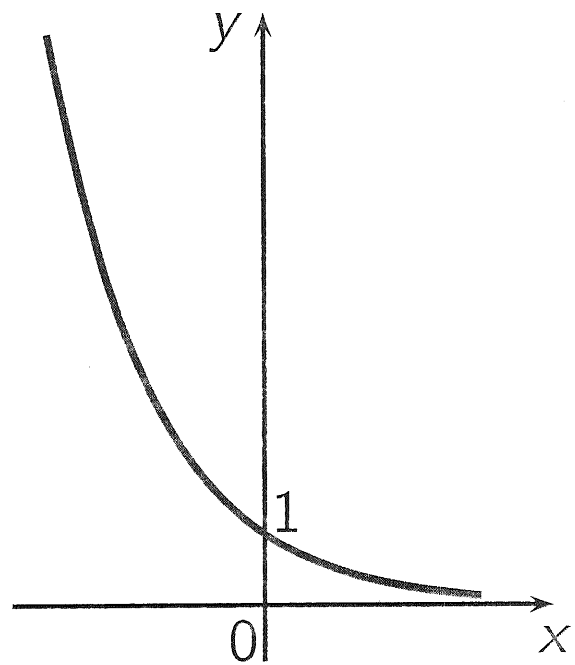
The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$
for $0 < a < 1$

Laws of Exponents

Let a and b be real numbers so that $a, b > 0$ and $a, b \neq 1$.

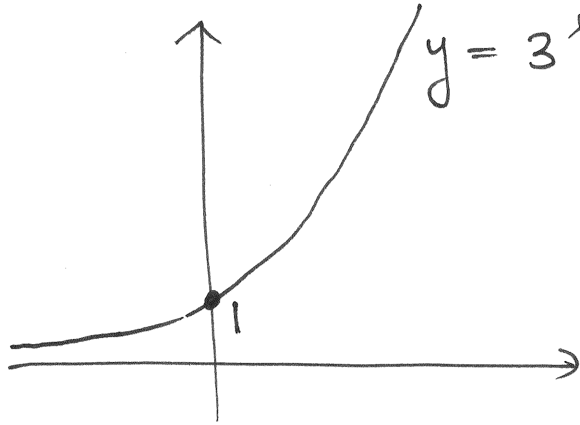
- $a^0 = 1$
- $a^u a^v = a^{u+v}$
- $\dfrac{a^u}{a^v} = a^{u-v}$ In particular, $\dfrac{1}{a^v} = \dfrac{a^0}{a^v} = a^{0-v} = a^{-v}$
- $(a^u)^v = a^{uv}$ In particular, $a^{1/n} = \sqrt[n]{a}$
- $(ab)^u = a^u b^u$ $\left(\dfrac{a}{b}\right)^u = \dfrac{a^u}{b^u}$

Example 1:

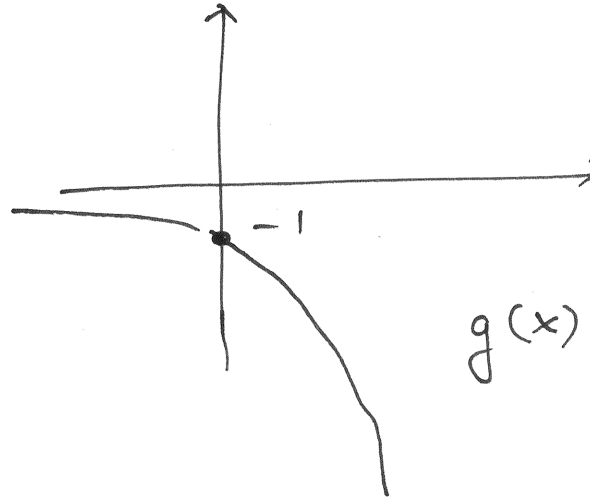
Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

$g(x) = -3^x$

$h(x) = 1 - 3^{-x}$

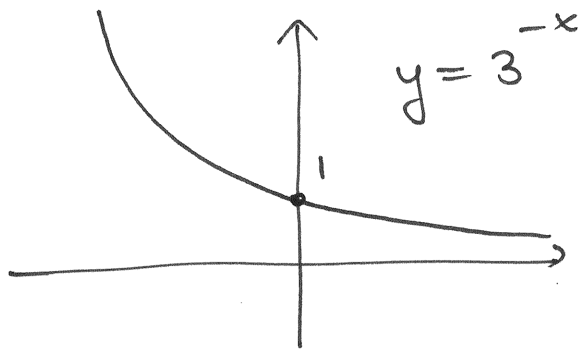


$y = 3^x$

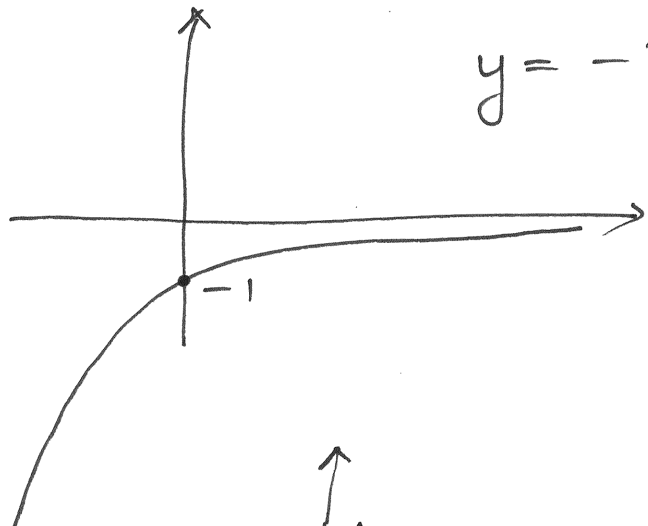


$g(x) = -3^x$

Let's construct the graph of $h(x) = 1 - 3^{-x}$ in steps:

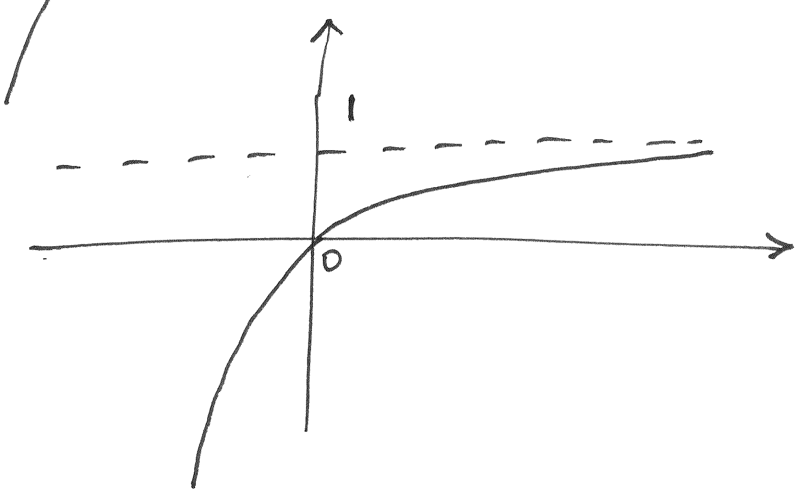


$y = 3^{-x} = \left(\frac{1}{3}\right)^x$



$y = -3^{-x}$

Finally, $y = 1 - 3^{-x}$



Why are exponential functions of interest?

Suppose that we study a population of 100 individuals and suppose that it grows annually at a 3% rate. Describe the population growth at time t.

(You can also consider \$ 100 in a bank growing annually at a 3%)

$P_0 = P(0) = 100$; growth rate $= 3\%$ or $r = 0.03$

$$P(1) = 100 + \underbrace{0.03 \cdot 100}_{\text{growth}} = 100 + 3 = 103 = 100\,(1.03)$$

$$P(2) = 103 + \underbrace{0.03 \cdot 103}_{\text{growth}} = 103\,(1 + 0.03) = 100(1.03)(1.03) = 100(1.03)^2$$

In general

$$\boxed{P(t) = 100\,(1.03)^t}$$

OR

$$\boxed{P(t) = P_0\,(1+r)^t}$$

The formula for compounded interest is

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}$$

(see earlier discussion with $n = 1$)

where P_0 = initial principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years.

If n becomes very large ($\equiv$ interest is compounded continuously) the above formula becomes

$$P(t) = P_0 \, e^{rt}$$

The Number 'e' (Euler's constant)

The most important base is the number denoted by the letter e.

The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n becomes very large.

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The Natural Exponential Function

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

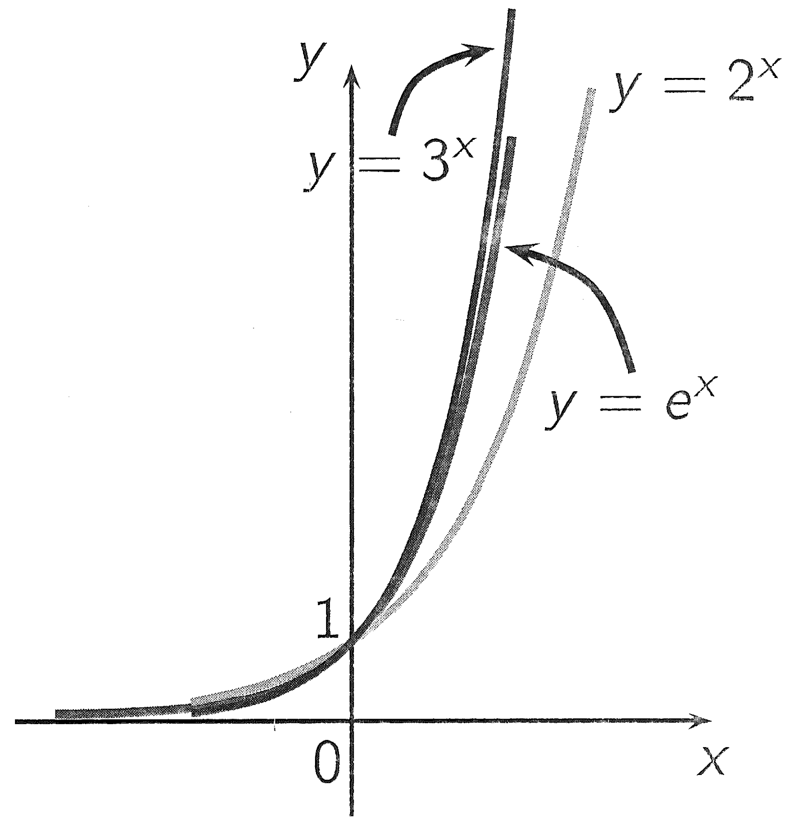
Note:

Sometimes we write

$$f(x) = \exp(x)$$

to denote the exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



Example 2:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\, e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

(*) Notice that $D(0) = 50\, e^{-0.2 \cdot 0} = 50 \underbrace{e^{0}}_{1}$

$= 50$

Thus 50 is the initial amount of drug administered to the patient.

(**) $D(3) = 50\, e^{-0.2 \cdot 3} = 50\, e^{-0.6} \cong 27.44$ mg

(***) It would have been more interesting to ask: How long do we need to wait so that the blood stream of the patient only has 25 mg left of drug?

$$25 = D(\bar{t}) = 50\, e^{-0.2\bar{t}} \iff \boxed{\frac{1}{2} = e^{-0.2\bar{t}}}$$

How do we solve for $\bar{t}$?

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $0 < a \neq 1$, is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by $\log_a x$.

Definition

Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x.$$

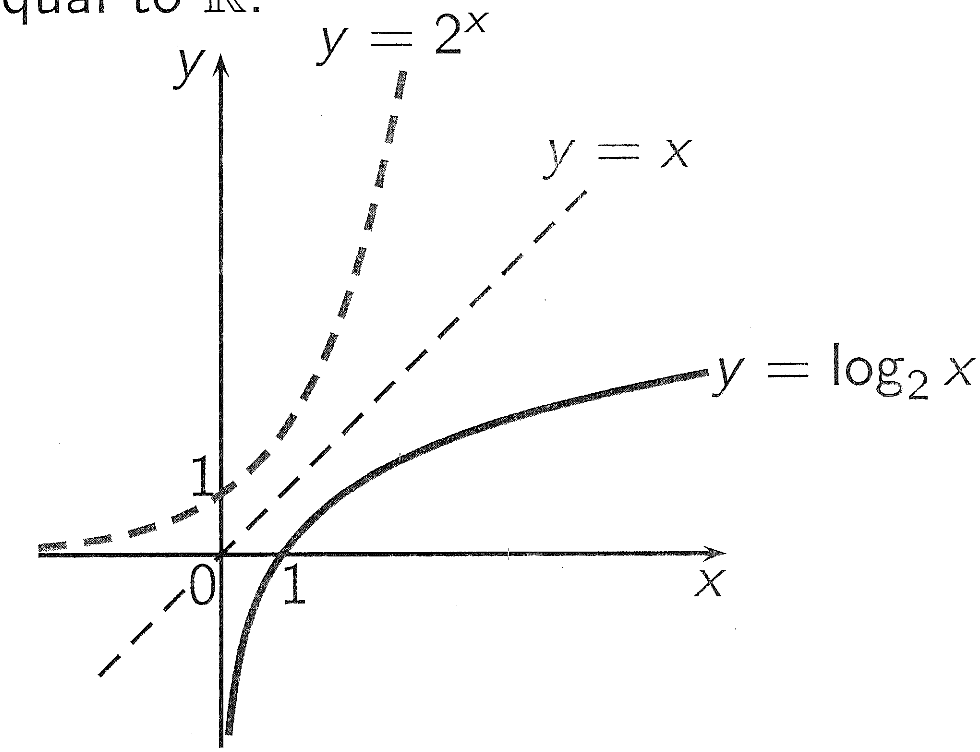
That is, $\log_a x$ is the exponent to which a must be raised to give x.

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Graphs of Logarithmic Functions

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Thus, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$.



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition

The logarithm with base e is called the **natural logarithm** and denoted:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \qquad \Longleftrightarrow \qquad e^y = x.$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Common Logarithms

Another convenient choice of base for the purposes of the Life Sciences is the number 10.

Definition

The logarithm with base 10 is called the **common logarithm** and denoted:

$$\boxed{\log x := \log_{10} x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \log x \qquad \Longleftrightarrow \qquad 10^y = x.$$

Properties of Natural Logarithms

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 10^x = x$
4. $10^{\log x} = x$

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C \log_a A$.

Proof of **Law 1.**: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:

$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u\, a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u+v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Expanding and Combining Logarithmic Expressions

Example 3:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

into a single logarithm.

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

$$= \left[\log_a b + \log_a (d^c)\right] - \left[\log_a (s^r) + \log_a t\right]$$

$$= \log_a (b d^c) - \log_a (s^r t)$$

$$= \log_a \left(\frac{b d^c}{s^r t}\right)$$

We used properties 1.–3. of Logarithms.

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Proof: Set $y = \log_b x$. By definition, this means that $b^y = x$. Apply now $\log_a(\cdot)$ to $b^y = x$. We obtain

$$\log_a(b^y) = \log_a x \qquad \rightsquigarrow \qquad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}.$$

Example: $\log_5 2 = \dfrac{\log 2}{\log 5} = \dfrac{\ln 2}{\ln 5} \approx 0.43068.$

Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x+2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7$$

$$\rightsquigarrow \quad (x+2)\log 3 = \log 7$$

$$\rightsquigarrow \quad x + 2 = \frac{\log 7}{\log 3}$$

$$\rightsquigarrow \quad x = \frac{\log 7}{\log 3} - 2 \approx -0.228756$$

Example 4: (Online Homework HW03, # 6)

Solve the given equation for x:

$$2^{5x-4} = 3^{10x-10}$$

$$2^{5x-4} = 3^{10x-10}$$

Take log of both sides (OR ln)

$$\log\left(2^{5x-4}\right) = \log\left(3^{10x-10}\right)$$

$$\iff (5x-4)\log 2 = (10x-10)\log 3$$

$$\iff (5\log 2)x - 4\log 2 = (10\log 3)x - 10\log 3$$

$$\iff (10\log 3)x - (5\log 2)x = 10\log 3 - 4\log 2$$

$$\iff \left[10\log 3 - 5\log 2\right]x = 10\log 3 - 4\log 2$$

$$\iff x = \frac{(10\log 3 - 4\log 2)}{(10\log 3 - 5\log 2)} = \frac{\log\left(\frac{3^{10}}{2^4}\right)}{\log\left(3^{10}/2^5\right)} \cong \underline{1.09216}$$

Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25 - x) = 3.$$

To solve for x, we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \quad \rightsquigarrow \quad 25 - x = 8 \quad \rightsquigarrow \quad x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \quad \rightsquigarrow \quad 25 - x = 2^3 \quad \rightsquigarrow \quad x = 17.$$

Example 5: (Online Homework HW03, # 5)

Solve the given equation for x:

$$\log_{10} x + \log_{10}(x + 21) = 2$$

$$\log_{10} x + \log_{10}(x+21) = 2$$

$$\iff \log_{10}\left[x(x+21)\right] = 2$$

$$\iff 10^{\log_{10}\left[x(x+21)\right]} = 10^2$$

$$\iff \boxed{x(x+21) = 100}$$

$$\iff x^2 + 21x - 100 = 0$$

$$\iff (x+25)(x-4) = 0$$

$$\iff \underline{x = -25,\ 4}$$

<u>HOWEVER</u>, $\log_{10}(-25) + \log_{10}(-25+21) = 2$ does not make any sense! So $\boxed{x = 4}$ is the only solution.

MA137 – Calculus 1 with Life Science Applications
Applications: Exponential Growth and Decay
(Sections 1.1 and 1.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

September 2, 2016

Exponential Models of Population Growth

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth r of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Exponential Growth Model

If n_0 is the initial size of a population that experiences **exponential growth**, then the population $n(t)$ at time t increases according to the model

$$\boxed{n(t) = n_0 e^{rt}}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Remark:

Biologists sometimes express the growth rate in terms of the **doubling-time** h, the time required for the population to double in size: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$2n_0 = n(h) = n_0 e^{rh}$$

we obtain

$$2 = e^{rh} \quad \rightsquigarrow \quad \ln 2 = rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the doubling-time h, we can also rewrite $n(t)$ as:

$$\boxed{n(t)} = n_0 e^{rt} = n_0 e^{\frac{\ln 2}{h} t} = n_0 e^{(t/h)\cdot \ln 2} = n_0 e^{\ln(2^{t/h})} \boxed{= n_0 2^{t/h}}.$$

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiations. Also in this situation, the rate of decay is proportional to the mass of the substance and is independent of environmental conditions.

This is analogous to population growth, except that the mass of radioactive material *decreases*.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance then the mass $m(t)$ remaining at time t is modeled by the function

$$\boxed{m(t) = m_0 e^{-rt}}$$

where r is the relative rate of decay of the radioactive substance.

Remark:

Physicists sometimes express the rate of decay in terms of the **half-life** h, the time required for half the mass to decay: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$\frac{1}{2} m_0 = m(h) = m_0 e^{-rh}$$

we obtain

$$\frac{1}{2} = e^{-rh} \quad \rightsquigarrow \quad \ln\frac{1}{2} = -rh \quad \rightsquigarrow \quad -\ln 2 = -rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the half-time h, we can also rewrite $m(t)$ as:

$$\boxed{m(t)} = m_0 e^{-rt} = m_0 e^{-\frac{\ln 2}{h} t} = m_0 e^{(-t/h)\cdot \ln 2} = m_0 e^{\ln(2^{-t/h})} \boxed{= m_0 \left(\frac{1}{2}\right)^{t/h}}.$$

Example 1: (Online Homework HW03, # 8)

A town has population 750 people at year $t = 0$.
Write a formula for the population, P, in year t if the town

(a) Grows by 70 people per year
(b) Grows by 12% per year
(c) Grows at a continuous rate of 12% per year.
(d) Shrinks by 14 people per year.
(e) Shrinks by 4% per year.
(f) Shrinks at a continuous rate of 4% per year.

(a) $P(t) = \underline{750 + 70t}$

(b) $P(t) = 750(1+0.12)^t = \underline{750(1.12)^t}$

(c) $P(t) = \underline{750e^{0.12t}}$

(d) $P(t) = \underline{750 - 14t}$

(e) $P(t) = 750(1-0.04)^t = \underline{750(0.96)^t}$

(f) $P(t) = \underline{750e^{-0.04t}}$

Example 2 (Frog Population):

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

(a) Which function models the population after t years?
(b) Find the projected frog population after 3 years.
(c) When will the frog population reach 600?
(d) When will the frog population double?

(a) $n(t) = 85\, e^{0.18t}$

(b) $n(3) = 85\, e^{0.18(3)} = 85\, e^{0.54} \cong 145.86$

(c) We need to find $\bar{t}$ so that

$$85 e^{0.18\bar{t}} = n(\bar{t}) = 600 \Rightarrow e^{0.18\bar{t}} = \frac{600}{85}$$

$$\Rightarrow \ln e^{0.18\bar{t}} = \ln\left(\frac{600}{85}\right) \Rightarrow 0.18\bar{t} = \ln\left(\frac{600}{85}\right)$$

$$\therefore \bar{t} = \frac{\ln(600/85)}{0.18} \cong 10.857 \text{ years}$$

(d) Let h denote the doubling time. That is

$$\cancel{85} e^{0.18h} = n(h) = 2 \cdot \cancel{85}$$

$$\Rightarrow e^{0.18h} = 2 \Rightarrow \ln e^{0.18h} = \ln 2$$

$$\Rightarrow 0.18h = \ln 2 \Rightarrow h = \frac{\ln 2}{0.18} \cong 3.85 \text{ years}$$

Example 3: (Online Homework HW03, # 14)

Assume that the number of bacteria follows an exponential growth model: $P(t) = P_0 e^{kt}$. The count in the bacteria culture was 100 after 15 minutes and 1800 after 35 minutes.

(a) What was the initial size of the culture?

(b) Find the population after 105 minutes.

(c) How many minutes after the start of the experiment will the population reach $14,000$?

(a) Using our information we have

$$P(15) = P_0 e^{15k} = 100$$

$$P(35) = P_0 e^{35k} = 1800$$

Thus $P_0 = \frac{100}{e^{15k}}$ and $P_0 = \frac{1800}{e^{35k}}$

$$\Longrightarrow \frac{100}{e^{15k}} = \frac{1800}{e^{35k}} \Longrightarrow 100 e^{35k} = 1800 e^{15k}$$

OR

$$\frac{e^{35k}}{e^{15k}} = \frac{1800}{100} \Longrightarrow e^{35k-15k} = 18$$

$$\therefore \boxed{e^{20k} = 18} \Longrightarrow 20k = \ln 18 \Longrightarrow \boxed{k = \frac{\ln 18}{20}}$$

Thus $k = 0.14518$

What about P_0? Since $P_0 = \frac{100}{e^{15k}}$, for example, we obtain $$P_0 = \frac{100}{e^{15\left(\frac{\ln 18}{20}\right)}} = \frac{100}{e^{3/4(\ln 18)}}$$

$$= \frac{100}{e^{\ln(18^{3/4})}} = \boxed{\frac{100}{18^{3/4}}} \cong \underline{11.4431508}$$

(b) $$\underline{\underline{P(t)}} = \frac{100}{18^{3/4}} \cdot e^{\left[\frac{\ln 18}{20}\right]t} = \frac{100}{18^{3/4}} e^{\left[t/20 \ln 18\right]} =$$

$$= \frac{100}{18^{3/4}} \cdot e^{\ln(18^{t/20})} = \frac{100}{18^{3/4}} \cdot 18^{t/20}$$

$$= 100 \cdot 18^{(t/20 - 3/4)} = \underline{\underline{100 \cdot 18^{\left(\frac{t-15}{20}\right)}}}$$

$$\therefore \underline{\underline{P(105)}} = 100 \cdot 18^{\left(\frac{105-15}{20}\right)} = \underline{100 \cdot 18^{4.5}}$$

$$\cong \underline{\underline{44,537,544.88}}$$

(c) Finally,

$$14{,}000 = 100 \cdot 18^{\left(\frac{\bar{t}-15}{20}\right)}$$

$$\Rightarrow \quad 140 = 18^{\frac{\bar{t}-15}{20}}$$

$$\Rightarrow \quad \ln(140) = \left(\frac{\bar{t}-15}{20}\right) \cdot \ln(18)$$

$$\Rightarrow \quad 20 \ln(140) = \bar{t} \cdot \ln(18) - 15 \cdot \ln(18)$$

$$\Rightarrow \quad \frac{20 \ln(140) + 15 \ln(18)}{\ln(18)} = \bar{t}$$

$$\cong \underline{49.193818 9} \quad \text{minutes}$$

Example 4:

The mass $m(t)$ remaining after t days from a 40-g sample of thorium-234 is given by:

$$m(t) = 40e^{-0.0277\,t}.$$

(a) How much of the sample will be left after 60 days?

(b) After how long will only 10-g of the sample remain?

(a) $m(60) = 40\, e^{-0.0277 \cdot 60} \cong 7.59036 \text{ mg}$

(b) $10 = m(\bar{t}) = 40\, e^{-0.0277 \cdot \bar{t}}$

we need to find $\bar{t}$.

$$\Longrightarrow \quad \frac{1}{4} = e^{-0.0277\,\bar{t}}$$

$$\Longrightarrow \quad \ln(1/4) = \ln e^{-0.0277\,\bar{t}} = -0.0277\,\bar{t}$$

$$\therefore \quad \bar{t} = \frac{\ln(1/4)}{-0.0277} = \frac{\ln 1 - \ln 4}{-0.0277} = \frac{\ln 4}{0.0277}$$

$$\cong 50.046 \text{ days}$$

From Neuhauser's Textbook, p. 27

[...] Carbon 14 is formed high in the atmosphere. It is radioactive and decays into nitrogen (N^{14}).

There is an equilibrium between atmospheric carbon 12 (C^{12}) and carbon 14 (C^{14}) — a ratio that has been relatively constant over a fairly long period.

When plants capture carbon dioxide (CO_2) molecules from the atmosphere and build them into a product (such as cellulose), the initial ratio of C^{14} to C^{12} is the same as that in the atmosphere.

Once the plants die, however, their uptake of CO_2 ceases, and the radioactive decay of C^{14} causes the ratio of C^{14} to C^{12} to decline.

Because the law of radioactive decay is known, the change in ratio provides an accurate measure of the time since the plants death.

Example 5: (Neuhauser, Problem # 64, p.37)

The half-life of C^{14} is 5730 years. Suppose that wood found at an archeological excavation site contains about 35% as much C^{14} (in relation to C^{12}) as does living plant material.

Determine when the wood was cut.

By a previous discussion

$$\boxed{m(t) = m_0 \left(\tfrac{1}{2}\right)^{t/5730}}$$

Hence we are seeking $\bar{t}$ such that

$$0.35\, m_0 = m(\bar{t}) = m_0 \left(\tfrac{1}{2}\right)^{\bar{t}/5730}$$

$$\therefore \quad 0.35 = \left(\tfrac{1}{2}\right)^{\bar{t}/5730} \implies \ln(0.35) = \frac{\bar{t}}{5730} \ln(1/2)$$

$$\therefore \quad \bar{t} = \frac{5730 \ln(0.35)}{\ln(1/2)} = \frac{5730 \ln(0.35)}{-\ln(2)} =$$

$$= \frac{5730 \cdot (-1) \ln(0.35)}{\ln(2)} = \frac{5730 \ln\left(\frac{1}{0.35}\right)}{\ln(2)} \cong 8678.50428 \text{ years}$$

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large.

Using Calculus, the following model can be deduced from this law:

The Model

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_S, then the temperature of the object at time t is modeled by the function

$$\boxed{T(t) = T_S + D_0 e^{-kt}}$$

where k is a positive constant that depends on the object.

Example 6 (Cooling Turkey):

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 minutes?

(b) When will the turkey cool to 100°F?

(a) $D_0 = 185 - 75 = 110$

hence the temperature of the turkey is given by

$$T(t) = 75 + 110\, e^{-kt}$$

So $T(30) = 75 + 110\, e^{-30k} = 150$

$$\Longrightarrow \quad e^{-30k} = \frac{150 - 75}{110} \Longrightarrow \quad k = \frac{\ln(75/110)}{-30}$$

$\therefore \quad k \cong 0.01276$

Hence $\boxed{T(t) = 75 + 110\, e^{-0.01276\, t}}$

(b) Check that $T(\bar{t}) = 100 \Longrightarrow$

$\bar{t} = 116$ minutes (almost 2 hours)

Interested in Forensic Pathology?

Newton's Law of Cooling is used in **homicide investigations** to determine the time of death. Immediately following death, the body begins to cool (its normal temperature is $98.6°F$). It has been experimentally determined that the constant in Newton's Law of Cooling is $k \approx 0.1947$, assuming time is measured in hours.

MA137 – Calculus 1 with Life Science Applications
Semilog and Double Log Plots
(Section 1.3)

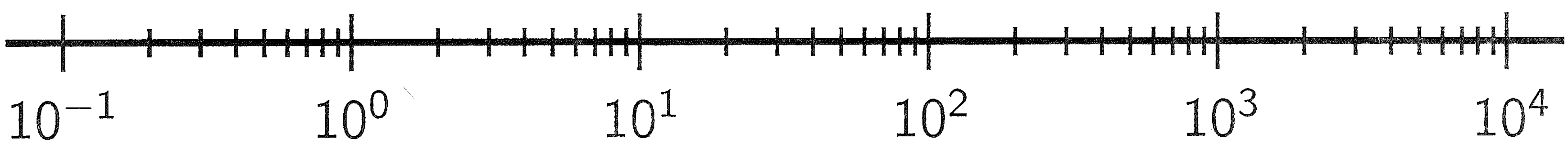
Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

September 7, 2016

Logarithmic Scales

- When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers.
- Quantities that are measured on logarithmic scales include
  - acidity of a solution (the **pH scale**),
  - earthquake intensity (Richter scale),
  - loudness of sounds (decibel scale),
  - light intensity,
  - information capacity,
  - radiation.
- In such cases, the equidistant marks on a logarithmic scale represent consecutive powers of 10.



The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, defined a more convenient measure:

$$\boxed{\mathrm{pH} = -\log[H^+]}$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M).

Solutions are defined in terms of the pH as follows:

those with $\mathrm{pH} = 7$ (or $[H^+] = 10^{-7}\mathrm{M}$) are *neutral*,

those with $\mathrm{pH} < 7$ (or $[H^+] > 10^{-7}\mathrm{M}$) are *acidic*,

those with $\mathrm{pH} > 7$ (or $[H^+] < 10^{-7}\mathrm{M}$) are *basic*.

Example 1 (Finding pH):

The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$M

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$M

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}$M

(a) Lemon juice $[H^+] = 5.0 \times 10^{-3}$ M

$$pH = -\log(5.0 \times 10^{-3}) = -\log(5) - \log(10^{-3})$$

$$= 3 - \log(5) = \underline{\underline{2.301}}$$

(b) Tomato juice $[H^+] = 3.2 \times 10^{-4}$ M

$$pH = -\log(3.2 \times 10^{-4}) = -\log(3.2) - \log(10^{-4})$$

$$= 4 - \log(3.2) = \underline{\underline{3.49485}}$$

(c) Seawater $[H^+] = 5.0 \times 10^{-9}$ M

$$pH = -\log(5.0 \times 10^{-9}) = -\log(5) - \log(10^{-9})$$

$$= 9 - \log(5) = \underline{\underline{8.301}}$$

Example 2 (Ion Concentration):

Calculate the hydrogen ion concentration of each substance from its pH reading.

(a) Vinegar: pH $= 3.0$

(b) Milk: pH $= 6.5$

(a) Vinegar : pH = 3.0

$$\Rightarrow \quad 3.0 = -\log[H^+] \quad \Rightarrow \quad \log[H^+] = -3$$

$$\Rightarrow \quad \boxed{[H^+] = 10^{-3}}$$

(b) Milk : pH = 6.5

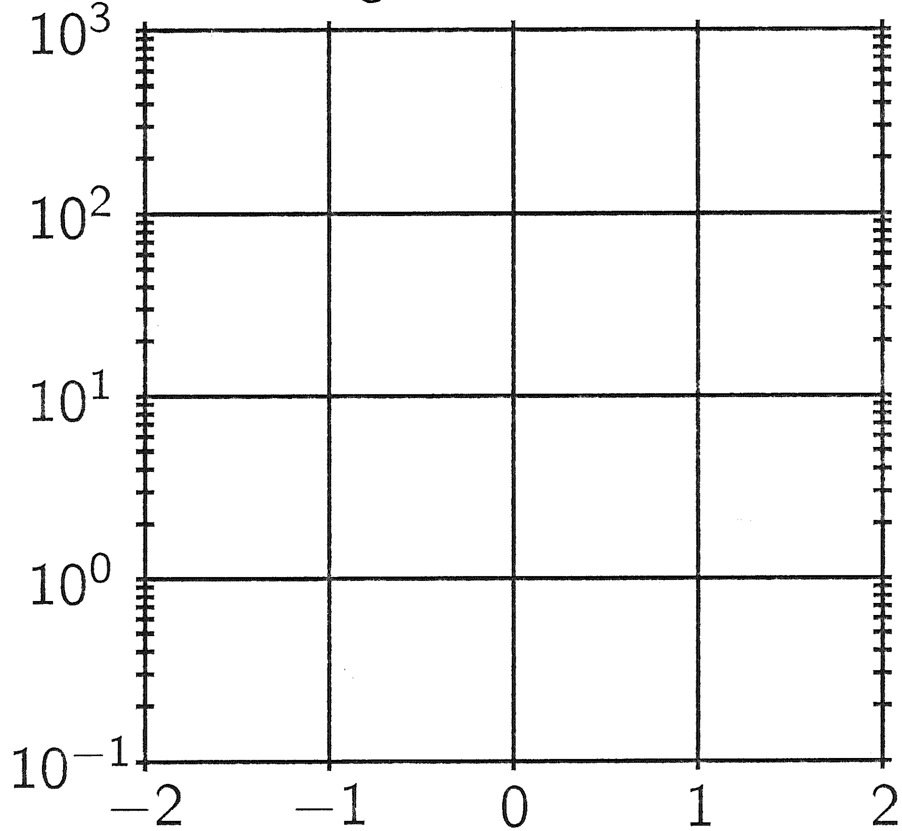
$$\Rightarrow \quad 6.5 = -\log[H^+] \quad \Rightarrow \quad \log[H^+] = -6.5$$

$$\Rightarrow \quad [H^+] = 10^{-6.5} = 10^{0.5} \cdot 10^{-0.5} \cdot 10^{-6.5}$$

$$= 3.2 \times 10^{-7}$$

Semilog Plots

- In biology its common to use a semilog plot to see whether data points are appropriately modeled by an exponential function.
- This means that instead of plotting the points (x, y), we plot the points $(x, \log y)$.
- In other words, we use a logarithmic scale on the vertical axis.



Graphs for a Science article

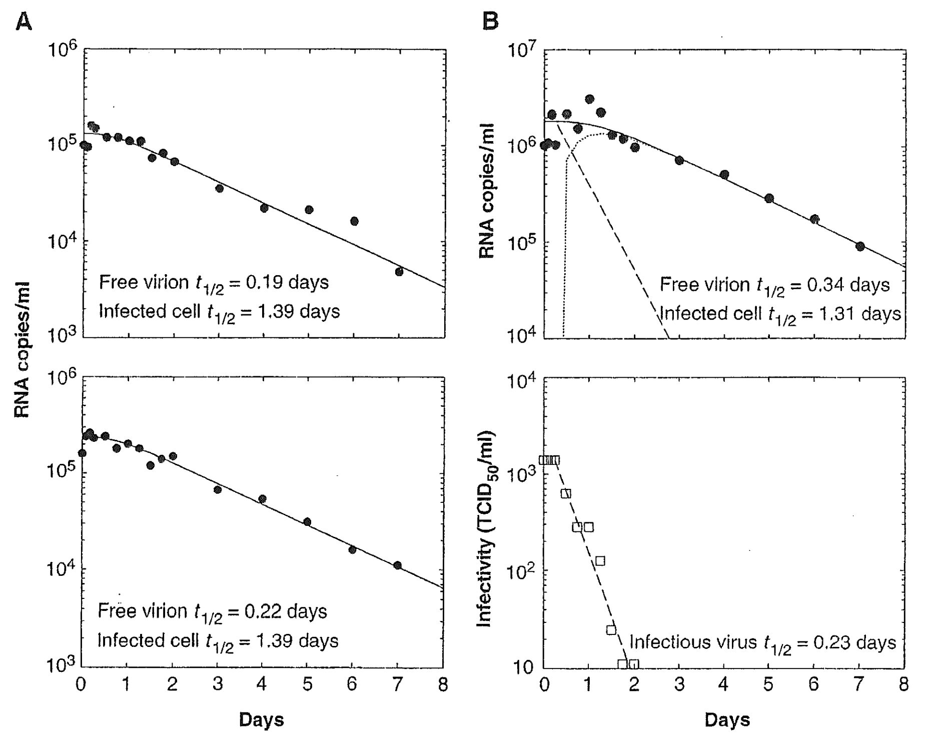


Fig. 1. (**A**) Plasma concentrations (copies per milliliter) of HIV-1 RNA (circles) for two representative patients (upper panel, patient 104; lower panel, patient 107) after ritonavir treatment was begun on day 0. The theoretical curve (solid line) was obtained by nonlinear least squares fitting of Eq. 6 to the data. The parameters c (virion clearance rate), δ (rate of loss of infected cells), and V_0 (initial viral load) were simultaneously estimated. To account for the pharmacokinetic delay, we assumed $t = 0$ in Eq. 6 to correspond to the time of the pharmacokinetic delay (if measured) or selected 2, 4, or 6 hours as the best-fit value (see Table 1). The logarithm of the experimental data was fitted to the logarithm of Eq. 6 by a nonlinear least squares method with the use of the subroutine DNLS1 from the Common Los Alamos Software Library, which is based on a finite difference Levenberg-Marquardt algorithm. The best fit, with the smallest sum of squares per data point, was chosen after eliminating the worst outlying data point for each patient with the use of the jackknife method. (**B**) Plasma concentrations of HIV-1 RNA (upper panel; circles) and the plasma infectivity titer (lower panel; squares) for patient 105. (Top panel) The solid curve is the best fit of Eq. 6 to the RNA data; the dotted line is the curve of the noninfectious pool of virions, $V_{NI}(t)$; and the dashed line is the curve of the infectious pool of virions, $V_I(t)$. (Bottom panel) The dashed line is the best fit of the equation for $V_I(t)$ to the plasma infectivity data. $TCID_{50}$, 50% tissue culture infectious dose.

The graphs are taken from the article

HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time,

by Alan S. Perelson, Avidan U. Neumann, Martin Markowitz, John M. Leonard and David D. Ho,

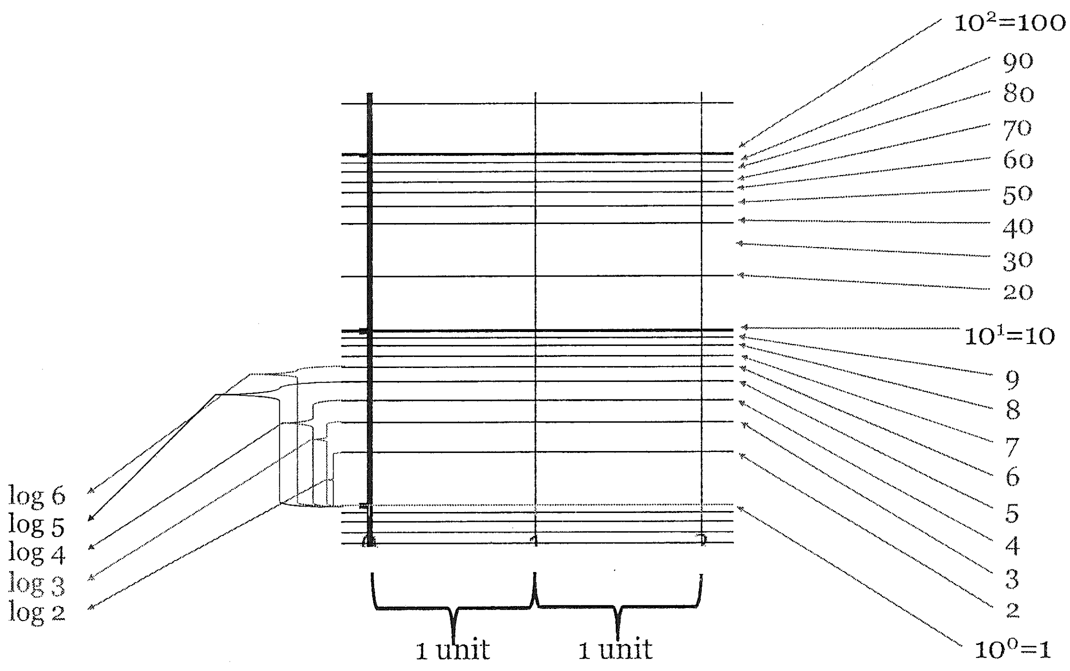
Science, New Series, Vol. 271, No. 5255 (Mar. 15, 1996), pp. 1582-1586.

David Ho was Time magazine's 1996 Man of the Year.

How to Read a Semilog Plot

You need remember is that the log axis runs in exponential cycles.
Each cycle runs linearly in 10's but the increase from one cycle to another is an increase by a factor of 10.
So within a cycle you would have a series of: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (this could also be 0.1-1, etc.).
The next cycle begins with 10 and progresses as 20, 30, 40, 50, 60, 70, 80, 90, 100.
The cycle after that would be 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000.

Below is a picture of semilog graph paper.



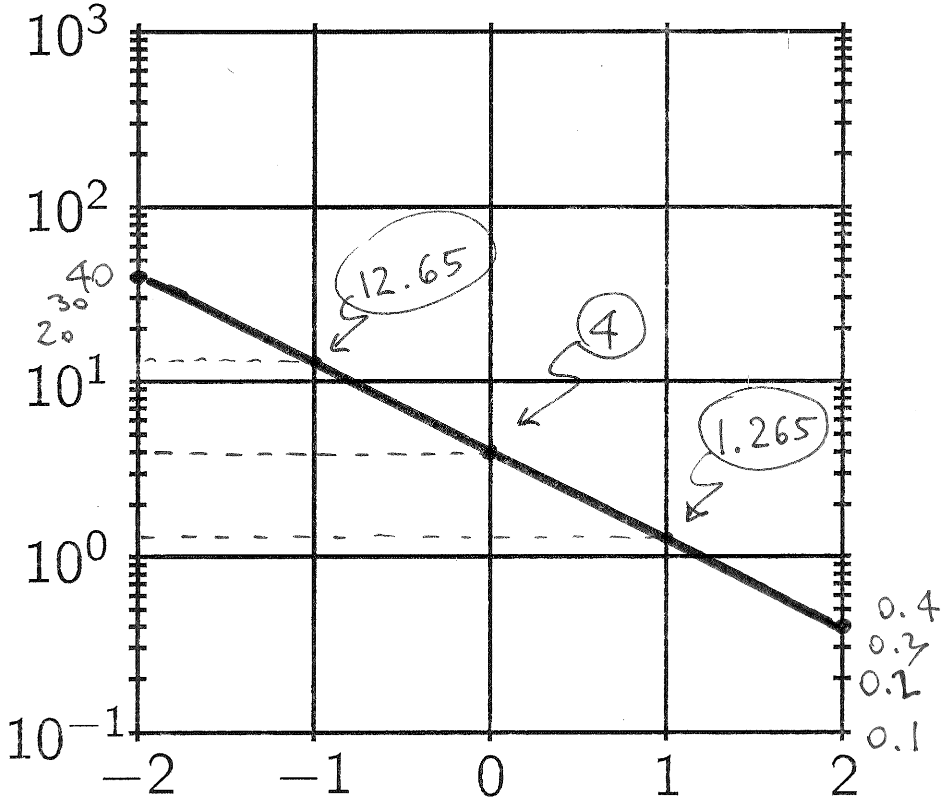
Number	log
100	2
90	1.9542
80	1.9031
70	1.8451
60	1.7782
50	1.6990
40	1.6021
30	1.4771
20	1.3010
10	1
9	0.9542
8	0.9031
7	0.8451
6	0.7782
5	0.6990
4	0.6021
3	0.4771
2	0.3010
1	0.0000

Example 3:

Suppose that x and y are related by the expression

$$y = 4 \cdot 10^{-x/2} \quad [= 4 \cdot (10^{-1/2})^x = 4 \cdot (0.316)^x].$$

Use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship in the semilog (or log-linear) plot.



Let's plot a few values of that function.
We can build the table

x	
−2	$4 \cdot 10^{-(-2/2)} = 4 \cdot 10 = 40$
−1	$4 \cdot 10^{-(-1/2)} = 12.65$
0	4
1	1.265
2	0.4

From $y = 4 \cdot 10^{-x/2}$

take $\log = \log_{10}$ of both sides:

$$\log y = \log\left(4 \cdot 10^{-x/2}\right)$$

$$= \log(4) + \log\left(10^{-x/2}\right)$$

$$= \log(4) + \left(-\tfrac{1}{2}\right) \cdot x$$

Set $Y = \log y$, so the above equation becomes

$$\boxed{Y = \left(-\tfrac{1}{2}\right) \cdot x + \log(4)}$$

slope of line

at $(\log 4)$ when the intercept is plotted

Lines in Semilog Plots

- If we start with an **exponential function** of the form $y = a \cdot b^x$ and take logarithms of both sides, we get

$$\log y = \log(a \cdot b^x) = \log a + \log b^x$$

$$\log y = \log a + x \log b$$

If we let $Y = \log y$, $M = \log b$, and $B = \log a$, then we obtain

$$Y = B + Mx,$$

i.e., the equation of a line with slope M and Y-intercept B.

- So if we obtain experimental data that we suspect might possibly be exponential, then we could graph a semilog scatter plot and see if it is approximately linear.

Conversely, suppose we have a straight line in a semilog plot:

$$Y = Mx + B \qquad \text{where } Y = \log y$$

Then from $\log y = Mx + B$ we obtain

$$10^{\log y} = 10^{Mx+B}$$

$$\Longleftrightarrow$$

$$y = 10^{Mx} \cdot 10^{B}$$

$$\Longleftrightarrow$$

$$y = \underbrace{(10^{B})}_{a} \cdot \underbrace{(10^{M})}_{b}{}^{x} = a \cdot b^{x}$$

where $a = 10^{B}$ $b = 10^{M}$

Example 4:

When $\log y$ is graphed as a function of x, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (0, 40) \qquad (x_2, y_2) = (2, 600)$$

on a log-linear plot. Determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

First method: a line in a semilog plot corresponds to an exponential function of the form $y = a \cdot b^x$

when $x = 0$ then $y = 40$
$x = 2$ then $y = 600$ $\Longrightarrow$ $\begin{cases} 40 = a \cdot \underbrace{b^0}_{=1} \Rightarrow \boxed{a = 40} \\ 600 = a\, b^2 \end{cases}$

$\therefore$ $a = 40$ and $600 = 40 \cdot b^2 \Longrightarrow b^2 = \frac{600}{40} = 15$

$\therefore$ $b = \sqrt{15} \cong \underline{3.873}$ $\therefore$ $\boxed{y = 40 \cdot (3.873)^x}$

Second method: in the $(x, \log y)$ plot we need to compute the equation of the line through $(0, \log 40)$ and $(2, \log 600)$

Slope of the line is $m = \dfrac{\log 600 - \log 40}{2 - 0}$

$$= \frac{\log\left(\frac{600}{40}\right)}{2} = \frac{1}{2}\log(15) = \boxed{\log(\sqrt{15})}$$

Hence the point-slope form of the line is

$$(\underbrace{Y}_{\log y} - \log 40) = \log(\sqrt{15}) \cdot (x - 0)$$

$$\therefore \quad \log(y/40) = \log(\sqrt{15}) \cdot x$$

OR

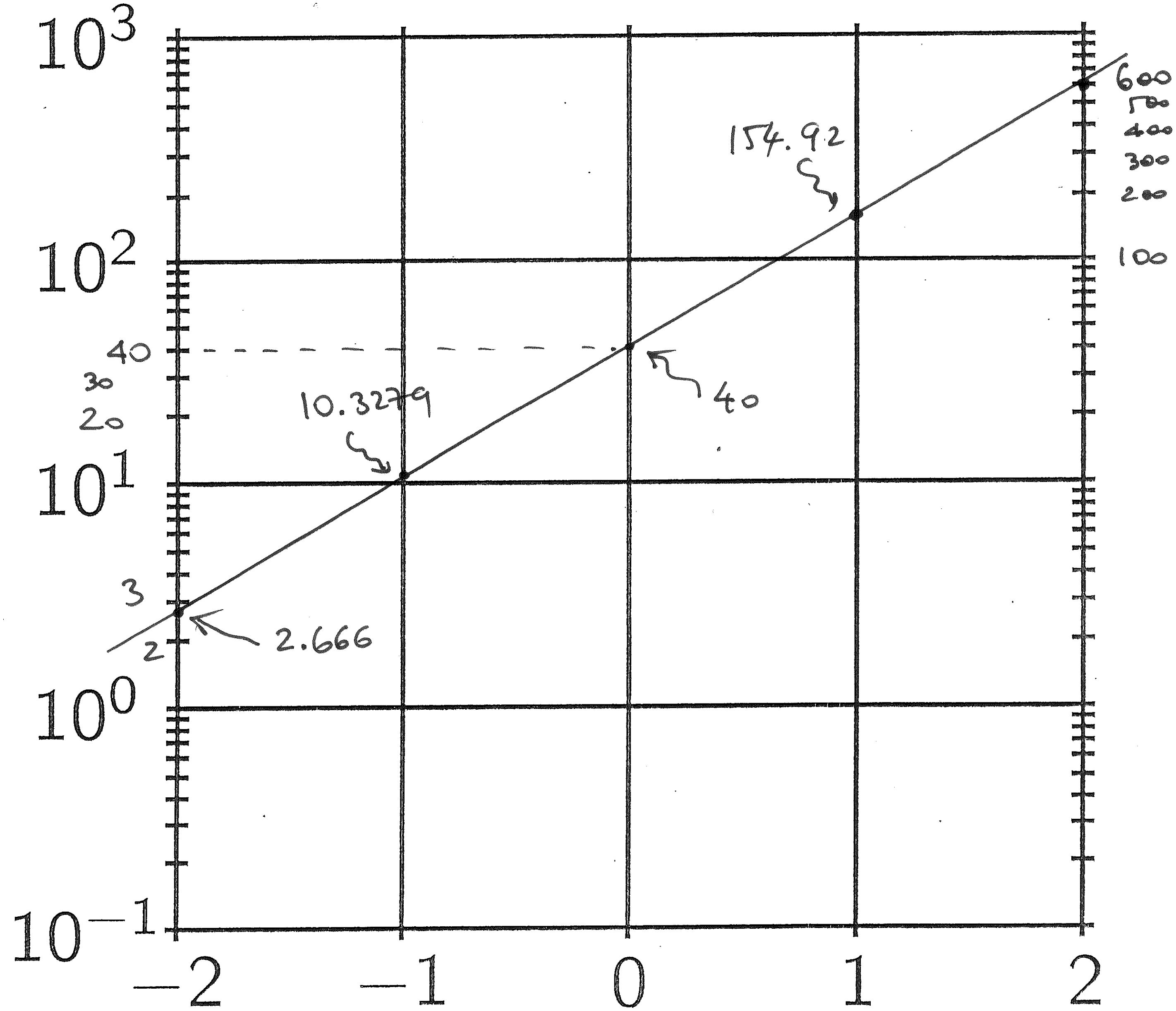
$$\log(y/40) = x \log(\sqrt{15})$$

$$= \log((\sqrt{15})^x)$$

$$\frac{y}{40} = (\sqrt{15})^x$$

OR

$$\boxed{y = 40(3.873)^x}$$



x	$40(3.873)^x$
-2	2.666
-1	10.3279
0	40
1	154.92
2	600

Example 5: (Problem # 46, Section 1.3, p. 53)

When $\log y$ is graphed as a function of x, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (1, 4) \qquad (x_2, y_2) = (6, 1)$$

on a log-linear plot. Determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

<u>First method</u>: since we obtain a straight line in a semi log plot, the functional relation between x and y is exponential: $y = a \cdot b^x$

Hence $x_1 = 1 \Rightarrow y_1 = 4$, $x_2 = 6 \Rightarrow y_2 = 1$ $\Longrightarrow$ $4 = a \cdot b^1$, $1 = a \cdot b^6$

Solve in both eq. for a: $\frac{4}{b} = a = \frac{1}{b^6}$

$\therefore \frac{4}{b} = \frac{1}{b^6} \Longrightarrow \frac{b^6}{b} = 1/4 \Longrightarrow b^5 = 1/4$

$\therefore b = \sqrt[5]{1/4} \cong 0.7578$ Now $a = \frac{4}{b} = \frac{4}{0.7578} \cong 5.278$

$$\therefore \boxed{y = 5.278\,(0.7578)^x}$$

2nd method
Let's compute the equation of the line in the semi log plot through $(1, \log 4)$ and $(6, \log 1)$

$$m = \frac{\log 1 - \log 4}{6 - 1} = \frac{-\log 4}{5} = \left(-\tfrac{1}{5}\right)\log 4 = \log\left(4^{-1/5}\right)$$

$$= \log\left(\frac{1}{\sqrt[5]{4}}\right)$$

Hence

$$\underbrace{Y}_{\log y} - \log 1 = \log\left(\frac{1}{\sqrt[5]{4}}\right)(x - 6)$$

(point-slope form)

$$\Rightarrow \log y = (x-6)\log\left(\frac{1}{\sqrt[5]{4}}\right)$$

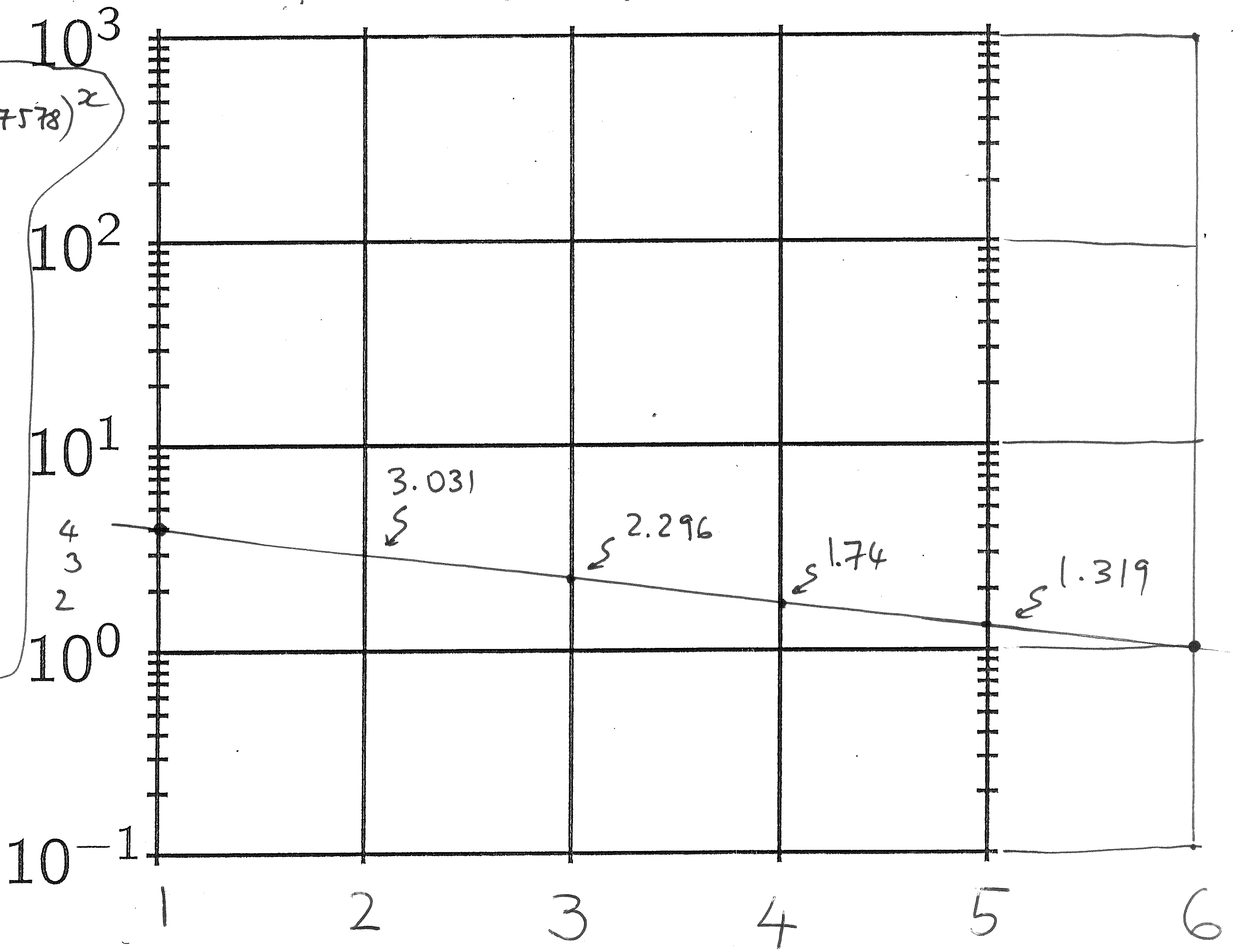
$$\log y = \log\left[\left(\frac{1}{\sqrt[5]{4}}\right)^{x-6}\right] \rightarrow y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x-6}$$

$$y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x} \cdot (4)^{6/5}$$

$$= \boxed{5.278\,(0.7578)^{x}}$$

Here is the plot of $y = 5.278(0.7578)^x$

x	$5.278(0.7578)^x$
1	4
2	3.031
3	2.296
4	1.74
5	1.319
6	1



Example 6: (Problem # 52, Section 1.3, p. 53)

Consider the relationship $y = 6 \times 2^{-0.9x}$ between the quantities x and y. Use a logarithmic transformation to find a linear relationship of the form

$$Y = mx + b$$

between the given quantities.

x	$y = 6 \cdot 2^{-0.9x}$
-2	20.89
-1	11.196
0	6
1	3.215
2	1.723

$$y = 6 \cdot 2^{-0.9x}$$

Take $\log = \log_{10}$ of both sides:

$$\log y = \log\left(6 \cdot 2^{-0.9x}\right)$$

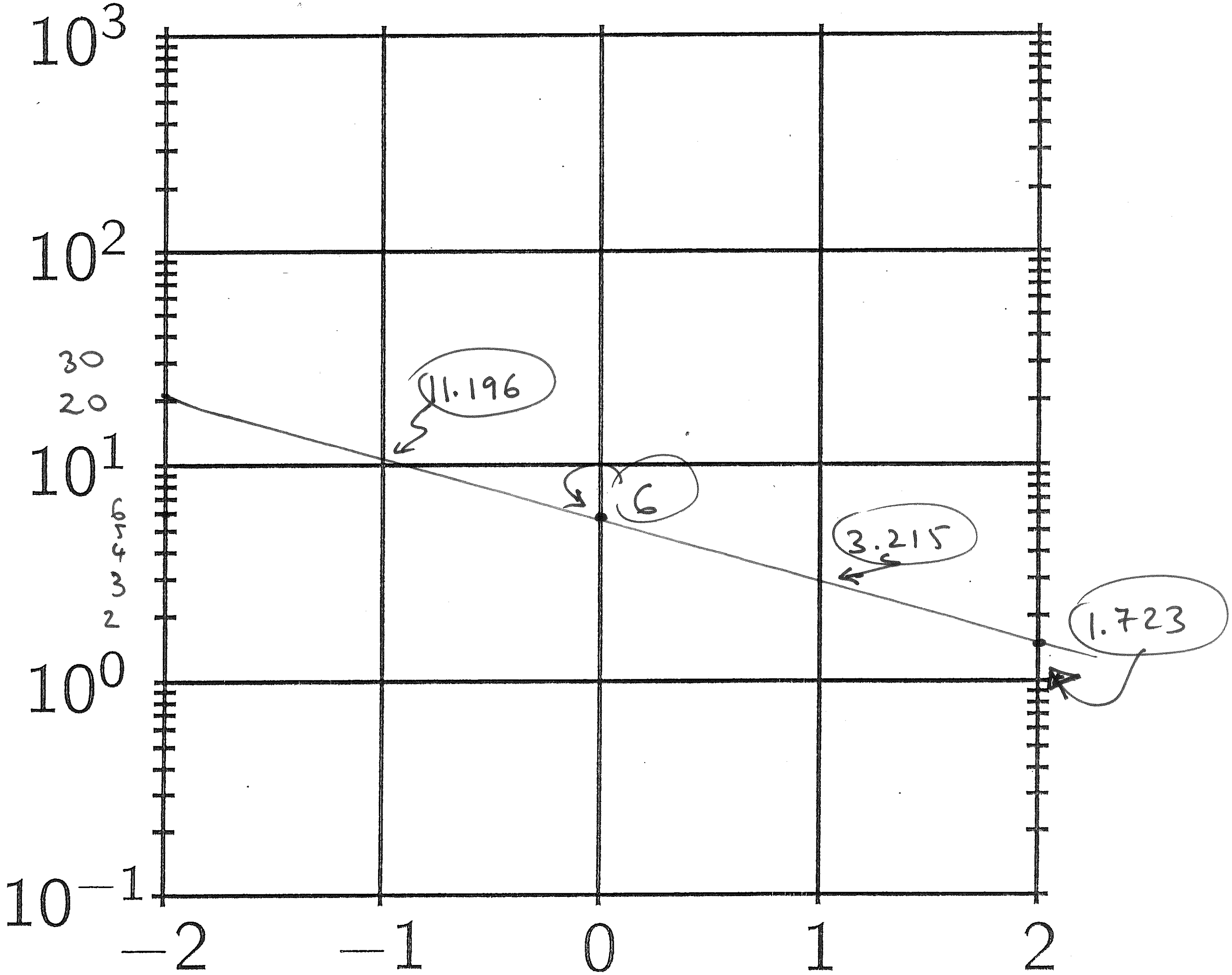
$$= \log 6 + \log\left(2^{-0.9x}\right)$$

$$= \left[(-0.9)\log 2\right] x + \log 6$$

$$\boxed{\log y = -0.27092x + 0.7782}$$

$\log y = Y$

approx. graph of $y = 6 \cdot 2^{-0.9x}$ in semilog plot



MA137 – Calculus 1 with Life Science Applications
Semilog and Double Log Plots
(Section 1.3)

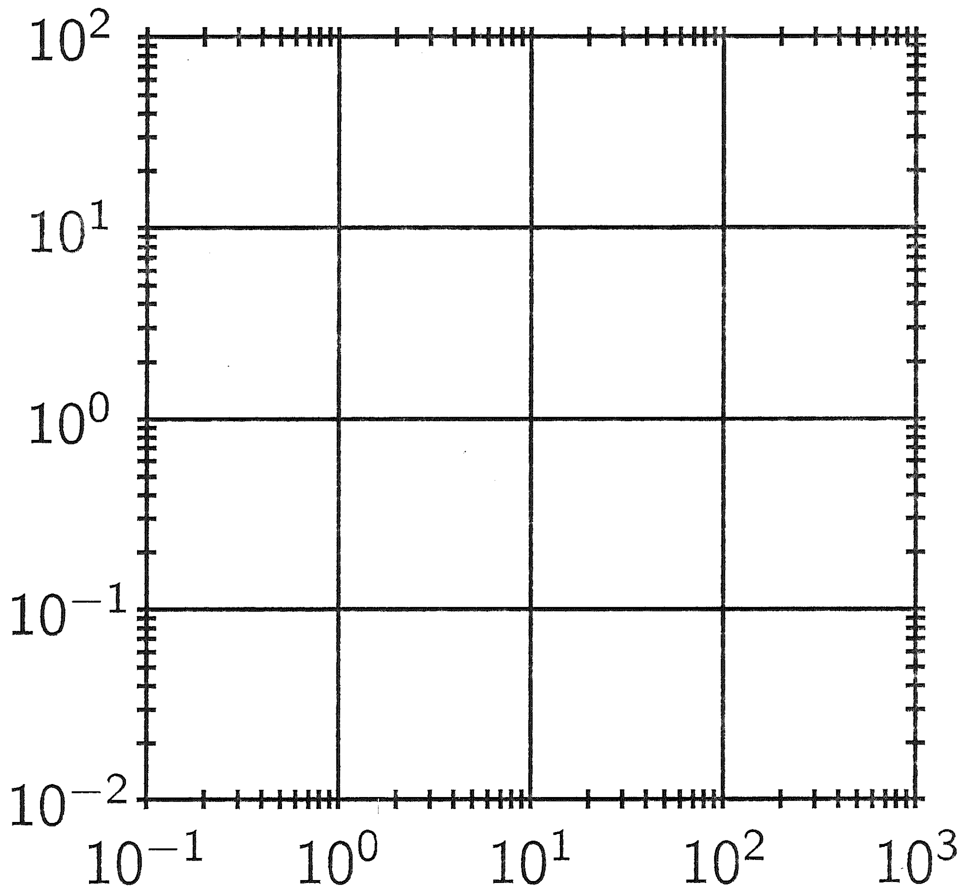
Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

September 9, 2016

Double-log (or Log-Log) Plots

- If we use logarithmic scales on both the horizontal and vertical axes, the resulting graph is called a log-log plot.



Lines in Double-Log Plots

- A log-log plot is used when we suspect that a power function might be a good model for our data.
- Recall that power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes). Finding such relationships is the objective of **allometry**.
- If we start with a **power function** $y = Cx^p$ and take logarithms of both sides, we get

$$\log y = \log(Cx^p) = \log C + \log x^p$$

$$\log y = \log C + p \log x$$

Let $Y = \log y$, $A = \log C$, and $X = \log x$. Then the latter equation becomes

$$Y = A + pX$$

We recognize that Y is a linear function of X, so the points $(\log x, \log y)$ lie on a straight line.

Conversely, suppose we have a straight line in a log-log plot:

$$Y = pX + B$$

when

$Y = \log y$

$X = \log x$

Hence we have

$$\log y = p \log x + B$$

trick!

$$\Longleftrightarrow$$

$$\log y = \log(x^p) + \log(10^B)$$

set $C = 10^B$.

$$\Longleftrightarrow$$

$$\log y = \log(10^B \cdot x^p)$$

$$\Longleftrightarrow \quad y = 10^B \cdot x^p$$

to get $y = C x^p$

Example 1: (Problem # 58, Section 1.3, p. 53)

When $\log y$ is graphed as a function of $\log x$, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (2, 5) \qquad (x_2, y_2) = (5, 2)$$

on a log-log plot. determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

1st method:

A line in a log-log plot corresponds to a power relation of the form: $y = Cx^p$.

Since (2, 5) and (5, 2) satisfy this relation we obtain:

$$5 = C\,2^p \quad \text{and} \quad 2 = C\,5^p$$

Thus $\frac{5}{2^p} = C = \frac{2}{5^p}$. This implies

$$\frac{5^p}{2^p} = 2/5 \quad \text{or} \quad \left(\frac{5}{2}\right)^p = 2/5$$

Take log of both sides and we get

$$\log\left[\left(\frac{5}{2}\right)^p\right] = \log(2/5) \rightsquigarrow p\log(2.5) = \log(0.4)$$

$$\Rightarrow p = \frac{\log(0.4)}{\log(2.5)} = \boxed{-1} \qquad \Rightarrow C = \frac{5}{2^{(-1)}} = \boxed{10}$$

Thus the functional relationship is $\boxed{y = \frac{10}{x}}$

2nd method:

$$m = \text{slope} = \frac{\log 5 - \log 2}{\log 2 - \log 5} = \frac{\log(5/2)}{\log(2/5)} = \boxed{-1}$$

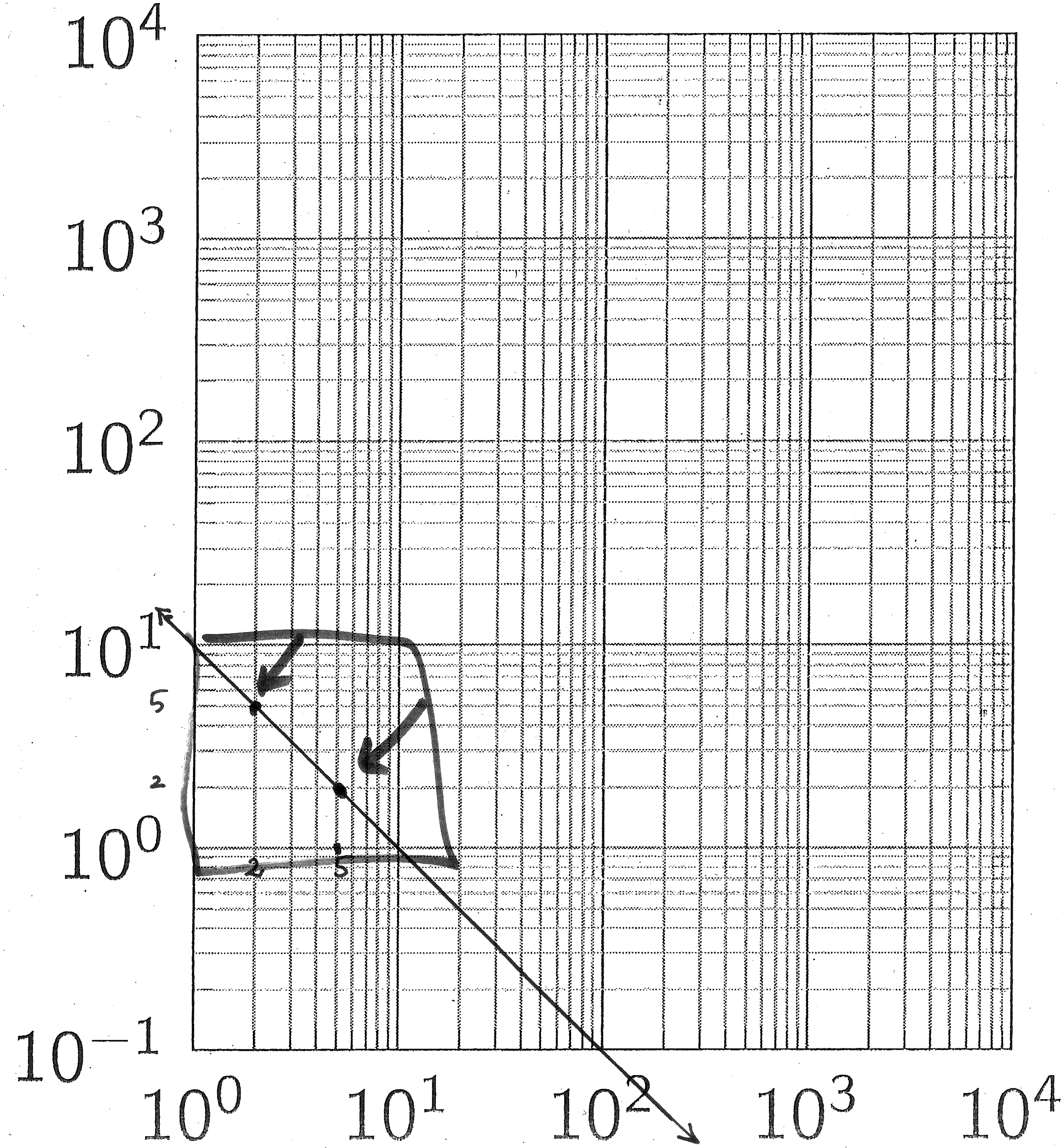
Hence the equation in point slope form is:

$$(\underbrace{\log y}_{Y} - \log 2) = -1\,(\underbrace{\log x}_{X} - \log 5)$$

$$\Rightarrow \log\left(\frac{y}{2}\right) = -\left(\log\left(\frac{x}{5}\right)\right)$$

$$\Rightarrow \log\left(\frac{y}{2}\right) = \log\left[\left(\frac{x}{5}\right)^{-1}\right] = \log\left(\frac{5}{x}\right)$$

$$\Rightarrow \frac{y}{2} = \frac{5}{x} \Longrightarrow y = \frac{10}{x}$$



$$y = \frac{10}{x}$$

$$= 10x^{-1}$$

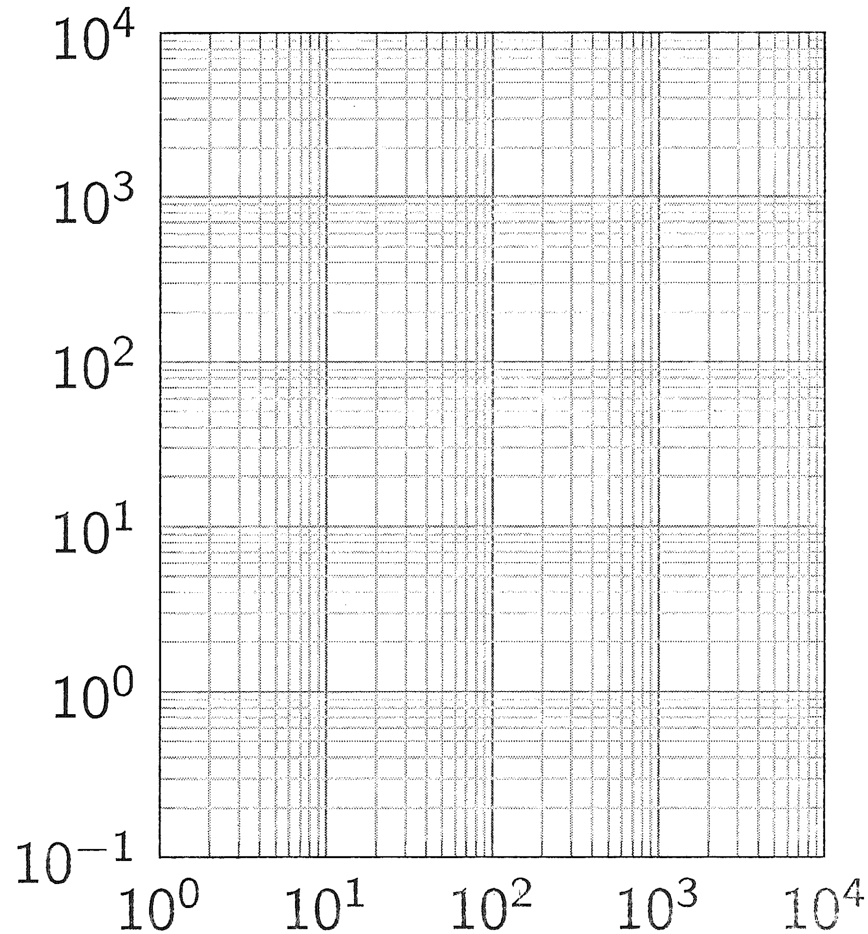
hyperbola

Example 2: (Exam 1, Fall 13, # 4)

There are several possible functional relationships between height and diameter of a tree. One particularly simple model is given by

$$H = AD^{3/4}$$

where A is a constant that depends on the species of tree, H is the height, and D is the diameter. If $A = 50$ plot this relationship in the double log plot below.



Is your graph a straight line? If so, what is its slope?

Consider the function $\underline{H = 50 D^{3/4}}$

We can construct the following table of values

D	$H = 50 D^{3/4}$
1	50
10	281.17
10^2	1,581.14
10^3	8,891.4
10^4	$50{,}000 = 5 \cdot 10^4$

In a log-log plot this power relationship becomes a straight line:

$$\log H = \log(50 D^{3/4})$$

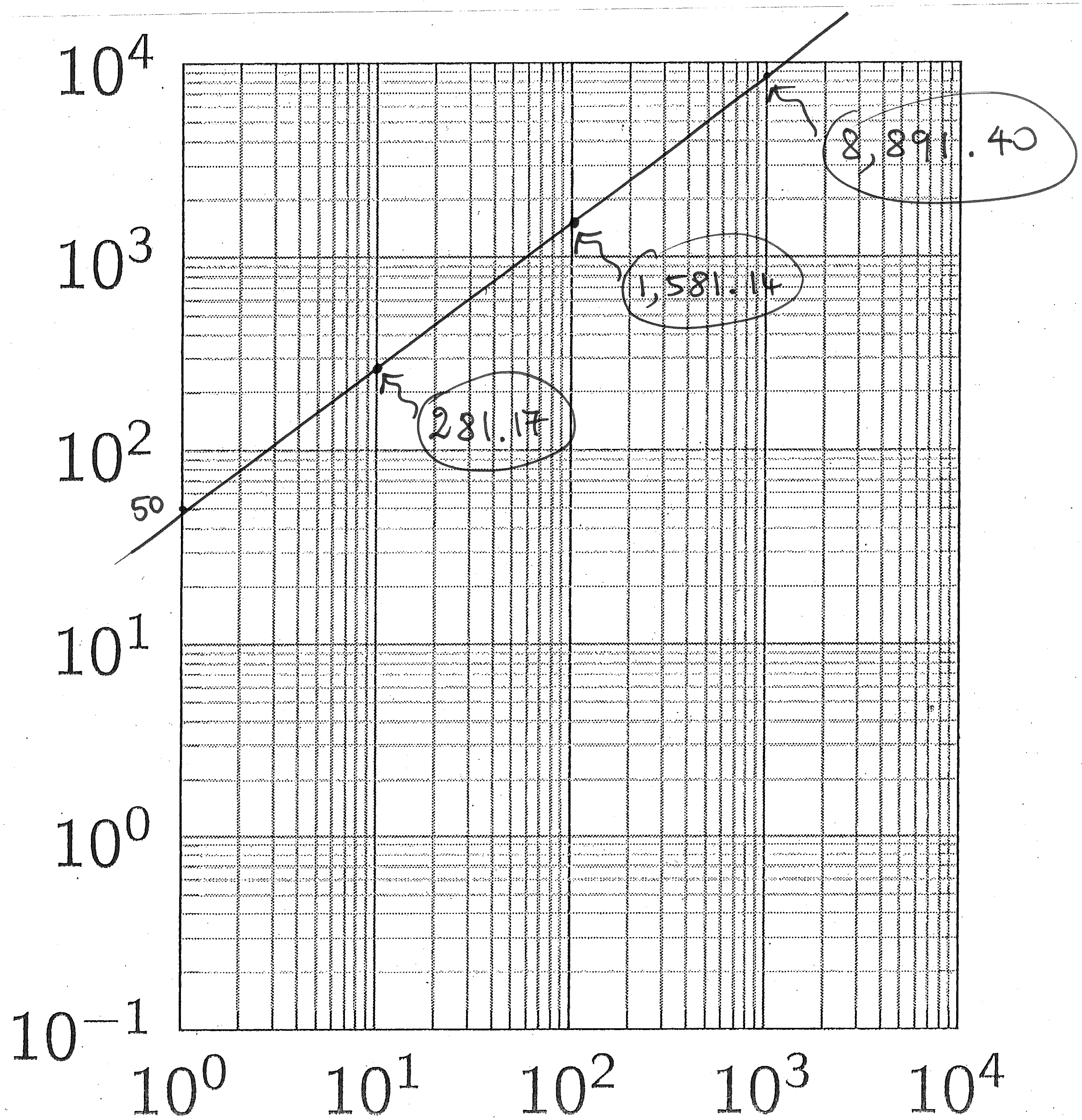
$$\Longleftrightarrow$$

$$\log H = \log 50 + \log(D^{3/4})$$

$$\Longleftrightarrow$$

$$\log(H) = 3/4 \log(D) + \log(50)$$

<u>slope is</u> $\boxed{3/4}$



10^4
10^3
10^2
10^1
10^0
10^-1
10^0
10^1
10^2
10^3
10^4
50
281.17
1,581.14
8,891.40

Example 3: (Problem # 74, Section 1.3, p. 54)

The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0.5	7.81
1	3.4
1.5	2.09
2	1.48
2.5	1.13

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

First, let's see if there is an exponential relationship among our data points. This means that in the semi log plot we have a straight line.

	x	y	$\log y$
→	0.5	7.81	0.893
	1	3.4	0.531
→	1.5	2.09	0.32
	2	1.48	0.17
→	2.5	1.13	0.053

Let's compute the slope of the line between two pairs of points of the form $(x, \log y)$

→ $(0.5, 0.893)$ & $(1.5, 0.32)$ ⟶ slope $m = \dfrac{0.32 - 0.893}{1} \cong -0.573$

→ $(1.5, 0.32)$ & $(2.5, 0.053)$ ⟶ slope $m = \dfrac{0.053 - 0.32}{1} \cong -0.267$

Since we do not get similar values, these points do not lie on a straight line.

Let's see if the points of the form $(\log x, \log y)$ lie on a straight line in a log-log plot:

$\log x$	$\log y$
-0.301	0.893
0	0.531
0.176	0.32
0.301	0.17
0.398	0.053

Pick: $(-0.301, 0.893)$ & $(0.176, 0.32)$

$$\text{slope} = \frac{0.32 - 0.893}{0.176 - (-0.301)} \cong -1.201$$

Pick $(0, 0.531)$ & $(0.398, 0.053)$

$$\text{slope} = \frac{0.053 - 0.531}{0.398 - 0} \cong -1.201$$

it seems that we can choose as a slope $\boxed{-1.20}$

The equation of the line in point slope form is (we choose the simplest point $(0, 0.531)$)

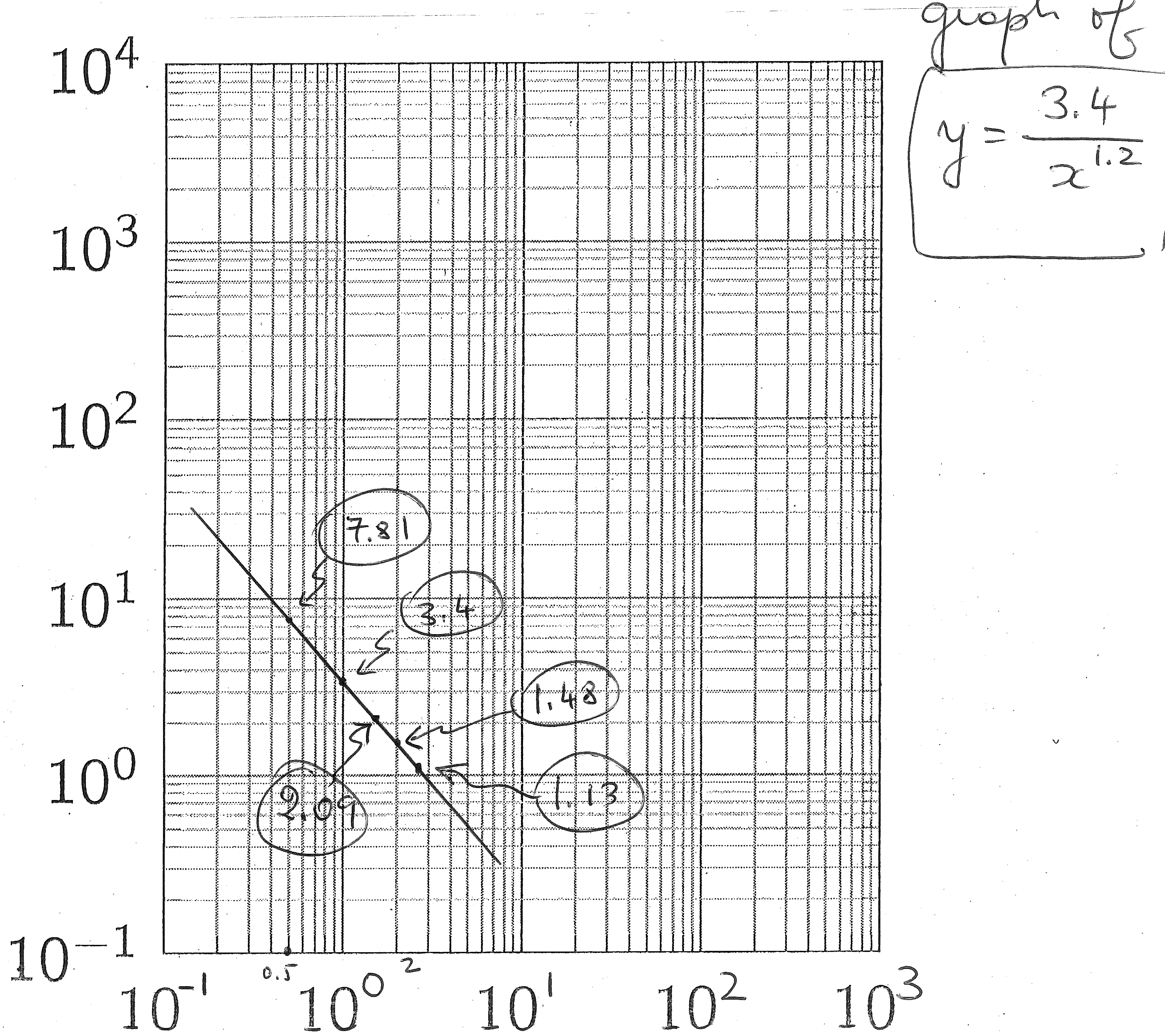
$$(\log y - 0.531) = -1.2(\log x - 0)$$

$$\log y - \log 10^{0.531} = -1.2 \log x$$

$$\Longleftrightarrow$$

$$\log\left(\frac{y}{10^{0.531}}\right) = \log x^{-1.2} \Longleftrightarrow \frac{y}{10^{0.531}} = x^{-1.2}$$

$$\therefore \quad y = 10^{0.531} x^{-1.2} \quad \text{OR} \quad \boxed{y = \frac{3.4}{x^{1.2}}}$$



graph of
$y = \frac{3.4}{x^{1.2}}$
7.81
3.4
1.48
1.13
2.09
0.5
2
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-1}
10^0
10^1
10^2
10^3

Example 4 (Forgetting):

Ebbinghaus's Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1),$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve the equation for P.

(b) Use Ebbinghaus's Law of Forgetting to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$.

(a) $\log P = \log P_0 - c \log(t+1)$

We want to solve for P:

$$\log P = \log P_0 - \log\left[(t+1)^c\right]$$

$$\Longleftrightarrow$$

$$\log P = \log\left[\frac{P_0}{(t+1)^c}\right]$$

Hence $10^{\log P} = 10^{\log\left[\frac{P_0}{(t+1)^c}\right]}$

$$\Longrightarrow \boxed{P(t) = \frac{P_0}{(t+1)^c}}$$

(b) With our data $P_0 = 80$ $c = 0.3$

$$P(t) = \frac{80}{(t+1)^{0.3}}$$

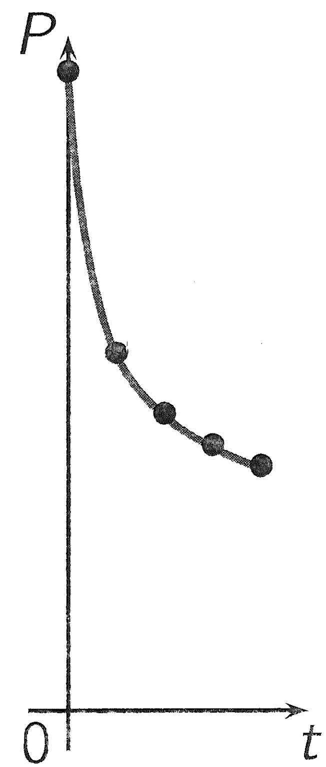
hence $P(24) = \frac{80}{(24+1)^{0.3}} \cong 30.46$

2 years in months

Comment (about Example 4)

Below is the graph of the function $P = 80/(t+1)^{0.3}$ in standard coordinates:

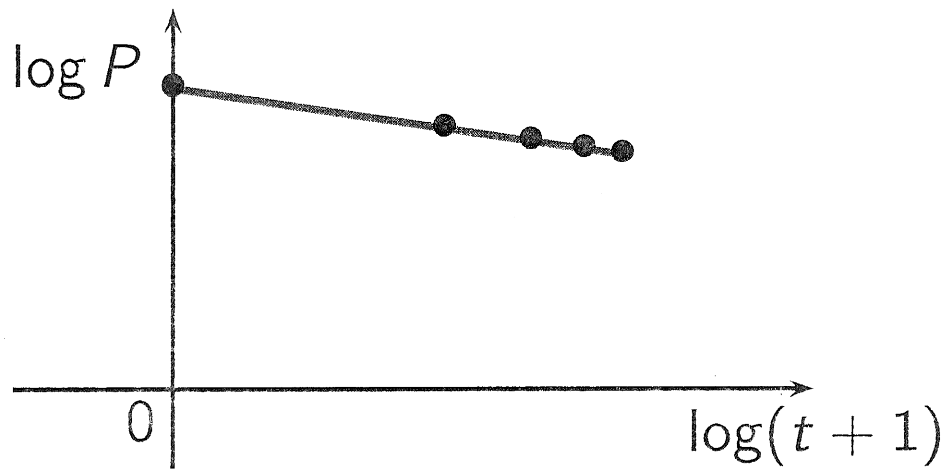
t	$P = 80/(t+1)^{0.3}$
0	80
6	44.62
12	37.06
18	33.072
24	30.458



Comment (cont.d)

Below is the graph of $\log P = \log 80 - 0.3 \log(t+1)$ in a log-log plot:

t	$\log(t+1)$	$\log P = \log 80 - 0.3 \log(t+1)$
0	0	1.903
6	0.845	1.650
12	1.114	1.569
18	1.279	1.519
24	1.398	1.484



Example 5 (Biodiversity):

Some biologists model the number of species S in a fixed area A (such as an island) by the **Species-Area relationship**

$$\log S = \log c + k \log A,$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part **(a)** to show that if $k = 3$ then doubling the area increases the number of species eightfold.

(a) $\log S = \log c + k \log A$

$\iff$

$\log S = \log c + \log [A^k]$

$\iff$

$\log S = \log [c A^k]$

$\iff 10^{\log S} = 10^{\log[cA^k]} \iff \boxed{S = cA^k}$

(b) Suppose $k = 3$, i.e. $\underline{S = cA^3}$

For $A = a_0$ we get that $S(a_0) = c a_0^3$.

However if we double the area, i.e. $A = 2a_0$,

we get $S(2a_0) = c(2a_0)^3 = 8 \underbrace{c a_0^3} = 8 S(a_0)$

i.e. doubling the area increases the number of species eightfold.