MA138 – Calculus 2 with Life Science Applications

Course Introduction & Section 6.3 (Applications of integration)

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January 11 & 13, 2017

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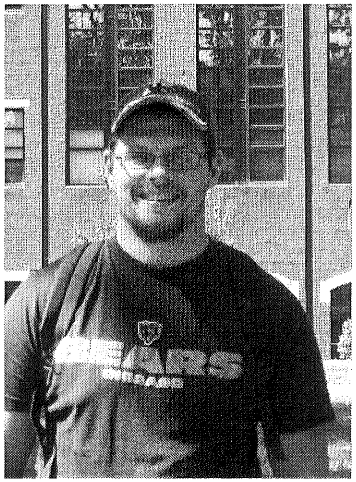
002 TR 11:00-11:50am – CB 341



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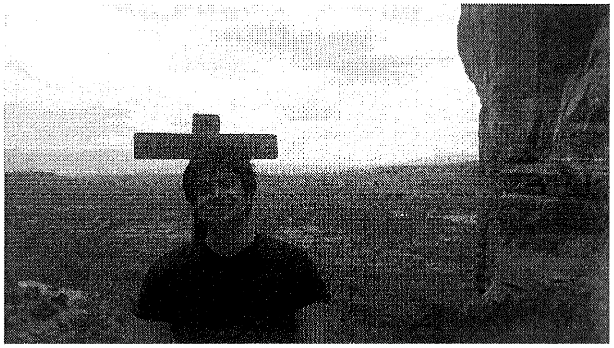
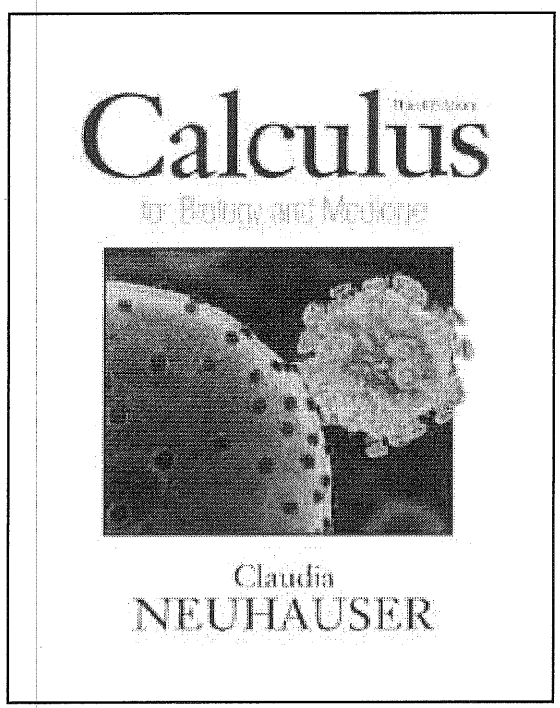


Textbook



Title: Calculus for Biology and Medicine

Author: Claudia Neuhauser

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Course Outline for MA 138

Ch. 6: Applications of integration

Ch. 7: Integration techniques and computational methods

Ch. 8: Differential equations

Ch. 9: Linear algebra and analytic geometry

Ch. 10: Multivariable calculus

Ch. 11: Systems of differential equations

Grading

You will be able to obtain a **maximum of 500 points** in this class, divided as follows:

- Three 2-hour exams, 100 points each;
- Final exam, 100 points;
- Homework, 50 points;
- Weekly quizzes, 50 points.

Your final grade for the course will be based on the total points you have earned as follows:

A: 450-500	B: 400-449	C: 350-399	D: 300-349	E: 0-299
$\geq 90\%$	$\geq 80\%$	$\geq 70\%$	$\geq 60\%$	$< 60\%$

Exams (Regular and Alternate)

Regular Exams will be given on

- Tuesday, February 7 — 5:00-7:00 pm
- Tuesday, March 7 — 5:00-7:00 pm
- Tuesday, April 11 — 5:00-7:00 pm
- Wednesday, May 3 — 8:30-10:30 pm

Alternate Exams for Exams 1-3 are given on the same days as the regular exams from 7:30-9:30 pm (January 7, March 7, April 11).

Review Sessions for exams 1-3 will be held on Monday February 6, March 6 and April 10 from 6:00-8:00 pm.

Homework

- The homework has two components: an online and handwritten homework component. Each will count as half of the final homework grade. The online problems cover the more routine aspects of the class. The written homework problems are usually more conceptual and are often motivated by problems from the Life Sciences.
- The online homework (WeBWorK) can be accessed through

  https://webwork.as.uky.edu/webwork2/MA138S17/
- Your username is your **Link Blue user ID** (use capital letters!) and your password is **your <u>8 digit</u> student ID number**.
- You can try online problems as many times as you like.
  The system will tell you if your answer is correct or not.
  You can email the TA a question from each of the problem. TAs will always do their best to respond within 24 hours.
- **Don't wait until the last minute!**

¿Minoring in Mathematics?

To obtain a **minor in Mathematics**, a student who has completed MA 137/138 Calculus I and II must complete the following:

1. MA 213 – Calculus III (4 credits)
2. MA 322 – Matrix Algebra and Its Applications (3 credits)
3. Six additional credit hours of Mathematics courses (=two courses) numbered greater than 213. Possible courses include: MA 214, MA 261, MA 320, MA 321, **MA 327 (Introduction to game theory)**, MA 330, MA 341, MA 351, MA 361, or any 400 level math course
4. We are also in the process of establishing a new cross-listed course by Fall 2017 at the upper level in Mathematics.
   MA 337/BIO 337: Mathematical Modeling in the Life Sciences

Thus you need 13 additional credit hours in Mathematics classes.

Section 6.3: Applications of Integration

We are interested in the following three applications of integrals:

(1) **average** of a continuous function on $[a, b]$;

(2) **area between curves**;

(3) **cumulative change**.

Average Values

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{avg}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

In general, let's try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $c_1, \ldots, c_n$ in successive subintervals and calculate the average of the numbers $f(c_1), \ldots, f(c_n)$:

$$\frac{f(c_1) + \cdots + f(c_n)}{n}$$

Since $\Delta x = (b-a)/n$, we can write $1/n = \Delta x/(b-a)$ and the average value becomes

$$\frac{f(c_1)\Delta x + \cdots + f(c_n)\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x.$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. More precisely,

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Average of a Continuous Function on $[a, b]$

Assume that $f(x)$ is a continuous function on $[a, b]$. The average value of f on the interval $[a, b]$ is defined to be

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

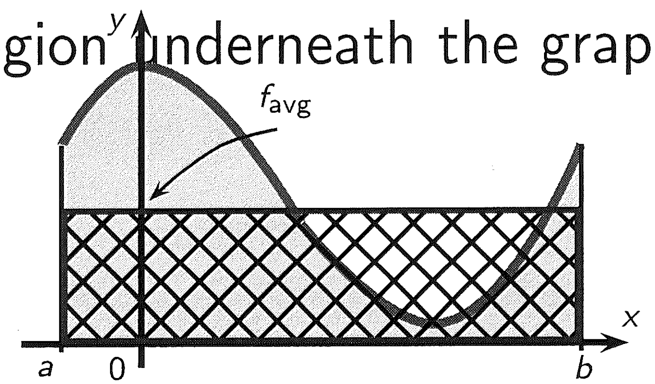
Geometric Meaning

Mean Value Theorem for Definite Integrals

Assume that $f(x)$ is a continuous function on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

That is, when f is continuous, there exists a number c such that $f(c) = f_{\text{avg}}$. If f is a continuous, positive valued function, f_{avg} is that number such that the rectangle with base $[a, b]$ and height f_{avg} has the same area as the region underneath the graph of f from a to b.



Example 1 (Online Homework #14)

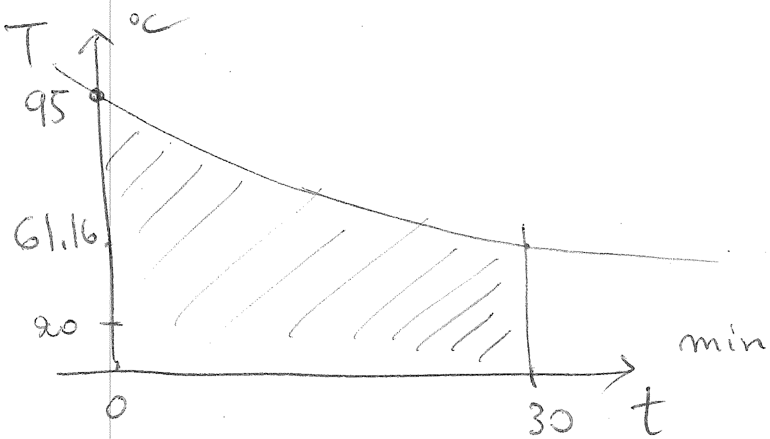
If a cup of coffee has temperature 95°C in a room where the temperature is 20°C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-t/50}.$$

What is the average temperature (in degrees Celsius) of the coffee during the first half hour?

Notice that the graph of $T(t) = 20 + 75e^{-t/50}$

Looks like:



We want to compute

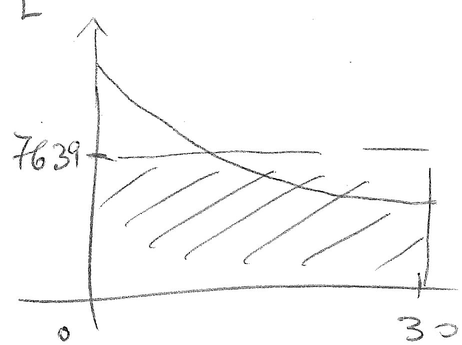
$$T_{avg} = \frac{1}{30-0} \cdot \int_0^{30} \left(20 + 75e^{-t/50}\right) dt$$

$$= \frac{1}{30}\left[20t + 75e^{-t/50} \cdot (-50)\right]_0^{30} =$$

$$= \frac{1}{30}\left[\left(20 \cdot 30 - 3750e^{-30/50}\right) - \left(0 - 3750\right)\right]$$

$$= \frac{1}{30}\left[600 - 3750\left(e^{-3/5} - 1\right)\right] = \frac{1}{30}\left[4350 - 3750e^{-3/5}\right]$$

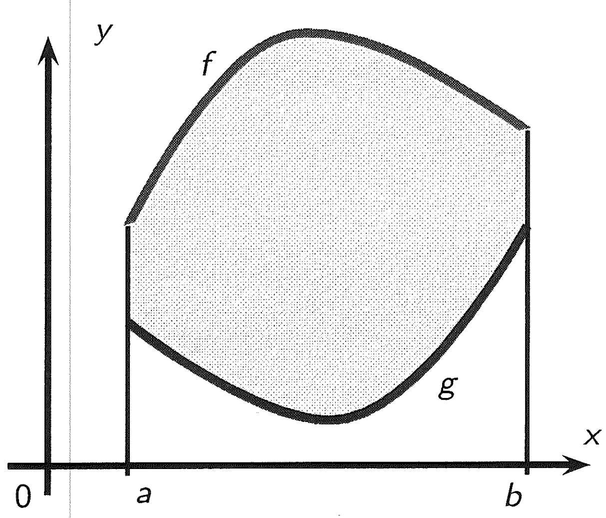
$$= 145 - 125e^{-3/5} \approx \boxed{76.39 \ °C}$$



Area Between Curves

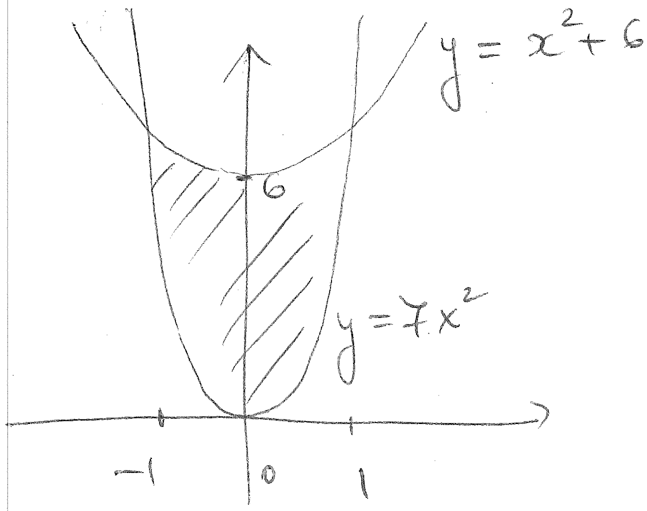
Assume f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the linesx $= a$, $x = b$, is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



Example 2 (Online Homework #2)

Find the area of the region enclosed by the two functions $y = 7x^2$ and $y = x^2 + 6$.



first of all we need to find the intersection points of

$$y = 7x^2 \quad \text{and} \quad y = x^2 + 6$$

$$x^2 + 6 = 7x^2 \iff$$

$$6x^2 = 6 \iff x^2 = 1$$

$$\iff x = \pm 1$$

Thus the area we seek is

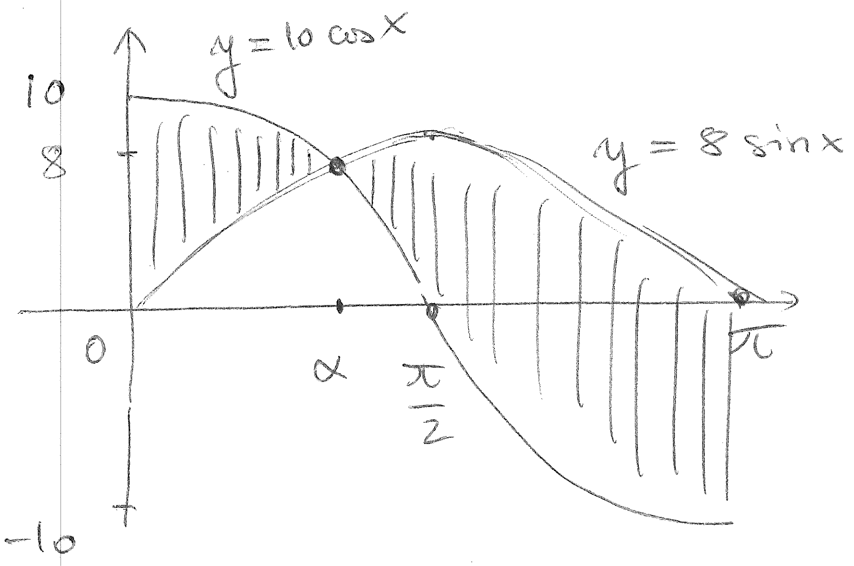
$$\int_{-1}^{1} \left[(x^2 + 6) - (7x^2)\right] dx = \int_{-1}^{1} (6 - 6x^2)\, dx =$$

$$= \text{by symmetry} = 2\int_0^1 6(1 - x^2)\, dx = \left[2(6x - 2x^3)\right]_0^1$$

$$= \left[(12 - 4) - (0)\right] = \boxed{8}$$

Example 3 (Online Homework #3)

Find the area between $y = 8\sin x$ and $y = 10\cos x$ over the interval $[0, \pi]$. Sketch the curves if necessary.



α is the angle such that

$$10\cos\alpha = y = 8\sin\alpha$$

OR $\quad \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{10}{8} = \boxed{\dfrac{5}{4}}$

Thus the area we want is:

$$\int_0^\alpha (10\cos x - 8\sin x) + \int_\alpha^\pi (8\sin x - 10\cos x)\,dx =$$

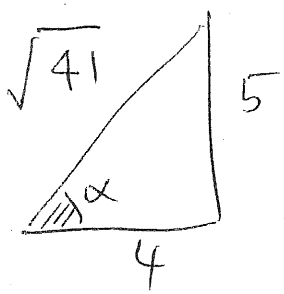
$$= \Big[10\sin x + 8\cos x\Big]_0^\alpha + \Big[-8\cos x - 10\sin x\Big]_\alpha^\pi =$$

$$= \left(10\sin\alpha + 8\cos\alpha \cancel{-8}\right) + \left(\cancel{8} + 8\cos\alpha + 10\sin\alpha\right)$$

$$= 20\sin\alpha + 16\cos\alpha$$

$= 20 \sin\alpha + 16 \cos\alpha$

Now $\tan\alpha = \frac{10}{8} = 5/4$ so



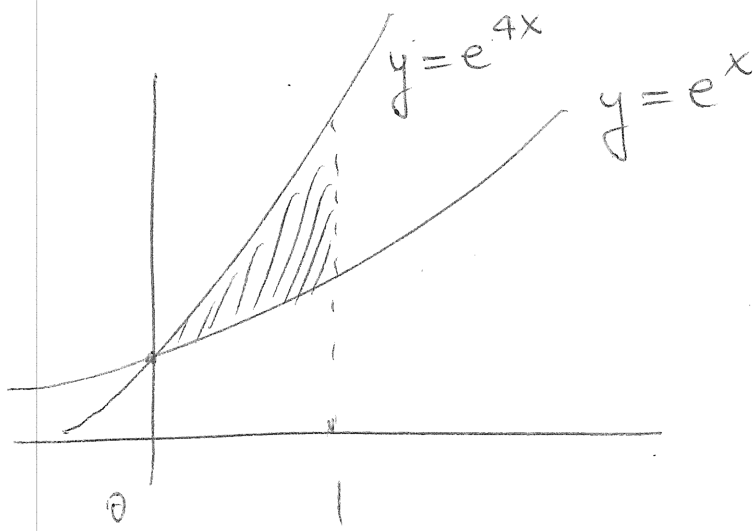
$\cos\alpha = \frac{4}{\sqrt{41}}$ $\sin\alpha = \frac{5}{\sqrt{41}}$

$\therefore$ Area $= 20 \cdot \frac{5}{\sqrt{41}} + 16 \cdot \frac{4}{\sqrt{41}} = \frac{164}{\sqrt{41}} = \frac{4 \cdot 41}{\sqrt{41}}$

$= \boxed{4\sqrt{41}} \simeq 25.6125$

Example 4 (Online Homework #4)

Find the area between $y = e^x$ and $y = e^{4x}$ over $[0, 1]$.



$$\text{Area} = \int_0^1 (e^{4x} - e^x)\,dx = \frac{1}{4}e^{4x} - e^x \Big]_0^1 =$$

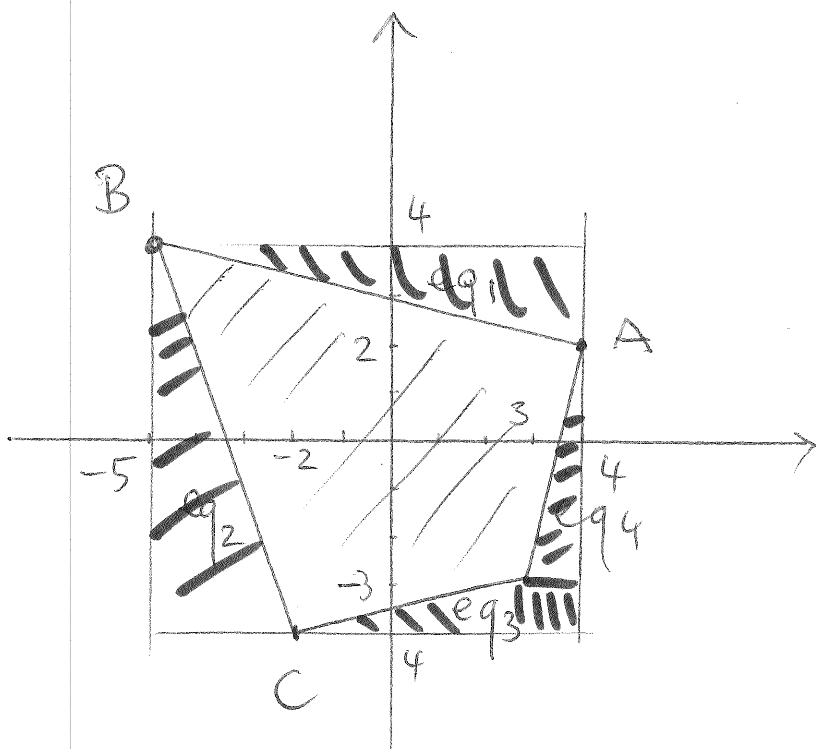
$$= \left[\frac{1}{4}e^4 - e\right] - \left[\frac{1}{4}e^0 - e^0\right]$$

$$= \frac{1}{4}e^4 - e - \left(\frac{1}{4} - 1\right) = \boxed{\frac{1}{4}e^4 - e + \frac{3}{4}}$$

$$= \frac{e^4 - 4e + 3}{4} \cong \underline{11.6813}$$

Example 5 (Online Homework #6)

Find the area of the quadrangle with vertices $(4, 2)$, $(-5, 4)$, $(-2, -4)$, and $(3, -3)$.



A (4,2)

B (-5,4)

C (-2,-4)

D (3,-3)

one can certainly compute the equations of the 4 lines and do:
$$\int_{-5}^{-2} (eq_1 - eq_2)\,dx + \int_{-2}^{3} (ep_1 - ep_3)\,dx + \dots$$

OR $8 \cdot 9 = 72$ = area of the big rectangle

– minus the area of the 4 triangles and one square

i.e. $$72 - \left(\frac{8 \cdot 3}{2} + \frac{5 \cdot 1}{2} + \frac{5 \cdot 1}{2} + \frac{9 \cdot 2}{2} + 1 \right)$$

$$= 72 - 12 - 5 - 9 - 1 = 72 - 27 = \boxed{45}$$

Example 6 (Online Homework #7)

Consider the area between the graphs $x + y = 14$ and $x + 6 = y^2$.

This area can be computed in two different ways using integrals.

- First of all it can be computed as a sum of two integrals

$$\int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$

where $a =$ ____, $b =$ ____, $c =$ ____, and $f(x) =$ ____ $g(x) =$ ____.

- Alternatively this area can be computed as a single integral

$$\int_\alpha^\beta h(y)\,dy$$

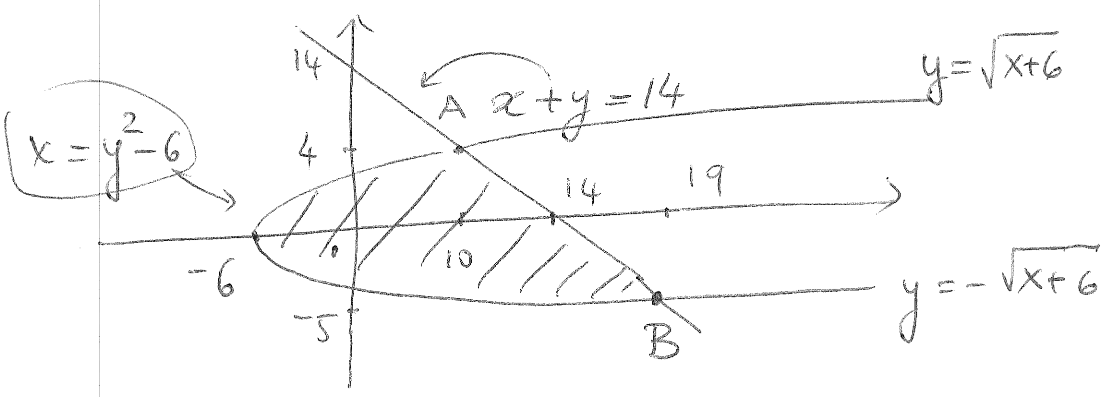
where $\alpha =$ ____, $\beta =$ ____, and $h(y) =$ ____.

The intersection points of $\boxed{x+y=14}$ and $\boxed{x+6=y^2}$ are given by

$$\begin{cases} x+y=14 \\ x+6=y^2 \end{cases} \iff \underline{14-y} = x = \underline{y^2-6} \iff y^2+y-20=0$$

$$\iff (y+5)(y-4)=0 \quad \therefore y = 4, -5 \quad \text{hence} \quad x = 10, 19$$

$\therefore$ $A(10, 4)$ $B(19, -5)$. The graph is:



We can compute the area in 2 ways:

(1)
$$\int_{-6}^{10}\left[\sqrt{x+6}-\left(-\sqrt{x+6}\right)\right]dx + \int_{10}^{19}\left[(14-x)-\left(-\sqrt{x+6}\right)\right]dx$$

$$= \int_{-6}^{10} 2\sqrt{x+6}\,dx + \int_{10}^{19}\left(14-x+\sqrt{x+6}\right)dx =$$

$$= \left[\tfrac{4}{3}(x+6)^{3/2}\right]_{-6}^{10} + \left[14x - \tfrac{1}{2}x^2 + \tfrac{2}{3}(x+6)^{3/2}\right]_{10}^{19} =$$

$$= \left[\frac{4}{3} \cdot 64 - 0 \right] + \left[\left(14 \cdot 19 - \frac{19^2}{2} + \frac{2}{3} \cdot 125 \right) - \left(140 - 50 + \frac{2}{3} \cdot 64 \right) \right]$$

$$= \frac{2}{3} \cdot 64 + 266 - \frac{361}{2} + 125 \frac{2}{3} - 90$$

$$= \frac{2}{\cancel{3}} \left(\cancel{189}^{63} \right) + 266 - 180 - \tfrac{1}{2} - 90 = 126 + 266 - 270 - \tfrac{1}{2}$$

$$= 392 - 270 - \tfrac{1}{2} = 122 - \tfrac{1}{2} = \underline{\underline{121.5}}$$

<u>Second</u> way:

$$\int_{-5}^{4} \left[(14 - y) - (y^2 - 6) \right] dy = \int_{-5}^{4} (20 - y - y^2)\, dy =$$

$$= \left[20y - \frac{1}{2} y^2 - \frac{1}{3} y^3 \right]_{-5}^{4} = \left[\left(20 \cdot 4 - \frac{1}{2}(4)^2 - \frac{1}{3}(4)^3 \right) - \right.$$

$$\left. - \left(20(-5) - \frac{1}{2}(-5)^2 - \frac{1}{3}(-5)^3 \right) \right] = \left[\left(80 - 8 - \frac{64}{3} \right) - \left(-100 - \frac{25}{2} + \frac{125}{3} \right) \right]$$

$$= 80 - 8 + 100 - \frac{64}{3} + \frac{25}{2} - \frac{125}{3} = 172 - \frac{\cancel{189}^{63}}{\cancel{3}} + \frac{25}{2} = 109 + 12.5 = \underline{\underline{121.5}}$$

Example 7 (Online Homework #5)

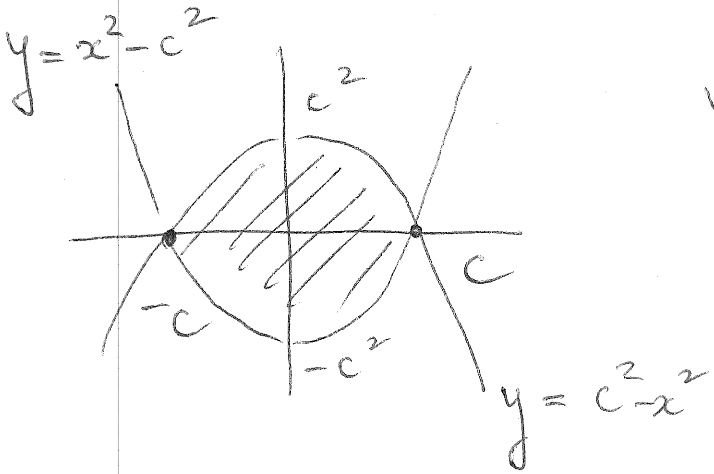
Find the value(s) of c such that the area of the region bounded by the parabolae $y = x^2 - c^2$ and $y = c^2 - x^2$ is 1944.

$y = x^2 - c^2$ and $y = c^2 - x^2$

intersect at $x^2 - c^2 = c^2 - x^2$

$\iff 2x^2 = 2c^2 \iff x^2 = c^2$

OR $x = \pm c$. Their graph is



We want

$$\boxed{\int_{-c}^{c} \left[(c^2 - x^2) - (x^2 - c^2)\right] dx = 1944}$$

$$\int_{-c}^{c} (2c^2 - 2x^2)\, dx = 1944$$

by symmetry

$$2\int_{0}^{c} (2c^2 - 2x^2)\, dx = 1944$$

$$\iff \int_{0}^{c} (c^2 - x^2)\, dx = \frac{1944}{4} = 486$$

$$\therefore \int_0^c (c^2 - x^2)\,dx = 486$$

$$\left. c^2x - \frac{1}{3}x^3 \right]_0^c = 486$$

$$\Longleftrightarrow \quad c^3 - \frac{1}{3}c^3 - (0) = 486$$

$$\Longleftrightarrow \quad \frac{2}{3}c^3 = 486$$

$$\Longleftrightarrow \quad c^3 = 3 \cdot 243$$

$$c^3 = 729$$

$$\therefore \boxed{c = 9}$$

$y = c^2 - x^2$; $y = x^2 - c^2$

But notice that $\boxed{c = -9}$ also works!

Cumulative Change

Suppose that we have a population whose size at time t is given by $N(t)$. Suppose further that its rate of growth is given by the initial value problem

IVP: $$\frac{dN}{dt} = f(t) \qquad N(0) = N_0.$$

Then, by Part I of the Fundamental Theorem of Calculus we have that

$$N(t) = \int_0^t f(u)\,du + C$$

represents all antiderivatives of $f(t)$ [or dN/dt].

Now, $N(0) = \underbrace{\int_0^0 f(u)\,du}_{=0} + C = C$ so $C = N_0 = N(0)$. Therefore

$$N(t) = \int_0^t f(u)\,du + N_0 \qquad \text{or} \qquad \boxed{N(t) - N(0) = \int_0^t f(u)\,du}.$$

More generally, the IVP: $\dfrac{dN}{dt} = f(t) \qquad N(a) = N_a$ has solution

$$\boxed{N(t) - N(a) = \int_a^t f(u)\,du = \int_a^t \frac{dN}{du}\,du}.$$

That is

$$\left\{ \begin{array}{c} \text{cumulative change} \\ \text{on the interval } [a,t] \end{array} \right\} = \int_a^t \left\{ \begin{array}{c} \text{instantaneous rate of} \\ \text{change at time } u \end{array} \right\} du$$

Similarly, if $p(t)$ is the position function of an object at time t, then

$$\frac{dp}{dt} = v(t) \qquad p(a) = p_a$$

gives $\rightsquigarrow$ $$\boxed{\underbrace{p(b) - p(a)}_{\text{distance traveled on } [a,b]} = \int_a^b v(t)\,dt = \int_a^b \frac{dp}{dt}\,dt}.$$

Example 8 (Problem #18, Section 6.3, page 321)

Suppose the change in biomass $B(t)$ at time t during the interval $[0, 12]$ follows the equation

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right).$$

How does the biomass at time $t = 12$ compare to the biomass at time $t = 0$?

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right)$$

Thus

$$B(12) - B(0) = \int_0^{12} \frac{dB}{dt}\,dt = \int_0^{12} \cos\left(\frac{\pi}{6}t\right) dt$$

$$= \frac{6}{\pi} \sin\left(\frac{\pi}{6}t\right)\Big]_0^{12}$$

$$= \frac{6}{\pi} \cdot \left(\sin\left(\frac{\pi}{6} \cdot 12\right) - \sin\left(\frac{\pi}{6} \cdot 0\right) \right)$$

$$= \frac{6}{\pi} \cdot \left(\sin(2\pi) - \sin(0) \right) = 0$$

Thus $\boxed{B(12) - B(0) = 0}$ OR $\underline{B(12) = B(0)}$

There is no change in biomass

Example 9 (Problem #22, Section 6.3, page 322)

If $\dfrac{dw}{dx}$ represents the rate of change of the weight of an organism of age x, explain what

$$\int_3^5 \frac{dw}{dx}\,dx$$

means.

$$\int_3^5 \frac{dw}{dx}\, dx = w(5) - w(3)$$

i.e. it represents the change in weight between age 3 and 5