MA 137 – Calculus 1 with Life Science Applications
Limits
(Section 3.1)

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Computing a limit means computing what happens to the value of a function as the variable in the expression gets closer and closer to (but does not equal) a particular value.

Intuitive Definition

Let f be a function of x. The expression $\lim_{x \to c} f(x) = L$ means that as x gets closer and closer to c, through values both smaller and larger than c, but not equal to c, then the values of $f(x)$ get closer and closer to the value L.

Note 1: If $\lim_{x \to c} f(x) = L$ and L is a finite number, we say that the limit exists and that $f(x)$ **converges** to L as x tends to c.
If the limit does not exist, we say that $f(x)$ **diverges** as x tends to c.

Note 2: when finding the limit of $f(x)$ as x approaches c, we do not simply plug c into $f(x)$. (OK...often we do!)
In fact, we will see examples in which $f(x)$ is not even defined at $x = c$.
The value of $f(c)$ is irrelevant when we compute the value of $\lim_{x \to c} f(x)$.

Example 1:

Compute $\lim_{x \to 2} \frac{x^2+8}{x+2}$.

x gets close to 2 from the left				
x	1.8	1.9	1.99	1.999
$f(x) = \frac{x^2+8}{x+2}$				

x gets close to 2 from the right				
2.001	2.01	2.1	2.2	x
				$f(x) = \frac{x^2+8}{x+2}$

Using a calculator or even better an Excel spreadsheet we have that

x	1.8	1.9	1.99	1.999
$f(x)$	2.9579	2.9769	2.9975	2.9998

for $f(x) = \dfrac{x^2+8}{x+2}$

and

x	2.001	2.01	2.1	2.2
$f(x)$	3.0003	3.0025	3.0268	3.0571

so it seems that the values of $f(x)$ approach 3 as x approaches 2 : $\lim\limits_{x \to 2} \dfrac{x^2+8}{x+2} = \boxed{3}$

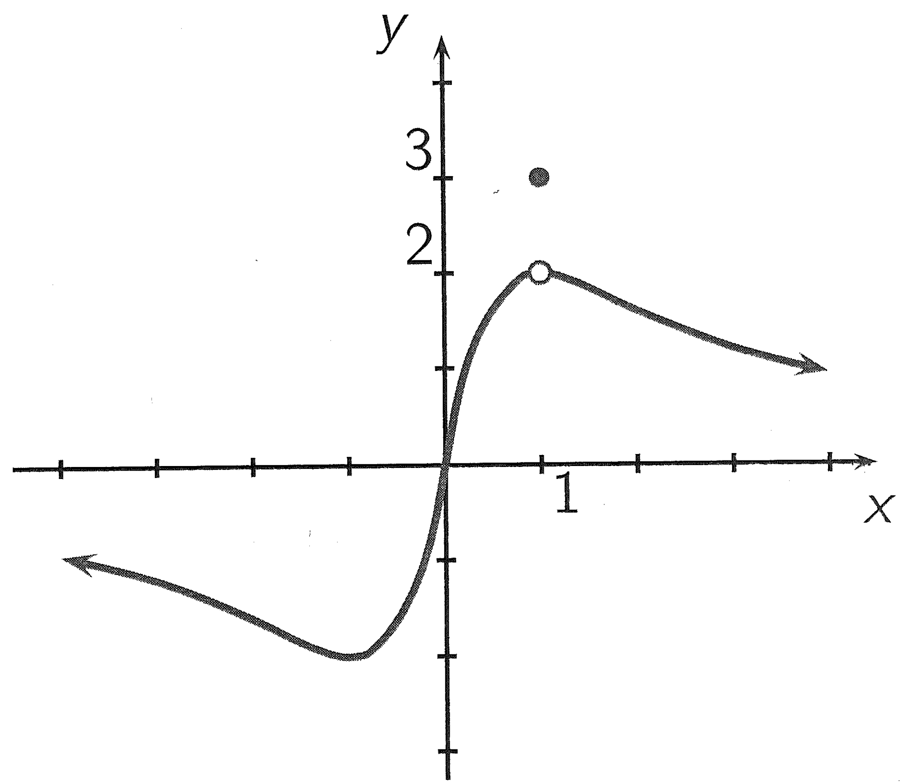
Note that in this case : $f(2) = \dfrac{2^2+8}{2+2} = \dfrac{12}{4} = \boxed{3}$

Example 2:

The graph of the function

$$g(x) = \begin{cases} \dfrac{4x}{x^2+1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

is shown to the right.



Compute $\lim\limits_{x \to 1} g(x)$.

x	0.8	0.9	0.99	1.001	1.1	1.2
$g(x)$	1.95121	1.98895	1.9999	1.9999	1.99095	1.96721

$$g(x) = \begin{cases} \dfrac{4x}{x^2+1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

notice that from the table of values:

x	0.8	0.9	0.99	1.001	1.1	1.2
$g(x)$	1.95121	1.98895	1.9999	1.9999	1.99095	1.96721

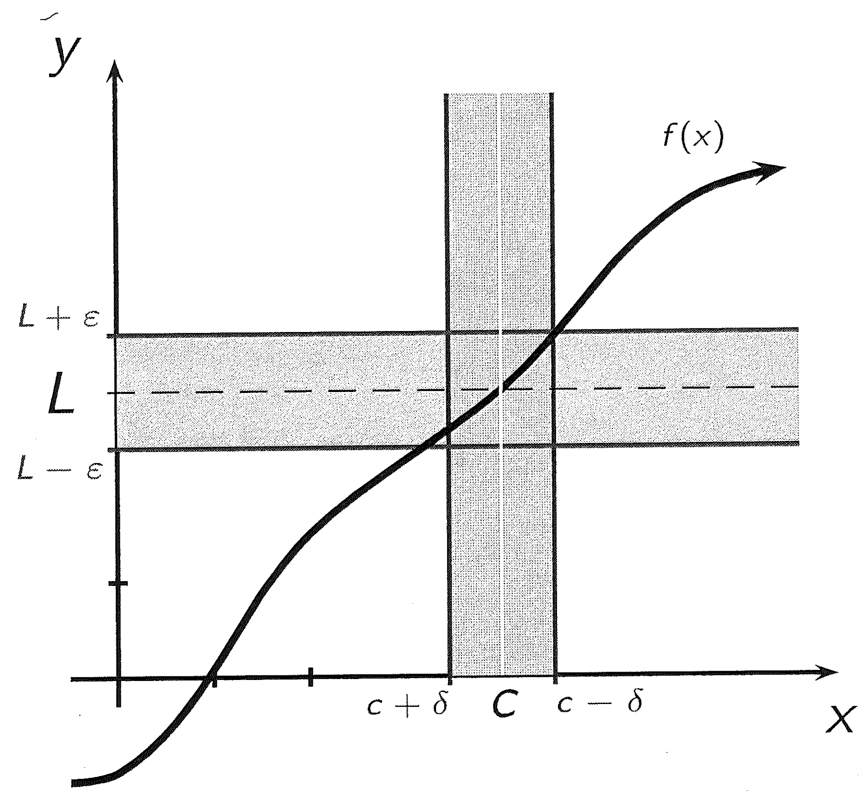
it seems that the values of $g(x)$ tend to 2 as x approaches 1:

$\boxed{\lim_{x \to 1} g(x) = 2}$. Note $\boxed{g(1) = 3}$

so $\boxed{\lim_{x \to 1} g(x) \neq g(1)}$

Formal Definition

The statement $\lim_{x \to c} f(x) = L$ means that, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.



The formal definition says that no matter how small we choose the horizontal strip about $y = L$, we can always choose a small strip about $x = c$ so that

whenever $c - \delta < \underline{\underline{x}} < c + \delta \qquad x \neq c$

<u>then</u> $L - \varepsilon < \underline{\underline{f(x)}} < L + \varepsilon$

<u>i.e.</u> if the values of x is sufficiently close to c then the corresponding value $f(x)$ is close to L.

Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim_{x \longrightarrow c} f(x)$ and $\lim_{x \longrightarrow c} g(x)$ exist and a is a constant, then

1. $\lim_{x \longrightarrow c} [f(x) + g(x)] = [\lim_{x \longrightarrow c} f(x)] + [\lim_{x \longrightarrow c} g(x)]$
2. $\lim_{x \longrightarrow c} [a\, f(x)] = a[\lim_{x \longrightarrow c} f(x)]$
3. $\lim_{x \longrightarrow c} [f(x) \cdot g(x)] = [\lim_{x \longrightarrow c} f(x)] \cdot [\lim_{x \longrightarrow c} g(x)]$
4. $\lim_{x \longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$, provided $\lim_{x \longrightarrow c} g(x) \neq 0$

Theorem (Substitution Theorem 1)

If $p(x)$ is a polynomial, then $\lim\limits_{x \longrightarrow c} p(x) = p(c)$.

Proof: A polynomial is a sum of terms, say $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$.

The result now follows from the Limit Laws:

$$\lim_{x \longrightarrow c} p(x) = \lim_{x \longrightarrow c} \left[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right]$$

(the limit of the sum is the sum of the limits)

$$= \lim_{x \longrightarrow c} [a_n x^n] + \lim_{x \longrightarrow c} [a_{n-1}x^{n-1}] + \cdots + \lim_{x \longrightarrow c} [a_2 x^2] + \lim_{x \longrightarrow c} [a_1 x] + \lim_{x \longrightarrow c} [a_0]$$

(each of the terms is a product and the limit of the product is the product of the limits)

$$= \lim_{x \longrightarrow c} [a_n] \lim_{x \longrightarrow c} [x^n] + \lim_{x \longrightarrow c} [a_{n-1}] \lim_{x \longrightarrow c} [x^{n-1}] + \cdots + \lim_{x \longrightarrow c} [a_2] \lim_{x \longrightarrow c} [x^2] + \lim_{x \longrightarrow c} [a_1] \lim_{x \longrightarrow c} [x] + \lim_{x \longrightarrow c} [a_0]$$

(each of these terms is either a constant or a power of x)

$$= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_2 c^2 + a_1 c + a_0 = p(c)$$

Theorem (Substitution Theorem 2)

If $f(x)$ is a rational function, that is $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)} = f(c),$$

provided $q(c) \neq 0$.

The usual issue is that we often have to compute limits when these conditions are not met.

Example 3:

(a) Compute $\displaystyle\lim_{x\to 1} \frac{x^2 - 2x + 1}{x + 1}$.

(b) Suppose $\displaystyle\lim_{x\to 3} f(x) = -2$ and $\displaystyle\lim_{x\to 3} g(x) = 4$. Determine

$$\lim_{x\to 3}\left[(x+1)\cdot f(x)^2 + \frac{x+2}{g(x)}\right]$$

(a) $$\lim_{x\to 1} \frac{x^2-2x+1}{x+1} = \frac{\lim_{x\to 1}(x^2-2x+1)}{\lim_{x\to 1}(x+1)} =$$

$$= \frac{\lim_{x\to 1}(x^2) + \lim_{x\to 1}(-2x) + \lim_{x\to 1} 1}{\left(\lim_{x\to 1} x\right) + \lim_{x\to 1} 1} =$$

$$= \frac{\left[\lim_{x\to 1} x\right]^2 - 2\left(\lim_{x\to 1} x\right) + 1}{1 + 1}$$

$$= \frac{1^2 - 2(1) + 1}{2} = \frac{0}{2} = \boxed{0}$$

this is Substitution Thm 1

(b)

$$\lim_{x \to 3} \left[(x+1) f(x)^2 + \frac{x+2}{g(x)} \right] =$$

$$= \lim_{x \to 3} \left[(x+1) f(x)^2 \right] + \lim_{x \to 3} \frac{x+2}{g(x)}$$

$$= \left[\lim_{x \to 3} (x+1) \right] \left[\lim_{x \to 3} f(x)^2 \right] + \frac{\lim_{x \to 3} (x+2)}{\lim_{x \to 3} g(x)}$$

$$= \left[\lim_{x \to 3} x + \lim_{x \to 3} 1 \right] \left[\underbrace{\lim_{x \to 3} f(x)}_{= -2} \right]^2 + \frac{\left[\lim_{x \to 3} x \right] + \lim_{x \to 3} 2}{\underbrace{\lim_{x \to 3} g(x)}_{= 4}}$$

$$= (3+1)(-2)^2 + \frac{3+2}{4} = 16 + \frac{5}{4} = \frac{69}{4} \cong 17.25$$

When Limits Fail to Exist

There are two basic ways that a limit can fail to exist.

(a) The function attempts to approach multiple values as $x \to c$.

Geometrically, this behavior can be seen as a jump in the graph of a function.

Algebraically, this behavior typically arises with piecewise defined functions.

(b) The function grows without bound as $x \to c$.

Geometrically, this behavior can be seen as a vertical asymptote in the graph of a function.

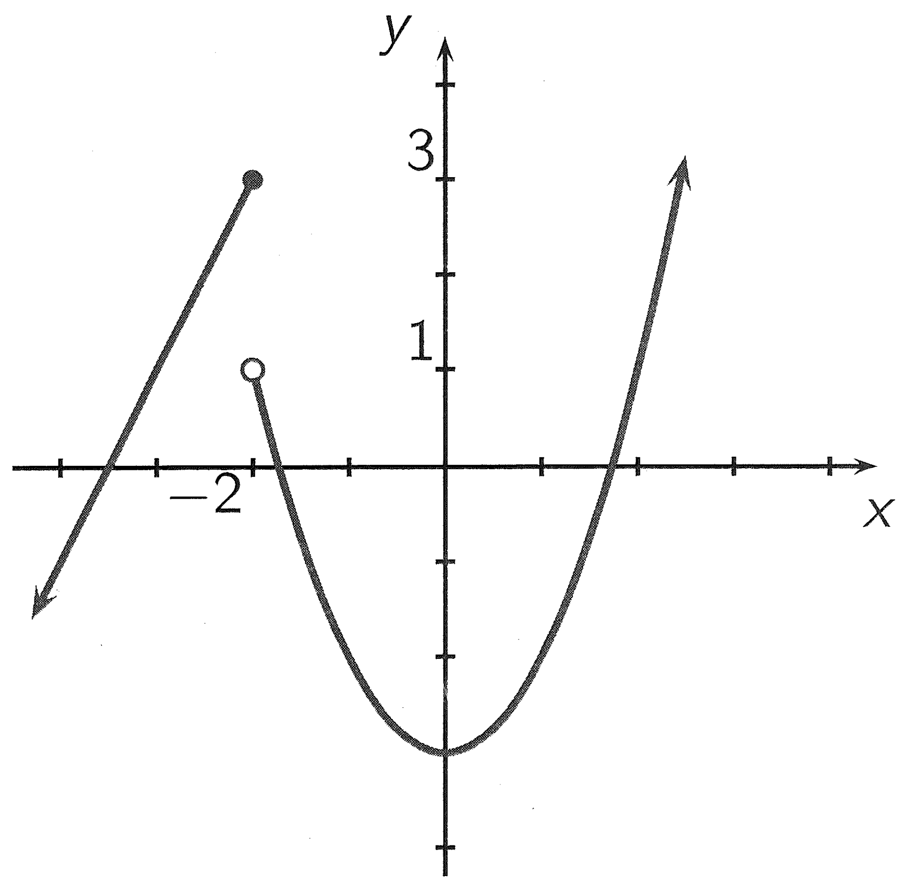
Algebraically, this behavior typically arises when the denominator of a function approaches zero.

Example 4:

The graph of the function

$$h(x) = \begin{cases} x^2 - 3 & \text{if } x > -2 \\ 2x + 7 & \text{if } x \leq -2 \end{cases}$$

is shown to the right.



Analyze $\lim_{x \to -2} h(x)$.

If we approach -2 from the left, the function $h(x)$ is defined by $2x+7$ hence

$$\lim_{\substack{x \to -2 \\ \text{from the left}}} h(x) = \lim_{\substack{x \to -2 \\ \text{from the left}}} (2x+7) = -4+7 = \boxed{3}$$

If we approach -2 from the right, the function $h(x)$ is defined by x^2-3 hence

$$\lim_{\substack{x \to -2 \\ \text{from the right}}} h(x) = \lim_{\substack{x \to -2 \\ \text{from the right}}} (x^2-3) = (-2)^2-3 = \boxed{1}$$

Since the values do not coincide $\lim_{x \to -2} h(x)$

DOES NOT EXIST !

One-sided Limits

The previous example brings us to the following notions:

One-sided limits

A one-sided limit expresses what happens to the values of an expression as the variable in the expression gets closer and closer to some particular value c from either the left on the number line (that is, through values less than c) or from the right on the number line (that is, through values greater than c).
The notation is:

$$\underbrace{\lim_{x \to c^-} f(x)}_{\text{limit from the left of } c} \qquad \underbrace{\lim_{x \to c^+} f(x)}_{\text{limit from the right of } c}$$

Fact: $\lim_{x \to c} f(x)$ exists if and only if

both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and have the same value.

Example 5:

(a) Analyze $\displaystyle\lim_{x\to1} \frac{5}{(x-1)^2}$.

(b) Analyze $\displaystyle\lim_{x\to1} \frac{2}{x-1}$.

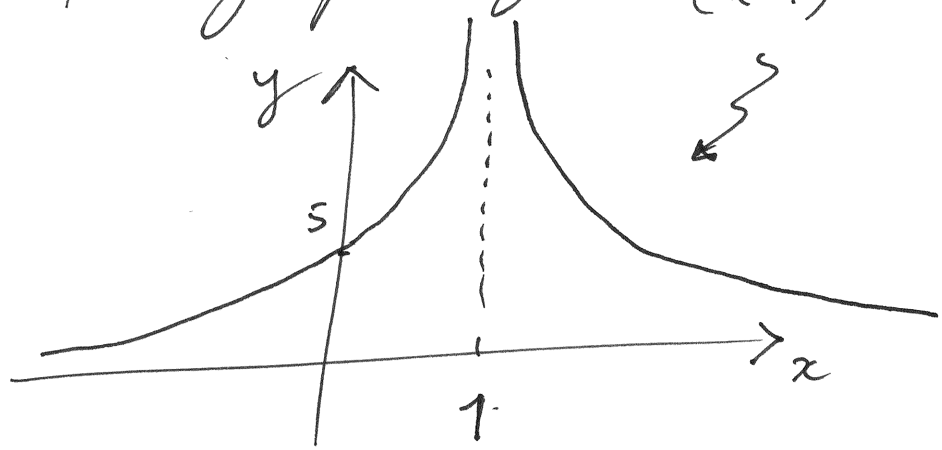
(c) Analyze the limit $\displaystyle\lim_{x\to0} \frac{2}{\sqrt{x}}$.

(a) $\lim_{x \to 1} \frac{5}{(x-1)^2}$ if we build a table of values nearby $x=1$ we obtain

x	0.9	0.99 ...	... 1.01	1.1
$5/(x-1)^2$	500	50,000 ...	... 50,000	50

it seems that $\lim_{x \to 1} \frac{5}{(x-1)^2} = +\infty$ i.e. D.N.E.

look at the graph of $\frac{5}{(x-1)^2}$



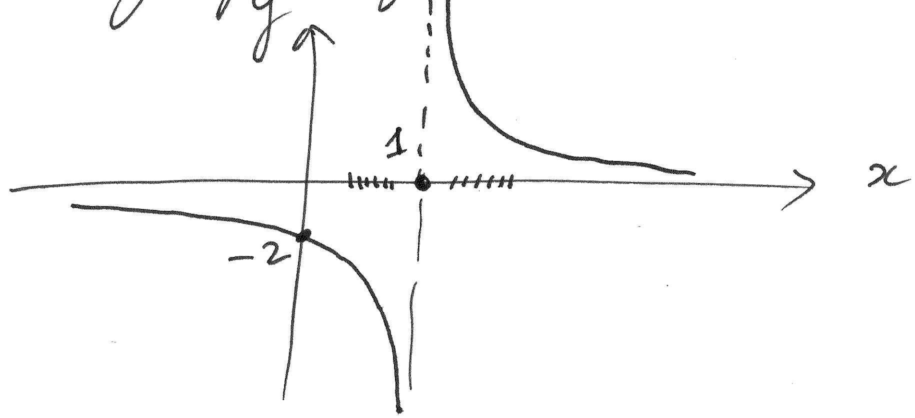
(b) $\lim_{x \to 1} \frac{2}{x-1}$ if we build a table of values nearby $x=1$ we obtain

x	0.9	0.99	0.999 ...	.... 1.001	1.01	1.1
$\frac{2}{x-1}$	-20	-200	-2000	2000	200	20

hence $\lim_{x \to 1^-} \frac{2}{x-1} = -\infty$ $\quad \lim_{x \to 1^+} \frac{2}{x-1} = +\infty$

no matter what $\lim_{x \to 1} \frac{2}{x-1}$ D.N.E

If you look at the graph of $\frac{2}{x-1}$ we have



(c) Analyze $\lim_{x \to 0} \frac{2}{\sqrt{x}}$.

First, the limit means $\lim_{x \to 0^+} \frac{2}{\sqrt{x}}$ as $\sqrt{x}$ is not defined for negative values of x.

Build a table of values:

x	... 0.001	0.01	0.1
$2/\sqrt{x}$	... 63.25	20	6.325

it seems $\lim_{x \to 0^+} \frac{2}{\sqrt{x}} = +\infty$ OR D.N.E.

Try to graph $2/\sqrt{x}$ and see how the graph looks like near 0^+.

The most interesting and important situation with limits is when a substitution yields $0/0$.

The result $0/0$ yields absolutely no information about the limit. It does not even tell us that the limit does not exist. The only thing it tells us is that we have to do more work to determine the limit.

Example 6:

Find the limit $\displaystyle\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$.

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \text{substitution Thm 2}$$

$$= \frac{\lim_{x \to 3} x^2 - 2x - 3}{\lim_{x \to 3} x - 3} = \frac{3^2 - 2(3) - 3}{3 - 3} = \frac{0}{0}$$

What now?

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} (x+1) = 3+1 = \boxed{4}$$

by substitution Thm I.

Example 7: (Online Homework HW07, #5)

Guess the value of the limit (if it exists) by evaluating the function at values close to where the limit is to be done.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{0}{0}$$

To make an educated guess, let's build a table of values:

x	-0.1	-0.01	-0.001	$-0.0001 \dashrightarrow$	$\dashleftarrow 0.0001$	0.001	0.01	0.1
$\frac{e^x - 1 - x}{x^2}$	0.48374	0.49834	0.49983	$0.49998 \dashrightarrow$	$\dashleftarrow 0.50002$	0.50017	0.50167	0.51709

it seems that $\boxed{\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = 0.5}$

Wow!!!

We will try to justify this answer... soon.

Example 8:

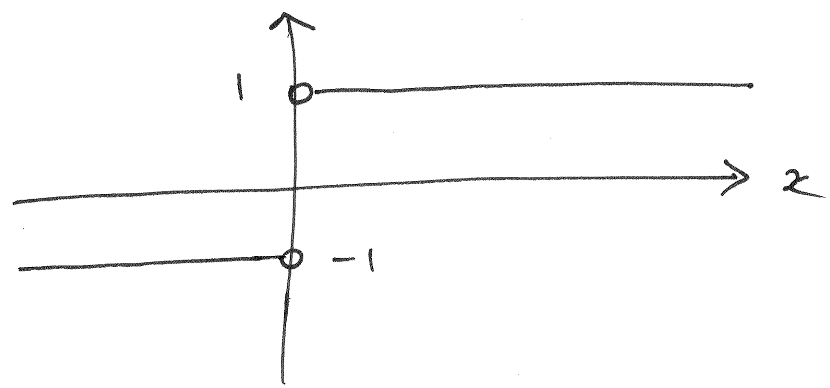
Find the limits

$$\lim_{x\to 0^+} \frac{|x|}{x} \qquad \lim_{x\to 0^-} \frac{|x|}{x} \qquad \lim_{x\to 0} \frac{|x|}{x}.$$

Graph $\frac{|x|}{x}$! For $x>0$ $|x|=x$ so

$\frac{|x|}{x} = \frac{x}{x} = 1$; For $x<0$ $|x| = -x$ so

$\frac{|x|}{x} = \frac{-x}{x} = -1$. Thus



Function is <u>not</u> defined when $\underline{\underline{x=0}}$

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

hence $\lim_{x\to 0} \frac{|x|}{x} = \underline{\underline{\text{D.N.E.}}}$

Example 9: (Neuhauser, Example 9, p. 97)

Find the limit $$\lim_{x \to 0} \frac{\sqrt{x^2 + 16} - 4}{x^2}$$

If we build a table of values around 0

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	0.12498	0.125	0.125	0.125	0.125	0.12434

0.0000001	0.00000001	....
0.17764	0	...

These # have been calculated with an Excel spreadsheet

So the calc. suggest $\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} = 0$?

HOWEVER even calculators may round up in the wrong way ....

$$\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} = \lim_{x\to 0} \frac{(\sqrt{x^2+16}-4)}{x^2} \cdot \frac{(\sqrt{x^2+16}+4)}{\sqrt{x^2+16}+4}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+16})^2 - 4^2}{x^2(\sqrt{x^2+16}+4)} =$$

$$= \lim_{x\to 0} \frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)} = \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+16}+4} = \frac{1}{\sqrt{0^2+16}+4} = \frac{1}{8}$$

$$= \underline{\underline{0.125}}$$

MA 137 – Calculus 1with Life Science Applications
Continuity
(Section 3.2)

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Intuitive Examples

(a) What is the main difference between the following two functions?

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases} \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

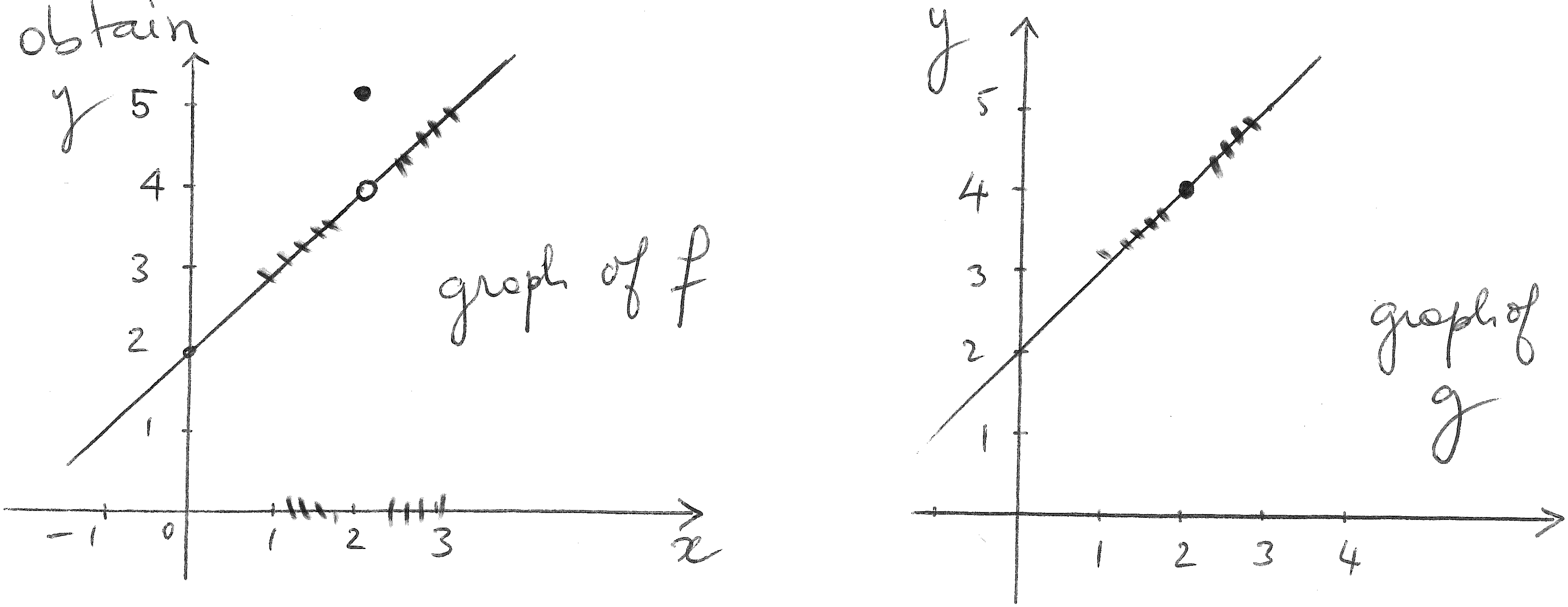
How does this difference translate when we graph f and g?

(b) What is the main difference between the following two functions?

$$\widetilde{f}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \widetilde{g}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

How does this difference translate when we graph $\widetilde{f}$ and $\widetilde{g}$?

(a) If we plot the functions f and g we obtain



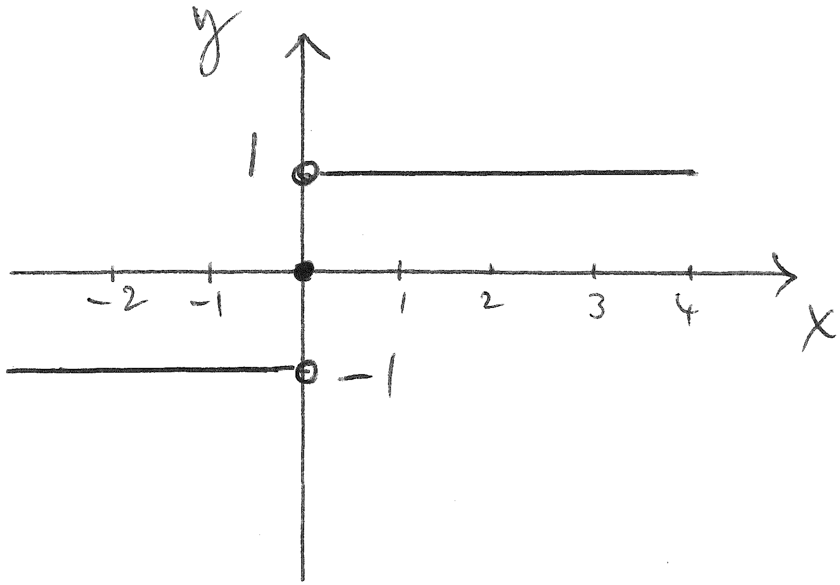
this is because you should realize that

$$\boxed{\frac{x^2-4}{x-2}} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \boxed{x+2}$$

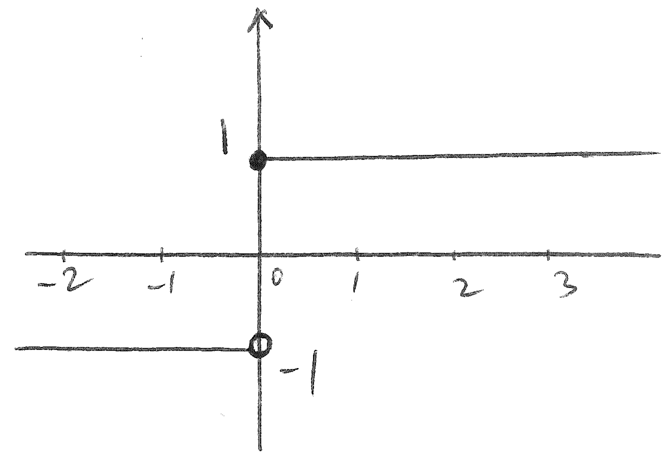
$(\text{for } x \neq 2)$

(b)

graph of $\tilde{f}$



graph of $\tilde{g}$



$$\tilde{f} = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$\tilde{g} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

The previous example, (**a**), suggests that the definition of g at $x = 2$ is the one that fills the "hole" in the graph!!

Unfortunately, in the second case, (**b**), there is no way that we can assign a value to either $\widetilde{f}$ or $\widetilde{g}$ such that the graphs have no jump at $x = 0$.

Intuitively, we define a function to be continuous if "we can draw its graph without lifting our pencil from the paper."

In other words, this means that there are "no holes" in the graph.

In the first case, (**a**), the discontinuity of f at $x = 2$ could be removed and we got g that is continuous at $x = 2$. In the other case, (**b**), the discontinuity at $x = 0$ is not removable.

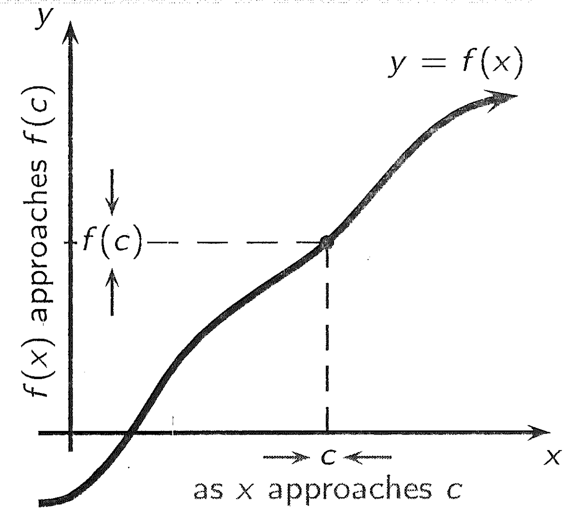
Continuity at a Point

Formal Definition

A function f is **continuous at a point** $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

To check whether f is continuous at $x = c$, we need to check the following conditions:

1. $f(x)$ is defined at $x = c$;
2. $\lim_{x \to c} f(x)$ exists;
3. $\lim_{x \to c} f(x)$ is equal to $f(c)$.



Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If any of those three conditions fails, f is **discontinuous** at $x = c$.

Continuity on an Interval

- A function f is continuous from the **right** at a point $x = c$ if
$$\lim_{x \to c^+} f(x) = f(c)$$
and f is continuous from the **left** at a point $x = c$ if
$$\lim_{x \to c^-} f(x) = f(c)$$
- We say that a function f is continuous on an **interval** I if f is continuous for all $x \in I$.
- If I is a closed interval, then continuity at the left (and, respectively, right) endpoint of the interval means continuous from the right (and, respectively, left).
- **Geometrically**, you can think of a function that is continuous at every point in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pencil from the paper.

Example 1:

Make a graph of the function $f(x) = \sqrt{9 - x^2}$.

For which values of x is the function f continuous?

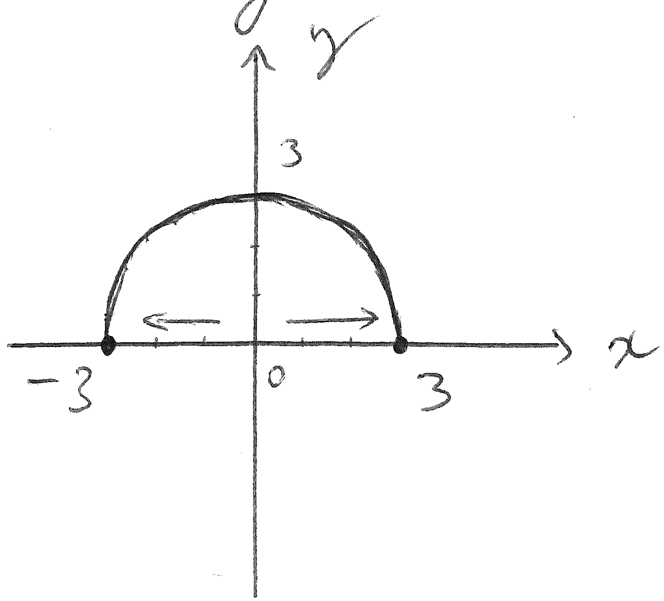
the graph of $f(x) = \sqrt{9-x^2}$

Corresponds to half of the circle of radius

3 centered at the origin

this is because $y = \sqrt{9-x^2} \geq 0$

$\iff y^2 = 9 - x^2 \iff \boxed{x^2 + y^2 = 9}$



the function is continuous for all $-3 < x < 3$

at $x=-3$ it is right continous

at $x=3$ it is left continuous

A Helpful Rewrite and a Few Comments

Our definition says that if $f(x)$ is continuous at a point $x = c$ then we can take the limit inside the function f:

$$\lim_{x \to c} f(x) = f(\lim_{x \to c} x).$$

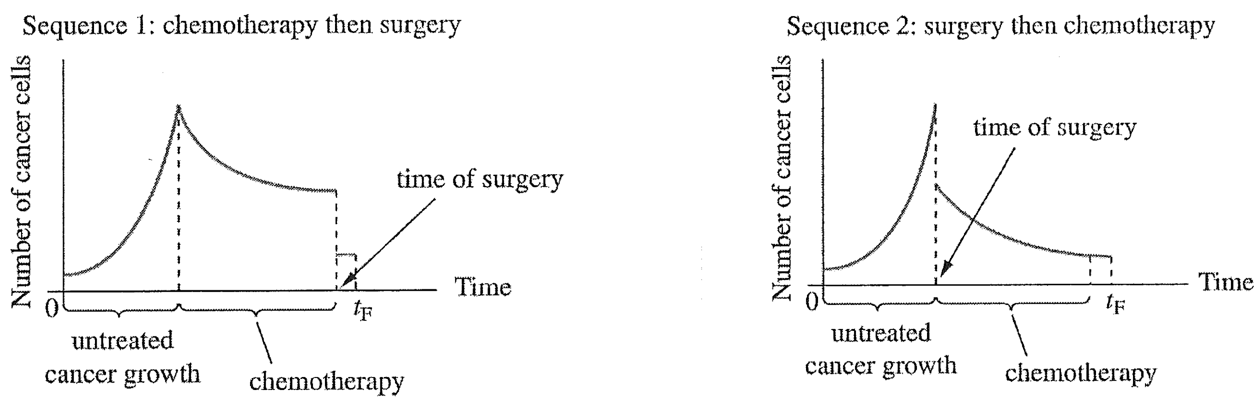
We like continuous functions is because they are...predictable.

- Assume you are watching a bird flying and then close your eyes for a second. When you open your eyes, you know that the bird will be somewhere around the location where you last saw it.
- As you move up a mountain, flora is a discontinuous function of altitude. There is the 'tree line,' below which the dominant plant species are pine and spruce and above which the dominant plant species are low growing brush and grasses.

Chemotherapy and surgery are two frequently used treatments for many cancers. How should they be used—chemotherapy first and then surgery, or the other way around? It is a non trivial matter.

For the case of ovarian cancer, researchers built mathematical models that track number of cancer cells as chemotherapy then surgery or surgery then chemotherapy.

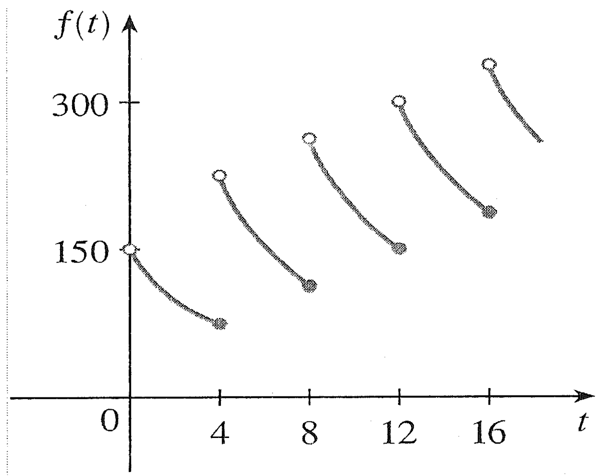
M. Kohandel et al.
Mathematical modeling of ovarian cancer treatments: Sequencing of surgery and chemotherapy
Journal of Theoretical Biology **242**, 62-68, 2006



Both functions have a discontinuity at moment of surgery (cancer cells removed). The size of the jump is highly relevant to decision about sequence of treatments applied.

Example 2:

A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours.



At what values of t does $f(t)$ have discontinuities?
What type of discontinuities does $f(t)$ have?

The discontinuities of f are jumps at $x = 4$, $x = 8$, $x = 12$, $x = 16$, ... that is every time there is an injection of the drug.

Example 3:

Let $f(x) = \lfloor x \rfloor$ be the function that associates to any value of x the greatest integer less than or equal to x.

Make a graph of the function $y = \lfloor x \rfloor$.

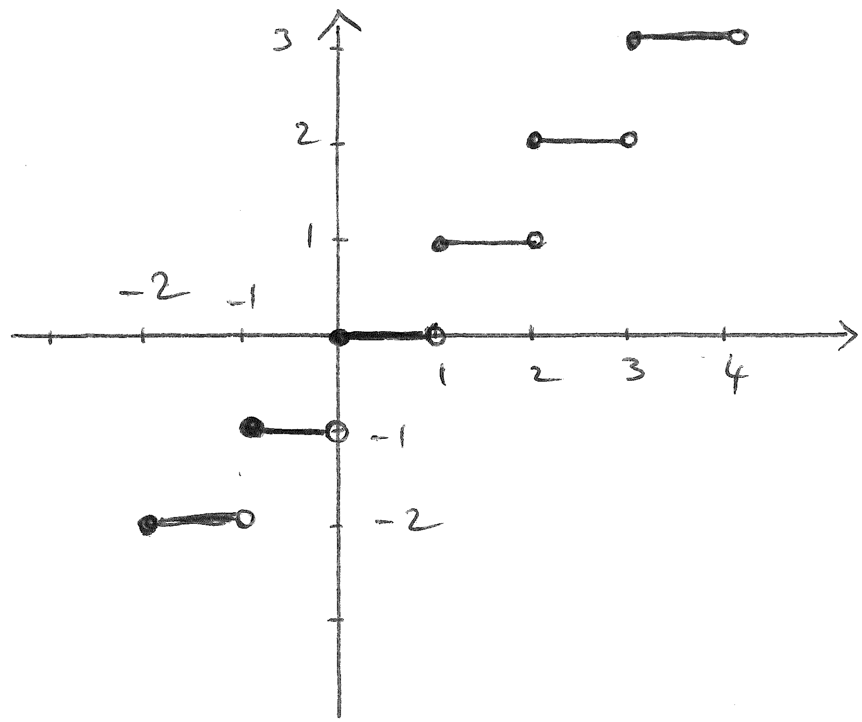
For which values of x is the function f continuous?

$f(x) = \lfloor x \rfloor =$ greatest integer less than or equal to x

So for example $\lfloor 1 \rfloor = 1$ $\lfloor 1.2 \rfloor = 1$

$\lfloor 1.99 \rfloor = 1$ but $\lfloor 2.1 \rfloor = 2$ $\lfloor 2.9 \rfloor = 2$

$\lfloor 3 \rfloor = 3$ etc...



the function is continuous whenever x is <u>not</u> an integer

Continuity and Operations on Functions

Using the limit laws, the following statements hold for combinations of continuous functions:

If α is a constant and the functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $\alpha \cdot f$
2. $f \pm g$
3. $f \cdot g$
4. f/g provided that $g(c) \neq 0$

Proof of 2.: Since f and g are continuous at $x = c$, it follows that

$$\lim_{x \to c} f(x) = f(c) \qquad \text{and} \qquad \lim_{x \to c} g(x) = g(c).$$

We can apply one of the rules of Limit Laws and find that

$$\lim_{x \to c}(f \pm g)(x) \overset{\text{def.}}{=} \lim_{x \to c}[f(x) \pm g(x)] \overset{\text{rule}}{=} [\lim_{x \to c} f(x)] \pm [\lim_{x \to c} g(x)] \overset{\text{cont.}}{=} f(c) \pm g(c) \overset{\text{def.}}{=} (f \pm g)(c).$$

Example 4: (Online Homework HW08, # 12)

If f and g are continuous functions with

$$f(3) = 5 \quad \text{and} \quad \lim_{x \to 3} [2f(x) - g(x)] = 4,$$

find $g(3)$.

f and g are continuous with

$f(3)=5$ and $\lim_{x \to 3} [2f(x) - g(x)] = 4$

Notice that $2f-g$ is also a continuous function. <u>So</u>!

$$4 = 2f(3) - g(3) \overset{\text{continuity}}{=} \lim_{x \to 3} [2f(x) - g(x)]$$

$$\Longrightarrow \quad 4 = 2 \cdot 5 - g(3)$$

So $g(3) = 10 - 4 = 6$

Catalogue of Continuous Functions

Many of the elementary functions are continuous *wherever they are defined*. Here is a list:

1. Polynomial functions[a]
2. Rational functions[a]
3. Power functions
4. Trigonometric functions
5. Exponential functions of the form a^x, $a > 0$ and $a \neq 1$
6. Logarithmic functions of the form $\log_a x, a > 0$ and $a \neq 1$

[a] For polynomials and rational functions, this statement follows immediately from the fact that certain combinations of continuous functions are continuous.

The phrase "wherever they are defined" helps us to identify points where a function might be discontinuous. For instance, the function $1/(x-3)$ is defined only for $x \neq 3$, and the function $\sqrt{x-2}$ is defined only for $x \geq 2$.

Example 5: (Online Homework HW08, # 17)

Consider the function $$f(x) = \begin{cases} b - 2x & \text{if } x < -4 \\[2ex] \dfrac{-96}{x - b} & \text{if } x \geq -4 \end{cases}$$

Find the two values of b for which f is a continuous function at $x = -4$.

Draw a graph of f.

$$f(x) = \begin{cases} b - 2x & x < -4 \\ \\ \dfrac{-96}{x-b} & x \geq -4 \end{cases}$$

We need to have $\displaystyle\lim_{x \to -4^-} f(x) = \lim_{x \to -4^+} f(x)$

I.e. $\displaystyle\lim_{x \to -4^-} (b - 2x) = \lim_{x \to -4^+} \frac{-96}{x-b}$

$$\iff \quad b - 2(-4) = \frac{-96}{-4-b}$$

$$\iff b + 8 = \frac{96}{4+b} \iff (b+8)(4+b) = 96$$

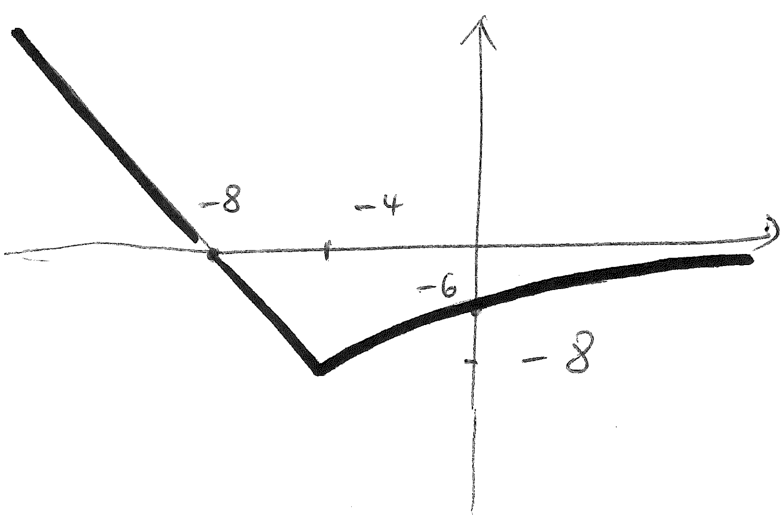
$$\iff b^2 + 12b - 64 = 0 \iff (b+16)(b-4) = 0$$

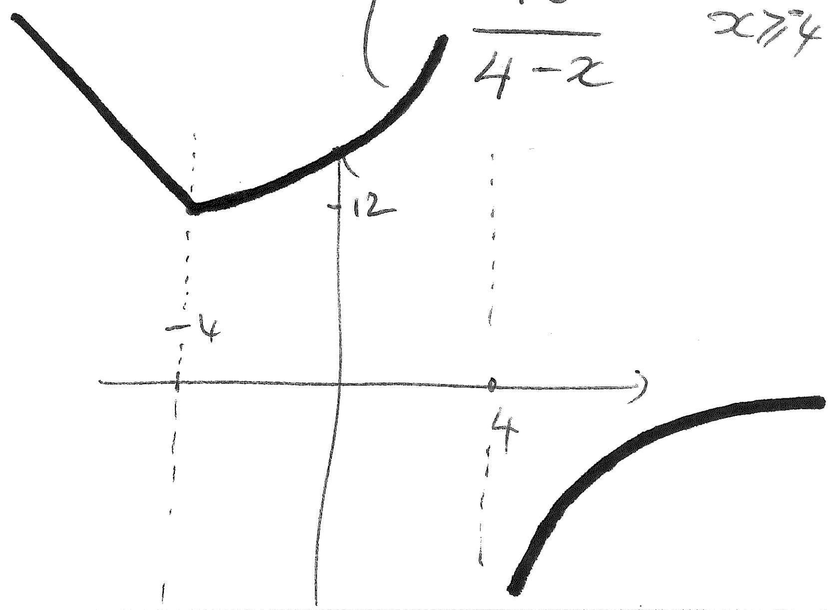
$$\therefore \; b = -16 \quad \text{and} \quad b = 4$$

We get 2 possibilities

$$f_1(x) = \begin{cases} -16 - 2x & x < -4 \\ \dfrac{-96}{x+16} & x \geq -4 \end{cases}$$

$$f_2(x) = \begin{cases} 4 - 2x & x < -4 \\ \dfrac{96}{4-x} & x \geq -4 \end{cases}$$





Continuity and Composition of Functions

Another way of combining continuous functions to get a new continuous function is to form their composition.

Theorem

If $g(x)$ is continuous at $x = c$ and $f(x)$ is continuous at $x = g(c)$, then $(f \circ g)(x)$ is continuous at $x = c$.
In particular,

$$\lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f[g(x)] = f[\lim_{x \to c} g(x)] = f(g(c)) = (f \circ g)(c)$$

In other words, the above theorem says that

"composition of continuous functions is continuous."

Example 6: (Online Homework HW08, # 10)

Use continuity to evaluate $\displaystyle\lim_{x\to 1} e^{x^2-3x+5}$.

the exponential function and any polynomial function are continuous so e^{x^2-3x+5} is also continuous because it is the composition of 2 continuous functions:

$$\lim_{x\to 1}\left(e^{x^2-3x+5}\right) = e^{\lim_{x\to 1}(x^2-3x+5)}$$

$$= e^{1^2-3(1)+5} = \boxed{e^3}$$

MA 137 – Calculus 1 with Life Science Applications
Limits at Infinity & Properties of Continuous Functions
(Sections 3.3 & 3.5)

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Asymptotic Behavior

When studying sequences $\{a_n\}$, say in the context of populations over time, we were interested in their long-term behavior: $\lim_{n \longrightarrow \infty} a_n$.

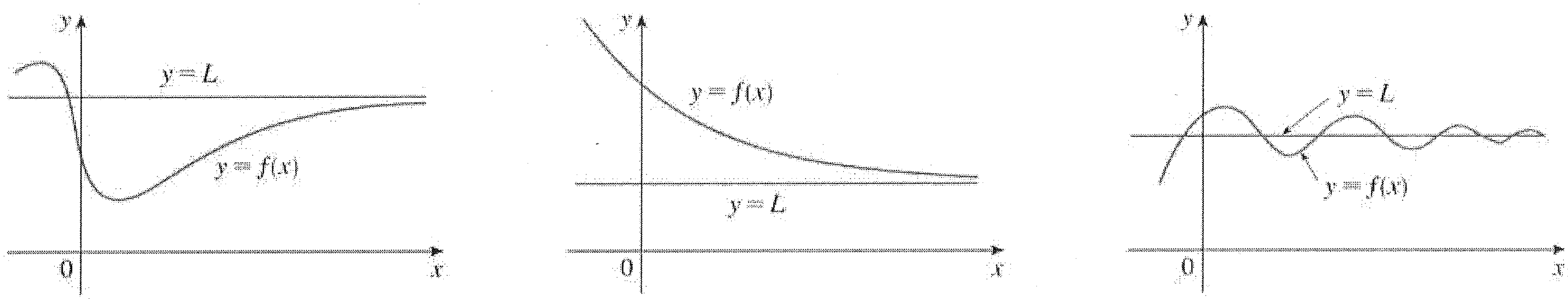
Now we do something similar: We ask what happens to function values $f(x)$ when x becomes large. Now, x is no longer restricted to be an integer.

Definition

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L whenever x is sufficiently large.



- A similar definition holds for limits where x tends to $-\infty$.
- The Limit Laws that we discussed earlier also hold as x tends to $\pm\infty$.
- It is easy to convince yourself that

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0.$$

From the Limit Laws it also follows that

$$\lim_{x \to \pm\infty} \frac{1}{x^p} = 0 \qquad \text{for any integer } p > 0.$$

Example 1:

Evaluate

$$\lim_{x\to\infty} \frac{1-x+2x^2}{3x-5x^2} \qquad \lim_{x\to\infty} \frac{1-x^3}{1+x^5}$$

$$\lim_{x\to\infty} \frac{1-x+2x^2}{3x-5x^2}$$ = if we use directly the Rules for Limits we obtain $= \frac{\infty}{-\infty}$

Thus we need to rewrite first the fraction by dividing top and bottom by x^2:

$$= \lim_{x\to\infty} \frac{\frac{1-x+2x^2}{x^2}}{\frac{3x-5x^2}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{1}{x} + 2}{\frac{3}{x} - 5}$$

= now we can use the Rules for Limits

$$= \frac{\left(\lim_{x\to\infty} \frac{1}{x^2}\right) - \left(\lim_{x\to\infty} \frac{1}{x}\right) + \lim_{x\to\infty} 2}{\left(\lim_{x\to\infty} \frac{3}{x}\right) - \lim_{x\to\infty} 5} = \frac{2}{-5}$$

$$= \boxed{-0.4}$$

As before

$$\lim_{x\to\infty} \frac{1-x^3}{1+x^5} = \frac{-\infty}{\infty} = \text{hence we rewrite}$$

$$= \lim_{x\to\infty} \frac{\frac{1-x^3}{x^3}}{\frac{1+x^5}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x^3} - 1}{\frac{1}{x^3} + x^2} =$$

$$= \frac{\left(\lim_{x\to\infty} \frac{1}{x^3}\right) - \lim_{x\to\infty} 1}{\left(\lim_{x\to\infty} \frac{1}{x^3}\right) + \lim_{x\to\infty} x^2} = \frac{-1}{\infty} = \boxed{0}$$

General Fact

If $f(x)$ is a rational function of the form $f(x) = p(x)/q(x)$, where $p(x)$ is a polynomial of degree $\deg(p)$ and $q(x)$ is a polynomial of $\deg(q)$, then:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L \neq 0 & \text{if } \deg(p) = \deg(q) \\ \text{does not exist} & \text{if } \deg(p) > \deg(q) \end{cases}$$

Here, L is a real number that is the ratio of the coefficients of the leading terms in the numerator and denominator.

The same behavior holds as $x \to -\infty$.

Example 2: (Online Homework HW09, # 5)

A function is said to have a **horizontal asymptote** if either the limit at infinity exists or the limit at negative infinity exists. Show that each of the following functions has a horizontal asymptote by calculating the given limit.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}}{8-7x} \qquad \lim_{x\to-\infty} \frac{\sqrt{x^2+2x}}{8-7x}.$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}}{8-7x} = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}\right)\frac{1}{x}}{(8-7x)\,\frac{1}{x}} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{\dfrac{x^2+2x}{x^2}}}{\dfrac{8}{x}-7} = \lim_{x\to\infty} \frac{\sqrt{1+2/x}}{\dfrac{8}{x}-7}$$

$$= \frac{\sqrt{1+\lim_{x\to\infty} 2/x}}{\left(\lim_{x\to\infty}\dfrac{8}{x}\right)-7} = \frac{1}{-7} = \boxed{-1/7}$$

we can bring inside the sqrt only positive values

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2x}}{8-7x} = \underline{\text{tricky point}} = \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+2x}\right)\left(-\frac{1}{x}\right)(-1)}{(8-7x)\,\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{1+\dfrac{2}{x}}}{8/x-7} = \frac{-\sqrt{1+\lim_{x\to-\infty}\dfrac{2}{x}}}{\left(\lim_{x\to-\infty}\dfrac{8}{x}\right)-7} = \boxed{\frac{1}{7}}$$

Rational functions are not the only functions that involve limits as $x \to \infty$ (or $x \to -\infty$).

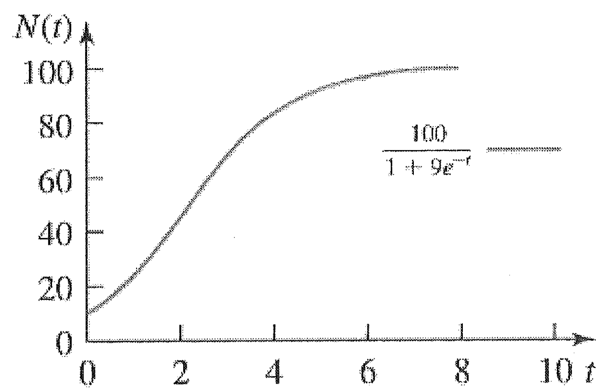
Many important applications in biology involve exponential functions. We will use the following result repeatedly — it is one of the most important limits:

$$\lim_{x \to \infty} e^{-x} = 0.$$

Another formulation of the same result is:

$$\lim_{x \to -\infty} e^{x} = 0.$$

For example: $$\lim_{t \to \infty} \frac{100}{1 + 9e^{-t}} = 100$$



Example 3: (Neuhauser, Example 3, p. 112)

The **logistic curve** describes the density of a population over time, where the rate of growth depends on the population size. It is characterized by the fact that *the per capita rate of growth decreases linearly with increasing population size.* If $N(t)$ denotes the size of the population at time t, then the logistic curve is given by

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}} \qquad \text{for} \quad t \geq 0.$$

The parameters K and r are positive numbers that describe the population dynamics.

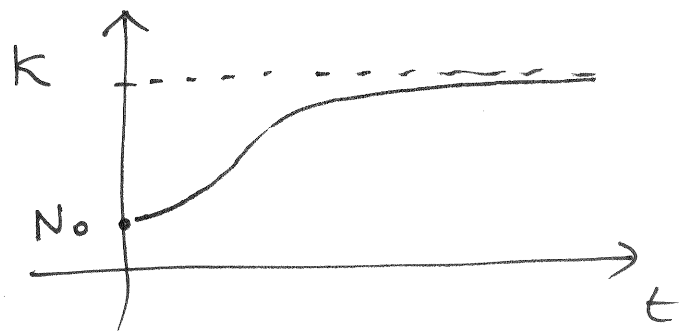
If we seek the long-term behavior of the population as it evolves in accordance with the logistic growth curve, we find that

$$\lim_{t \to \infty} N(t) = K.$$

That is, as $t \to \infty$, the population size approaches K, which is called the **carrying capacity** of the population.

logistic growth function

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$



eg: $N(t) = \dfrac{100}{1 + 9e^{-rt}}$ with $K = 100$

$N_0 = 10$

$r = 1 = 100\%$

Notice $N(0) = \dfrac{K}{1 + \left(\frac{K}{N_0} - 1\right) \underbrace{e^0}_{1}} = \dfrac{K}{\cancel{1} + \frac{K}{N_0} - \cancel{1}} = N_0$

Also $\lim\limits_{t \to \infty} \dfrac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} = \dfrac{K}{1 + \left(\frac{K}{N_0} - 1\right) \underbrace{\lim\limits_{t\to\infty} e^{-rt}}_{0}}$

$= \dfrac{K}{1 + 0} = \boxed{K}$ carrying capacity.

Example 4:

The **von Bertalanffy growth function**

$$L(t) = L_\infty - (L_\infty - L_0)e^{-kt}$$

where k is a positive constant, models the length L of a fish as a function of t, the age of fish. This model assume that the fish has a well defined length L_0 at birth $(t = 0)$.

Calculate $\displaystyle\lim_{t\to\infty} L(t)$.

$$L(t) = L_\infty - (L_\infty - L_0)e^{-kt}$$

$\underline{\underline{\text{eg.}}}$ $\quad L(t) = 10 - 8e^{-0.2t}$

$$L_\infty = 10$$
$$L_0 = 2$$
$$k = 0.2 = 20\%$$

Notice that

$$L(0) = L_\infty - (L_\infty - L_0)\underbrace{e^0}_{1} = \cancel{L_\infty} - \cancel{L_\infty} + L_0$$

$$= L_0$$

$$\lim_{t\to\infty} L_\infty - (L_\infty - L_0)e^{-kt} = L_\infty - (L_\infty - L_0)\underbrace{\lim_{t\to\infty} e^{-kt}}_{0}$$

$$= L_\infty$$

asymptotic length of the fish

A Spiritual Journey

- One morning, at sunrise, a (tibetan) monk began to climb a tall mountain from his monastery. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit he carried with him or to look at the flowers. He reached the temple at sunset.

  After several days of fasting and meditation, he began his journey back along the same path, starting at sunrise and again walking at varying speeds with many pauses along the way. He reached the bottom at sunset.

- I assert that there is at least one spot along the path the monk occupied at precisely the same time of day on both trips.?

- Is my assertion true? How do you decide?

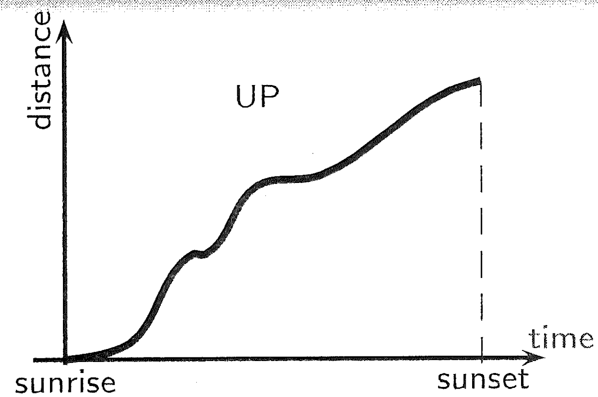
Visual Thinking: the monk and the mountain

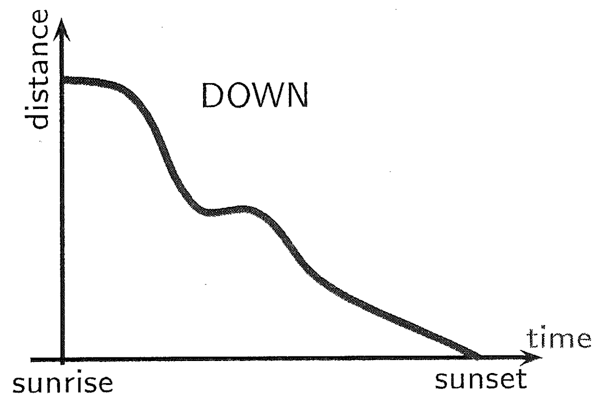
The monk travels along the same path on both days and his position is determined by the distance from the monastery.

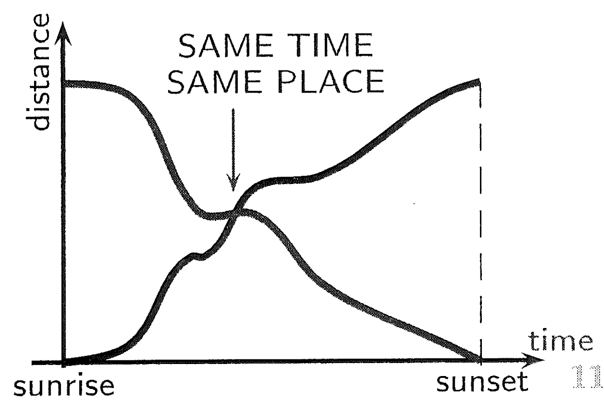
Position is a continuous function of time.

If we plot the path up the mountain in a time-distance coordinate system, then the curve goes from (sunrise, monastery) to (sunset, temple). Flat regions on the graph are rest times, dips arise from, say, retracing his steps to look at a flower. The path down the mountain is a curve from the point (sunrise, temple) to (sunset, monastery).

When the two paths are plotted on the same axes, it is obvious that the curves intersect — this is a point where the monk is at the same point at the same time on the two days.



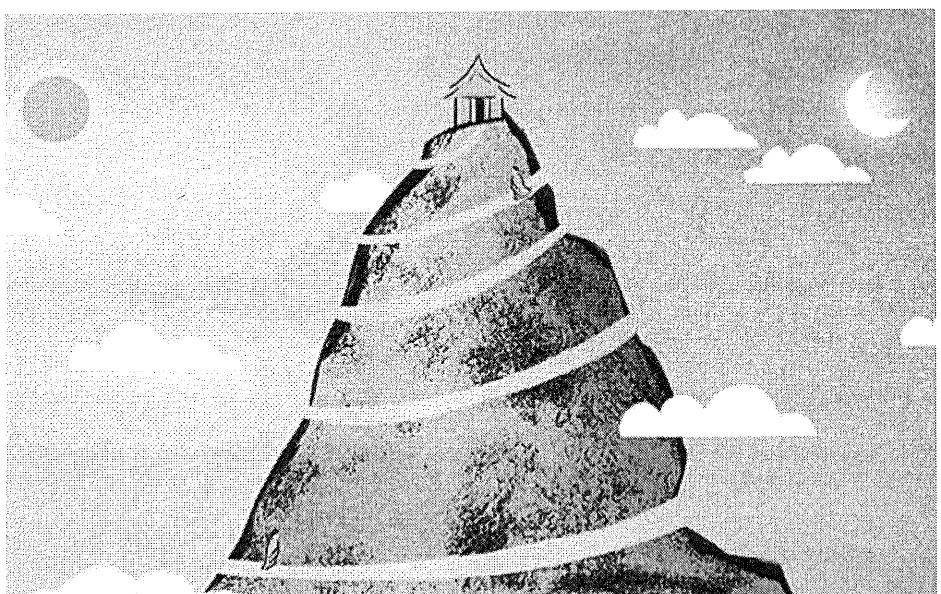




Two Monks and the Mountain

There is an insightful solution to the problem that is equivalent to the previous graphical one, but there is no need to graph a plot. One monk and two days does not make the solution as transparent as it could be. So we look at a similar, equivalent problem.

Suppose there are two monks and they both start at sunrise, one at the bottom of the mountain ($\equiv$ monastery), the other at the top ($\equiv$ temple). It seems obvious now that the two monks must meet somewhere along the path — at the same time and at the same place.



Mathematical Thinking

Let $dist_{\text{up}}(t)$ represent the distance the monk is away from the monastery on day 1, where t represents any time between sunrise and sunset. Note that $dist_{\text{up}}(\text{sunrise}) = 0$ and $dist_{\text{up}}(\text{sunset}) = d$, the distance from the monastery to the temple on the top of the mountain.

Let $dist_{\text{down}}(t)$ represent the distance the monk is away from the monastery on day 2, where t represents any time between sunrise and sunset. Note that $dist_{\text{down}}(\text{sunrise}) = d$ and $dist_{\text{down}}(\text{sunset}) = 0$.

Observe that these functions are continuous, since they correspond to the path that the monk is walking.

Now consider the function $f = dist_{\text{up}} - dist_{\text{down}}$.
Since $dist_{\text{up}}$ and $dist_{\text{down}}$ are continuous, so is the function f.

Observe that $f(\text{sunrise}) = -d$ and $f(\text{sunset}) = d$. Because f is continuous there must be a value t_0 between sunrise and sunset such that $f(t_0) = 0$.

$$f(t_0) = 0 \iff [dist_{\text{up}} - dist_{\text{down}}](t_0) = 0 \iff dist_{\text{up}}(t_0) = dist_{\text{down}}(t_0).$$

Thus t_0 is the time when the monk is at the same point at the same time on the two days.

The Intermediate Value Theorem (IVT)

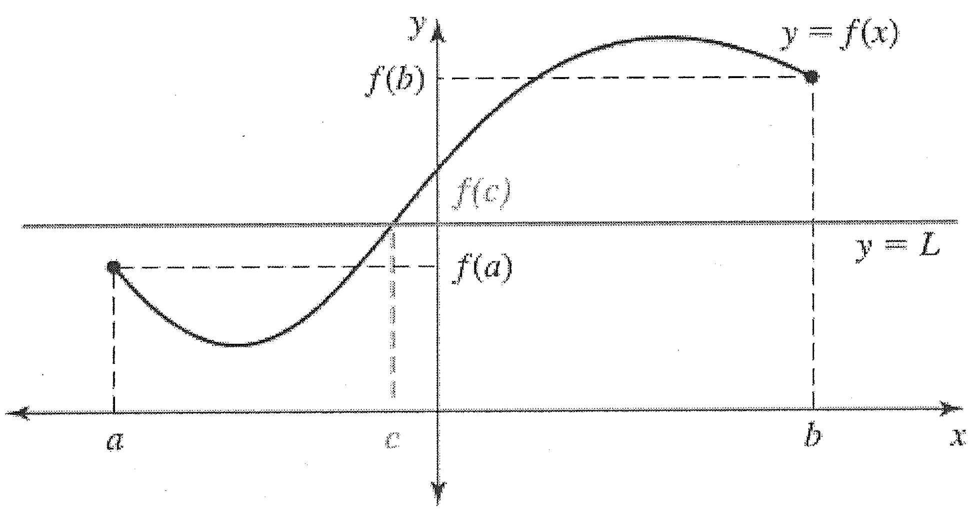
The previous story ($\equiv$ the monk and the mountain) represents an illustration of the content of the following result.

The Intermediate Value Theorem (B. Bolzano, 1817)

Suppose that f is **continuous** on the **closed** interval $[a, b]$.

If L is any real number with $f(a) < L < f(b)$ [or $f(b) < L < f(a)$],

then there exists **at least one** value $c \in (a, b)$ such that $f(c) = L$.



- In applying the Intermediate Value Theorem, it is important to check that f is continuous.
- Discontinuous functions can easily miss values; for example, the floor function misses all numbers that are not integers.
- The Intermediate Value Theorem gives us only the existence of a number c; it does not tell us how many such points there are or where they are located.
- As an application, the Theorem can be used to find approximate roots (or solutions) of equations of the form $f(x) = 0$.

Example 5:

Use the Intermediate Value Theorem to conclude that $e^{-x} = x$ has a solution in $[0, 1]$

$$e^{-x} = x \iff e^{-x} - x = 0$$

So consider the function $\boxed{f(x) = e^{-x} - x}$

It is a continuous function because it is the difference of <u>two</u> continuous functions.

In particular it is continuous for $x \in [0,1]$.

Notice that $f(0) = e^{-0} - 0 = e^{0} = \underline{\underline{1}}$

$f(1) = e^{-1} - 1 = \frac{1}{e} - 1 = -0.6321$

Thus there must be a value c in $(0,1)$ such that $f(c) = 0$ ∴ i.e. $e^{-c} - c = 0$

c is a root of our equation (by the $\underline{\underline{\text{IVT}}}$)

Example 6: (Online Homework HW09, #9)

Determine if the Intermediate Value Theorem implies that the equation $x^3 - 3x - 3.9 = 0$ has a root in the interval $[0, 1]$.

Determine if the Intermediate Value Theorem implies that the equation $x^3 - 3x + 1.2 = 0$ has a root in the interval $[0, 1]$.

(a) Consider $f(x) = x^3 - 3x - 3.9$ on $[0, 1]$

The function f is continuous on $[0, 1]$ because it is a polynomial.

$f(0) = -3.9$ $\qquad$ $f(1) = 1 - 3 - 3.9 = -5.9$

Hence the IVT does not apply. We cannot conclude whether there is a root of $x^3 - 3x - 3.9$ on $[0, 1]$

(b) Consider $g(x) = x^3 - 3x + 1.2$ on $[0, 1]$

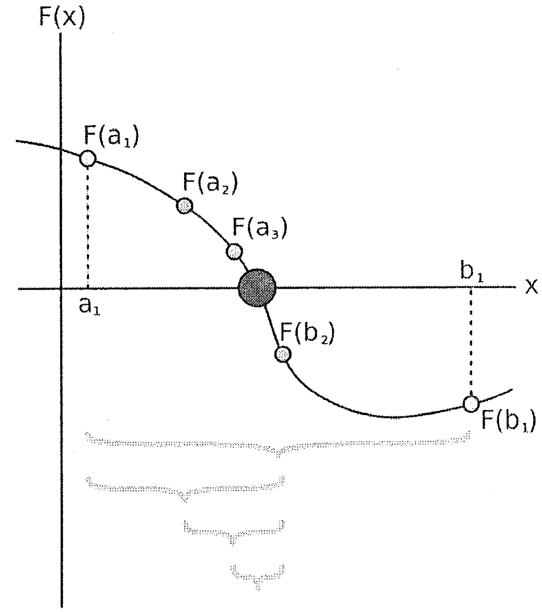
g is a continuous function: $g(0) = 1.2$

and $g(1) = 1 - 3 + 1.2 = -0.8$

Hence, by the IVT, there is a $c \in (0, 1)$ such that $g(c) = 0$ i.e. $\boxed{c^3 - 3c + 1.2 = 0}$

Bisection Method

The bisection method is used for numerically finding a root of the equation $f(x) = 0$, where f is a continuous function defined on an interval $[a, b]$ and where $f(a)$ and $f(b)$ have opposite signs.



In this method one repeatedly bisects an interval and then selects a subinterval in which a root must lie for further inspection.

At each step the method divides the interval in two by computing the midpoint $c = (a + b)/2$ of the interval and the value of the function $f(c)$ at that point.

Unless c is itself a root (which is very unlikely, but possible) there are only two possibilities: either $f(a)$ and $f(c)$ have opposite signs and $[a, c]$ contains a root, or $f(c)$ and $f(b)$ have opposite signs and $[c, b]$ contains a root.

In this way an interval that contains a zero of f is reduced in width by 50% at each step. The process is repeated until the interval is sufficiently small.

Example 7: (Online Homework HW09, #10)

Carry out three steps of the Bisection Method for $f(x) = 2^x - x^4$ as follows:

(a) Show that $f(x)$ has a zero in $[1, 2]$.

(b) Determine which subinterval, $[1, 1.5]$ or $[1.5, 2]$, contains a zero.

(c) Determine which interval, $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, or $[1.75, 2]$, contains a zero.

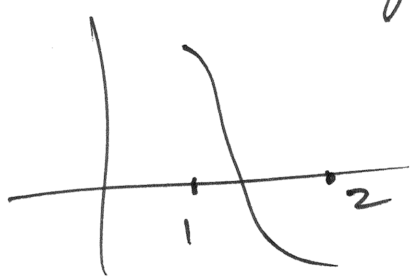
$f(x) = 2^x - x^4$

is a continuous function for all values of x as it is the difference of two continuous functions.

In particular it is continuous on $[1, 2]$.

Notice $\underline{\underline{f(1) = 1}}$ and $f(2) = 2^2 - 2^4 = \underline{\underline{-12}}$

Thus the graph of f must cross the x-axis at some point $c \in (1, 2)$



Consider the midpoint $x = 1.5$

$f(1.5) = 2^{1.5} - (1.5)^4 = -2.23$

Hence the root c lies in the interval $(1, 1.5)$

Let's keep computing middle points.

The next is $\frac{1+1.5}{2} = 1.25$

Now, $f(1.25) = 2^{1.25} - (1.25)^4 = -0.0629$

Hence the root $\boxed{c \in (1, 1.25)}$

WE KEEP GOING

The next midpoint is $\frac{1+1.25}{2} = 1.125$

and $f(1.125) = 0.5792$

Thus $c \in (1.125, 1.25)$.

Now, the next mid point is $\frac{1.125+1.25}{2} = \underline{\underline{1.1875}}$

and $f(1.1875) = 0.2890$ So $c \in (1.1875, 1.25)$

<u>ETC</u>

MA 137 – Calculus 1 with Life Science Applications
The Sandwich Theorem and Some Trigonometric Limits
(Section 3.4)

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The Sandwich (Squeeze) Theorem

Suppose we want to calculate $\lim_{x \to \infty} e^{-x} \cos(10x)$.

We soon realize that none of the rules we have learned so far apply. Although $\lim_{x \to \infty} e^{-x} = 0$, we find that $\lim_{x \to \infty} \cos(10x)$ does not exist as the function $\cos(10x)$ oscillates between -1 and 1.

We need to employ some other techniques. One of these techniques is to use the Squeeze (Sandwich) Theorem.

Sandwich (Squeeze) Theorem

Consider three functions $f(x)$, $g(x)$ and $h(x)$ and suppose for all x in an open interval that contains c (except possibly at c) we have

$$f(x) \leq g(x) \leq h(x).$$

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ then $\lim_{x \to c} g(x) = L$.

From the inequality

$$-1 \leq \cos(10x) \leq 1$$

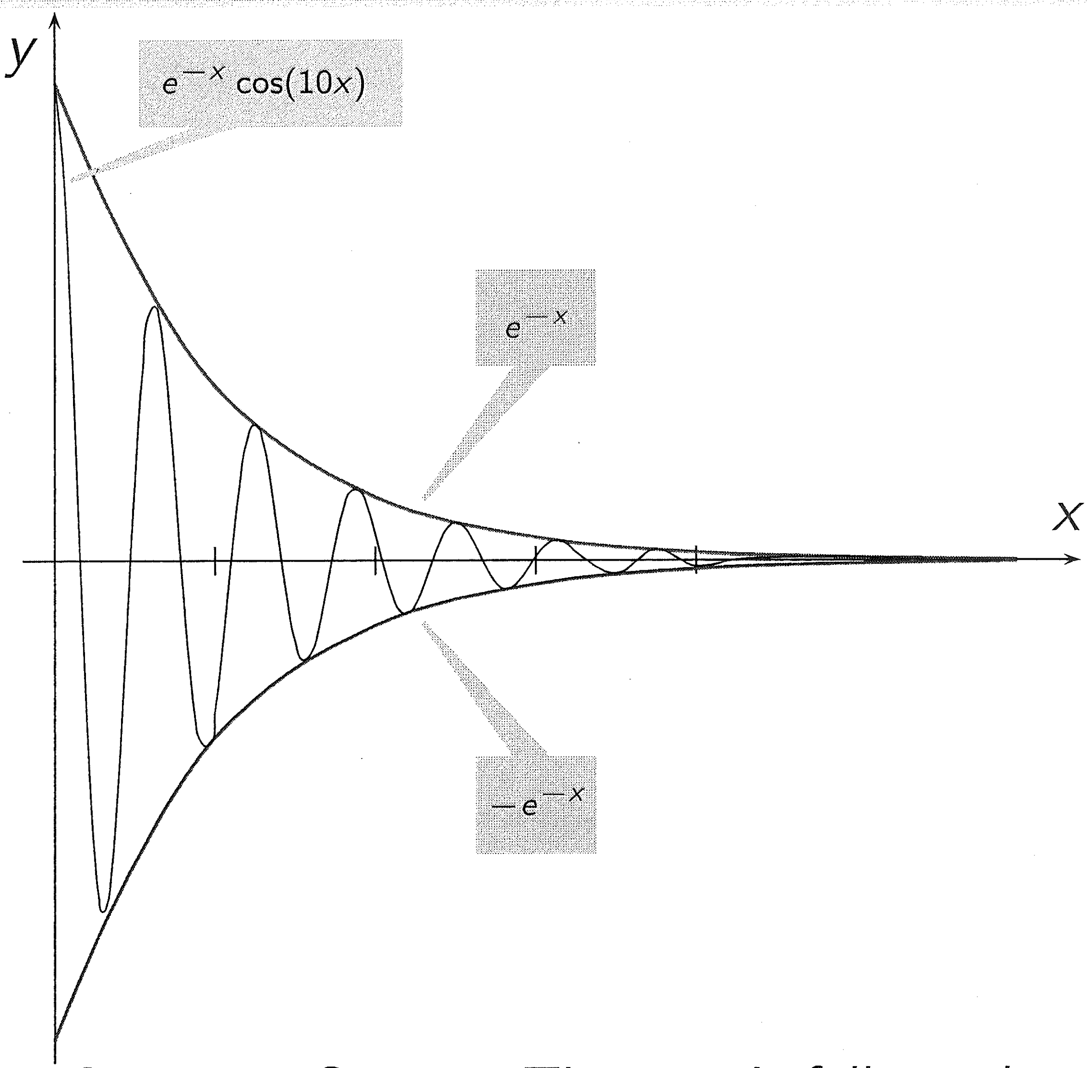
it follows that (as $e^{-x} > 0$, always)

$$-e^{-x} \leq e^{-x}\cos(10x) \leq e^{-x}$$

Then, since

$$\lim_{x\to\infty}(-e^{-x}) = 0 = \lim_{x\to\infty} e^{-x}$$

our function $g(x) = e^{-x}\cos(10x)$ is squeezed between the functions $f(x) = -e^{-x}$ and $h(x) = e^{-x}$, which both go to 0 as x tends to infinity.



So by the Squeeze Theorem it follows that

$$\lim_{x\to\infty} e^{-x}\cos(10x) = 0.$$

Example 1: (Online Homework HW10, # 2)

Suppose $$-8x - 22 \leq f(x) \leq x^2 - 2x - 13.$$

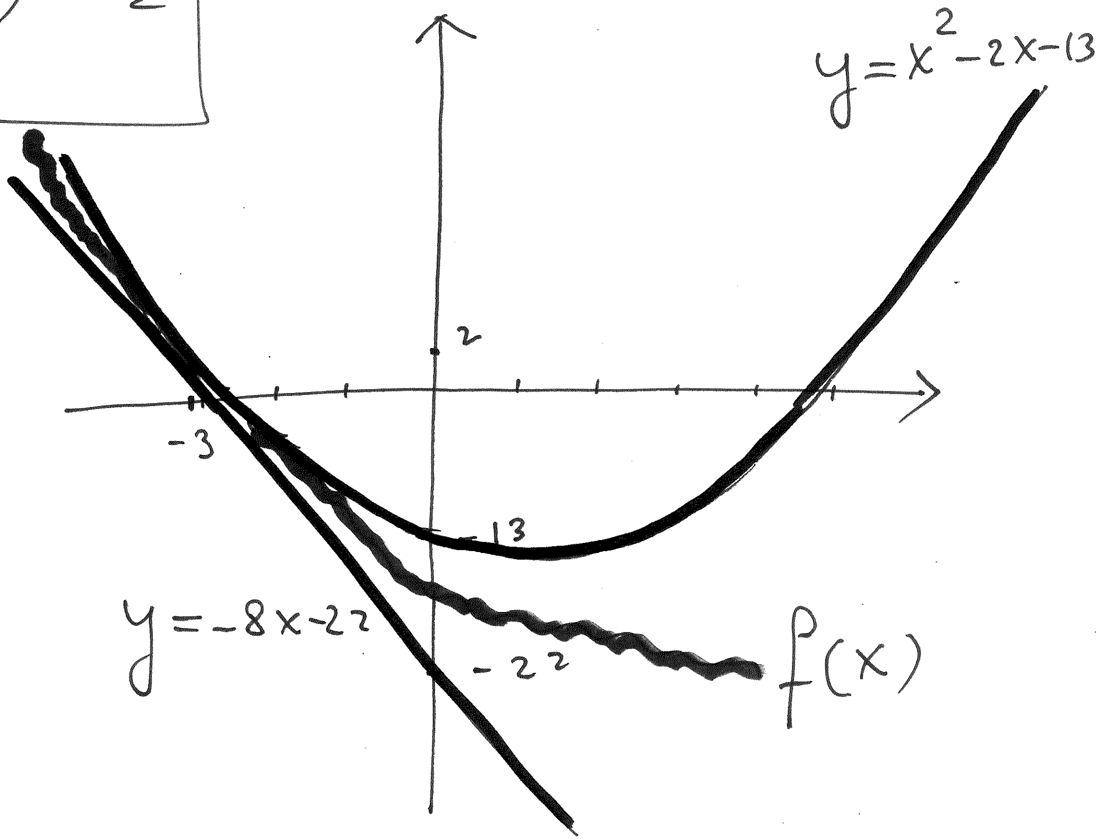
Use this to compute $\lim\limits_{x \to -3} f(x)$.

Note that $\lim_{x \to -3} (-8x - 22) \overset{\text{substitution theorem}}{=} -8(-3) - 22 = \boxed{2}$

Moreover $\lim_{x \to -3} (x^2 - 2x - 13) = (-3)^2 - 2(-3) - 13 = \boxed{2}$

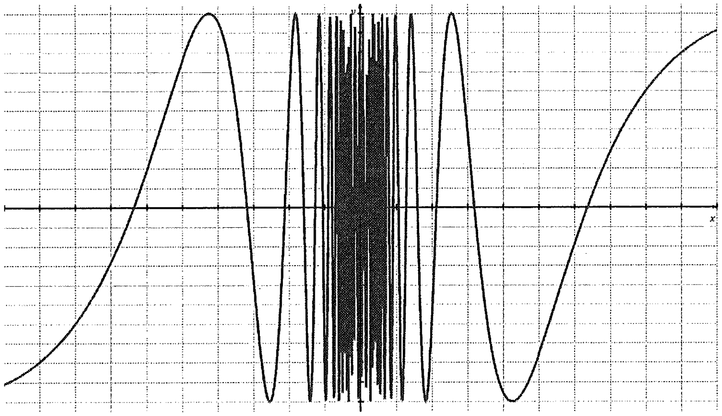
Hence by the sandwich (Squeeze) Theorem

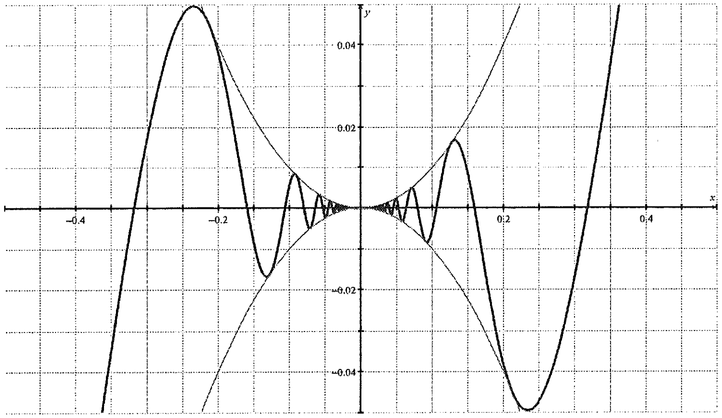
$$\boxed{\lim_{x \to -3} f(x) = 2}$$



Example 2: (Neuhauser, Example # 1, p. 114)

Find $\displaystyle\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.





Note that for all x: $\quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$.

Multiply everywhere by $x^2 \geq 0$ and the inequalities do not change:

$$\underline{\underline{-x^2}} \leq x^2 \sin\left(\frac{1}{x}\right) \leq \underline{\underline{x^2}}$$

Now $\lim\limits_{x \to 0} -x^2 = 0 = \lim\limits_{x \to 0} x^2$

Hence by the Sandwich Theorem we conclude

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

See the pictures on previous page!

Fundamental Trigonometric Limits

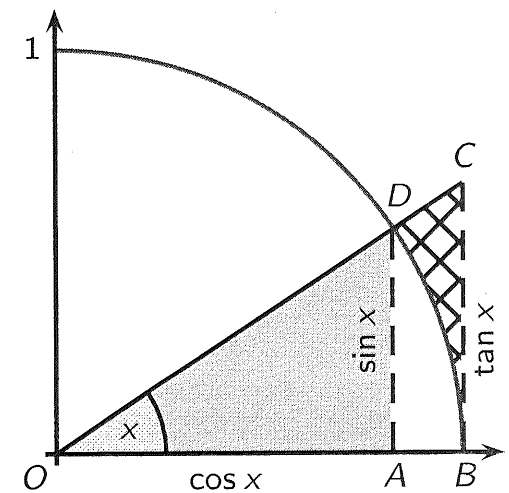
The following two trigonometric limits are important for developing the differential calculus for trigonometric functions:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- Note that the angle x is measured in radians.
- We will prove both statements.
- The proof of the first statement uses a nice geometric argument and the sandwich theorem.
- The second statement follows from the first.

Proof that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

- Since we are interested in the limit as $x \to 0$, we can restrict the values of x to values close to 0.
- We split the proof into two cases, one in which $0 < x < \pi/2$, the other in which $-\pi/2 < x < 0$.
- Since $f(x) = \sin x/x$ is an even function (indeed, it is the quotient of two odd functions!) we only need to study the case $0 < x < \pi/2$.



In this case, both x and $\sin x$ are positive.

We draw the unit circle together with the triangles OAD and OBC. The angle x is measured in radians. Since $\overline{OB} = 1$, we find that

$$\text{arc length of } BD = x \qquad \overline{OA} = \cos x \qquad \overline{AD} = \sin x \qquad \overline{BC} = \tan x.$$

Furthermore the picture illustrates that

$$\text{area of } OAD \le \text{area of sector } OBD \le \text{area of } OBC$$

The area of a sector of central angle x (in radians) and radius r is $\frac{1}{2}r^2x$.

Therefore, $\frac{1}{2}\cos x \cdot \sin x \leq \frac{1}{2} \cdot 1^2 \cdot x \leq \frac{1}{2} \cdot 1 \cdot \tan x.$

Dividing this pair of inequalities by $1/2 \sin x$ yields

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.$$

Solving now for $\sin x/x$ we obtain

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}.$$

We can now take the limit as $x \to 0^+$ and find that

$$\lim_{x\to 0^+} \cos x = 1 \qquad \lim_{x\to 0^+} \frac{1}{\cos x} = 1.$$

Finally the Sandwich Theorem yields $\lim_{x\to 0^+} \frac{\sin x}{x} = 0.$

By symmetry we also have that $\lim_{x\to 0^-} \frac{\sin x}{x} = 0.$

Proof that $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Multiplying both numerator and denominator of $f(x) = (1 - \cos x)/x$ by $1 + \cos x$, we can reduce the second statement to the first:

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \\
&= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1 + \cos x} \\
&= 1 \cdot 0 = 0
\end{aligned}$$

Example 3: (Online Homework HW10, # 7)

Evaluate $\displaystyle \lim_{\theta \to 0} \frac{\sin(4\theta)\sin(8\theta)}{\theta^2}$.

Using a different letter we have

$$\lim_{u \to 0} \frac{\sin(u)}{u} = 1$$

Hence we can rewrite our limit as:

$$\lim_{\theta \to 0} \frac{\sin(4\theta)\sin(8\theta)}{\theta^2} = \lim_{\theta \to 0} \left[\frac{\sin(4\theta)}{\theta}\right]\left[\frac{\sin(8\theta)}{\theta}\right]$$

We need to have both terms of the form $\frac{\sin(u)}{u}$

Hence we adjust as follows:

$$\lim_{\theta \to 0} \left[\frac{\sin(4\theta)}{4\theta}\right]\left[\frac{\sin(8\theta)}{8\theta}\right] \cdot 32 =$$

$$= 32 \left[\lim_{\underbrace{\theta \to 0}_{\text{also } 4\theta \to 0}} \frac{\sin 4\theta}{4\theta}\right] \cdot \left[\lim_{\underbrace{\theta \to 0}_{\text{also } 8\theta \to 0}} \frac{\sin(8\theta)}{8\theta}\right] = 32 \cdot 1 \cdot 1$$

$$\boxed{= 32}$$

Example 4: (Online Homework HW10, # 10)

Evaluate $\lim_{x \to 0} \dfrac{\tan(5x)}{\tan(6x)}$.

$$\lim_{x \to 0} \frac{\tan(5x)}{\tan(6x)} = \lim_{x \to 0} \frac{\sin(5x)}{\cos(5x)} \cdot \frac{1}{\frac{\sin(6x)}{\cos(6x)}} =$$

$$= \lim_{x \to 0} \frac{\sin(5x)}{\sin(6x)} \cdot \frac{\cos(6x)}{\cos(5x)} =$$ direct substitution would give $\frac{0}{0} \cdot \frac{1}{1} = \frac{0}{0}$

But divide top and bottom by x:

$$= \lim_{x \to 0} \left\{ \left[\frac{\sin(5x)}{x} \cdot \frac{x}{\sin(6x)} \right] \cdot \frac{\cos(6x)}{\cos(5x)} \right\}$$

= as before we re-adjust the coefficients:

$$= \lim_{x \to 0} \left\{ \left[\frac{\sin(5x)}{5x} \cdot \frac{6x}{\sin(6x)} \right] \cdot \frac{5}{6} \cdot \frac{\cos(6x)}{\cos(5x)} \right\} =$$

$$= \frac{5}{6} \cdot \left[\underbrace{\lim_{x \to 0} \frac{\sin(5x)}{5x}}_{1} \right] \left[\underbrace{\lim_{x \to 0} \frac{6x}{\sin(6x)}}_{1} \right] \cdot \underbrace{\lim_{x \to 0} \frac{\cos(6x)}{\cos(5x)}}_{1} = \boxed{\frac{5}{6}}$$

Example 5: (Neuhauser, Example 3(c), p. 118)

Evaluate $\displaystyle \lim_{x\to 0} \frac{\sec x - 1}{x \sec x}$.

$$\lim_{x\to 0} \frac{\sec x - 1}{x \sec x} = \lim_{x\to 0} \frac{\frac{1}{\cos x} - 1}{x \cdot \frac{1}{\cos x}} =$$

$$= \lim_{x\to 0} \frac{\frac{1-\cos x}{\cos x}}{\frac{x}{\cos x}} = \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

by the fundamental limit we showed earlier.

Example 6: (Online Homework HW10, # 13)

Evaluate $\displaystyle\lim_{x\to\pi/4} \frac{3(\sin x - \cos x)}{5\cos(2x)}$.

$$\lim_{x\to \pi/4} \frac{3(\sin x - \cos x)}{5\cos(2x)} = 3\left[\frac{\sin(\pi/4) - \cos(\pi/4)}{5\cos(2\cdot\frac{\pi}{4})}\right]$$

↑ substitution Theorem since trig functions are continuous

$$= \frac{3\left[\sqrt{2}/2 - \sqrt{2}/2\right]}{5\cos(\pi/2)} = \frac{0}{0}$$

What do we do? Try double angle formula

$$\lim_{x\to \pi/4} \frac{3(\sin x - \cos x)}{5\left[\cos^2 x - \sin^2 x\right]} = \lim_{x\to \frac{\pi}{4}} \frac{3(\sin x - \cos x)}{5\left[\cos x - \sin x\right]\cdot\left[\cos x + \sin x\right]}$$

can factor it!

Simplify (careful to the sign!)

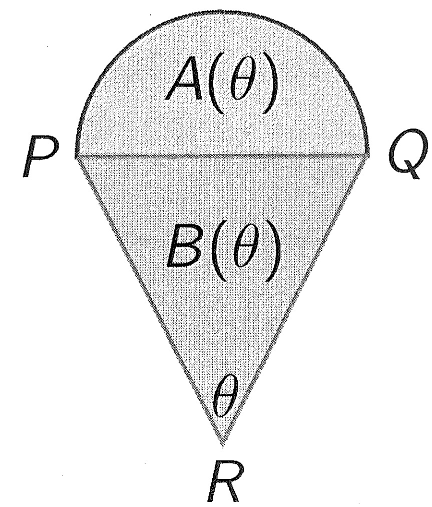
$$= \lim_{x\to \frac{\pi}{4}} \frac{-3}{5\left[\cos x + \sin x\right]} = \frac{-3}{5\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)} = -\frac{3}{5\sqrt{2}}$$

$$\boxed{\cong -0.424264}$$

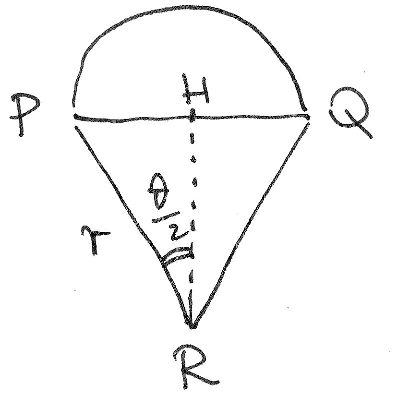
Example 7: (Online Homework HW10, # 14)

A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like an ice cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$



We need the height and base of the triangle and the radius of the circle:



Use right-triangle trigonometry

$\overline{RH} = r\cos(\theta/2)$

$\overline{PH} = r\sin(\theta/2)$

hence $PQ = 2r\sin\left(\frac{\theta}{2}\right)$

Area of triangle:

$$\widehat{PRQ} = B(\theta) = \frac{1}{2}\,\overline{PQ}\cdot\overline{RH} = \frac{1}{2}\left[2r\sin\left(\frac{\theta}{2}\right)\right]\left[r\cos(\theta/2)\right]$$

Area semicircle:

$$A(\theta) = \frac{1}{2}\pi\left(\overline{PH}\right)^2 = \frac{1}{2}\pi r^2\sin^2\left(\frac{\theta}{2}\right)$$

Hence $$\frac{A(\theta)}{B(\theta)} = \frac{\frac{1}{2}\pi \cancel{r^2} \sin^{\cancel{2}}(\theta/2)}{\cancel{r^2} \cancel{\sin(\tfrac{\theta}{2})} \cos(\tfrac{\theta}{2})} =$$

$$= \frac{1}{2}\pi \cdot \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{1}{2}\pi \tan(\theta/2)$$

Hence $$\lim_{\theta \to 0} \frac{A(\theta)}{B(\theta)} = \lim_{\theta \to 0} \frac{1}{2}\pi \tan(\theta/2) = 0$$

as tan is a continuous function and

$\underline{\tan(0) = 0}$.

Trigonometric and Exponential Functions

- We will sometimes use the double angle formulas

$$\begin{aligned} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \end{aligned} \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

which are special cases of the following addition formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

- What about $\sin(\alpha/2)$ and $\cos(\alpha/2)$? With some work

$$\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

(the sign ($+$ or $-$) depends on the quadrant in which $\dfrac{\alpha}{2}$ lies.)

- Is there a 'simple' way of remembering the above formulas?

Euler's Formula

Euler's formula states that, for any real number x,

$$\boxed{e^{ix} = \cos x + i \sin x,}$$

where i is the imaginary unit ($i^2 = -1$).

- For any α and β, using Euler's formula, we have

$$\begin{aligned}
\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} \\
&= e^{i\alpha} \cdot e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\
&= (\cos\alpha\cos\beta + i^2\sin\alpha\sin\beta) \\
&\qquad + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).
\end{aligned}$$

- Thus, by comparing the terms, we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

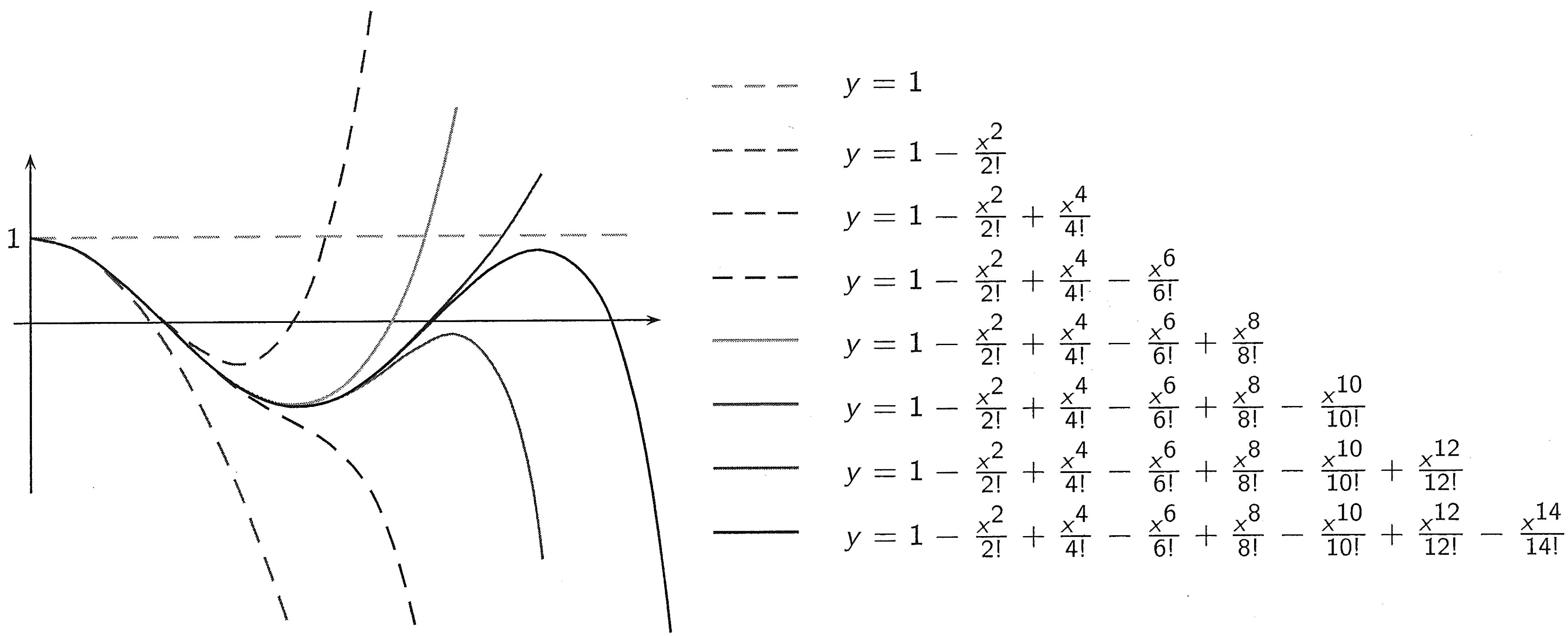
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Approximating $\cos x$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n \frac{x^{2n}}{(2n)!}.$$

As n increases, the graph of $T_{2n}(x)$ appears to approach the one of $\cos x$. This suggests that we can approximate $\cos x$ with $T_{2n}(x)$ as $n \to \infty$.

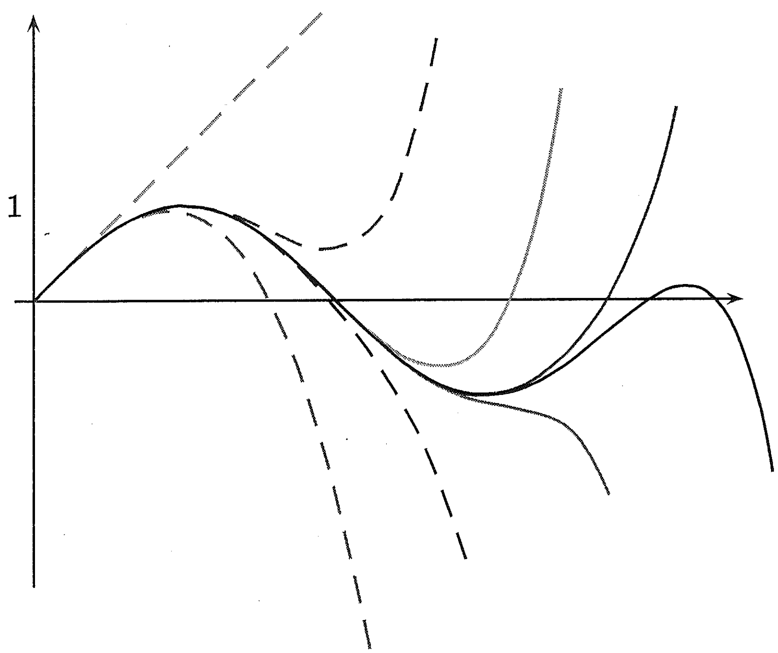


Approximating $\sin x$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

As n increases, the graph of $T_{2n+1}(x)$ appears to approach the one of $\sin x$. This suggests that we can approximate $\sin x$ with $T_{2n+1}(x)$ as $n \to \infty$.



$y = x$

$y = x - \frac{x^3}{3!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$

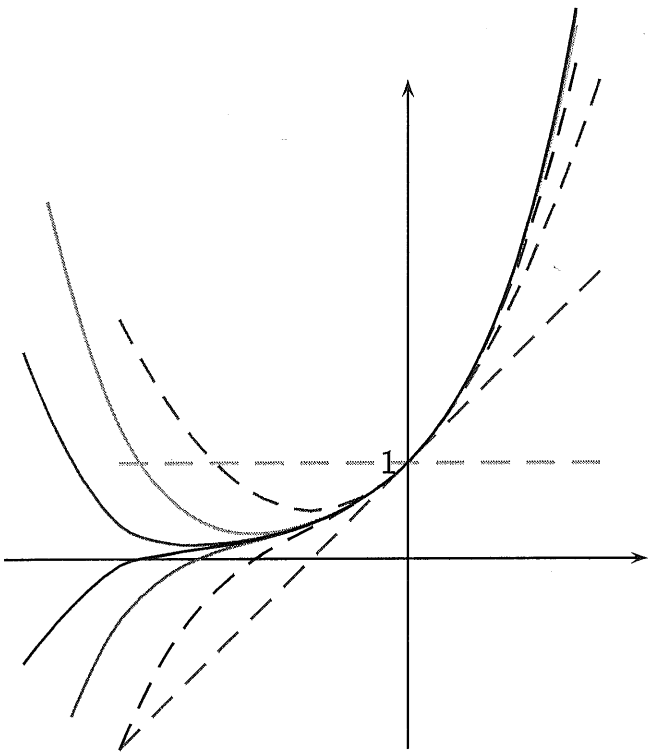
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$

Approximating e^x

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As n increases, the graph of $T_n(x)$ appears to approach the one of e^x. This suggests that we can approximate e^x with $T_n(x)$ as $n \to \infty$.



$y = 1$

$y = 1 + x$

$y = 1 + x + \frac{x^2}{2!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

Idea of Why Euler's Formula Works

To justify Euler's formula, we use the polynomial approximations for e^x, $\cos x$ and $\sin x$ that we just discussed. We start by approximating e^{ix}:

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

Curiosity: From Euler's formula with $x = \pi$ we obtain

$$e^{i\pi} + 1 = 0$$

which involves five interesting math values in one short equation.