

MA 138 – Calculus 2 with Life Science Applications  
**Solving Differential Equations**  
(Section 8.1)

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# Differential Equations ( $\equiv$ DEs)

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

For example

$$\blacksquare \frac{dy}{dx} + 6y = 7;$$

$$\blacksquare \frac{dy}{dt} + 0.2 t y = 6t;$$

$$\blacksquare \frac{dP}{dt} = \sqrt{P t};$$

$$\blacksquare xy' + y = y^2.$$

Differential equations can contain derivatives of any order; for example,

$$\blacksquare \frac{d^2y}{dx^2} + 6\frac{dy}{dx} = xy \quad \text{or} \quad y'' + 6y' - xy = 0$$

is a DE containing the first and second derivative of the function  $y = y(x)$ .

If a differential equation contains only the first derivative, it is called a **first-order differential equation**:  $\frac{dy}{dx} = h(x, y)$ .

DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

**exponential growth model:** 
$$\frac{dN}{dt} = rN \quad N(0) = N_0;$$

**logistic growth model:** 
$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) \quad N(0) = N_0;$$

**von Bertalanffy models:** 
$$\frac{dL}{dt} = k(L_\infty - L) \quad L(0) = L_0,$$

$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \quad W(0) = W_0;$$

**Solow's economic growth model:** 
$$\frac{dk}{dt} = sk^\alpha - \delta k \quad k(0) = k_0.$$

## Example 1

Consider the differential equation  $(t + 1)\frac{dy}{dt} - y + 6 = 0$ .

Which of the following functions

$$y_1(t) = t + 7 \quad y_2(t) = 3t + 21 \quad y_3(t) = 3t + 9$$

are solutions for all  $t$ ?

## Separable Differential Equations

We will restrict ourselves to first-order differential equations

$$\frac{dy}{dx} = h(x, y) \quad \text{of the form} \quad \frac{dy}{dx} = f(x)g(y).$$

That is, the right-hand side of the equation is the product of two functions, one depending only on  $x$ ,  $f(x)$ , the other only on  $y$ ,  $g(y)$ .

Such equations are called **separable differential equations**.

This type of differential equations includes two special cases:

■ **pure-time differential equations:**  $\frac{dy}{dx} = f(x)$  [i.e.,  $g(y) \equiv 1$ ]

■ **autonomous differential equations:**  $\frac{dy}{dx} = g(y)$  [i.e.,  $f(x) \equiv 1$ ]

(DEs of this form are frequently used in biological models.)

In order to solve the separable differential equation

$$\frac{dy}{dx} = f(x)g(y), \quad (*)$$

we divide both sides of (\*) by  $g(y)$  [assuming that  $g(y) \neq 0$ ]:

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

Now, if  $y = u(x)$  is a solution of (\*), then  $u(x)$  satisfies

$$\frac{1}{g[u(x)]} u'(x) = f(x).$$

If we integrate both sides with respect to  $x$ , we find that

$$\int \frac{1}{g[u(x)]} u'(x) dx = \int f(x) dx \quad \text{or} \quad \int \frac{1}{g(y)} dy = \int f(x) dx$$

since  $g[u(x)] = g(y)$  and  $u'(x)dx = dy$ .

## Example 1 (again)

Solve the differential equation  $(t + 1)\frac{dy}{dt} - y + 6 = 0$ .

## Example 2 (Online Homework # 2)

Solve the following initial value problem

$$\frac{dy}{dt} + 0.2ty = 6t$$

with  $y(0) = 4$ .



### Example 3 (Online Homework # 3)

Find the solution of the differential equation

$$\frac{dP}{dt} = \sqrt{Pt}$$

that satisfies the initial condition  $P(1) = 7$ .

## Example 4 (Online Homework # 5)

Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies the initial condition  $y(1) = -1$ .

## Pure-Time Differential Equations

In many applications, the independent variable represents time. If the rate of change of a function depends only on time, we call the resulting differential equation a **pure-time differential equation**. Such a differential equation is of the form

$$\frac{dy}{dx} = f(x), \quad x \in I, \quad y(x_0) = y_0,$$

where  $I$  is an interval and  $x$  represents time; the number  $x_0$  is in the interval  $I$ .

The solution can then be written as

$$y(x) = y_0 + \int_{x_0}^x f(u) du.$$

**Example 5** (Example # 1, Section 8.1, p. 392)

Suppose that the volume  $V(t)$  of a cell at time  $t$  changes according to

$$\frac{dV}{dt} = \sin t \quad \text{with} \quad V(0) = 3.$$

Find  $V(t)$ .

## Autonomous Differential Equations

Many of the differential equations that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not explicitly depend on  $x$ . These equations are called **autonomous differential equations**.

Formally, we can solve this autonomous differential equation by separation of variables. We begin by dividing both sides of the equation by  $g(y)$  and multiplying both sides by  $dx$ , to obtain

$$\frac{1}{g(y)} dy = dx.$$

Integrating both sides then gives  $\int \frac{1}{g(y)} dy = \int dx.$

## Example 6 (Online Homework # 1)

Find the particular solution of the differential equation

$$\frac{dy}{dx} + 6y = 7$$

satisfying the initial condition  $y(0) = 0$ .

**Example 7** (Problem # 35, Section 8.1, p. 405)

Find the general solution of the differential equation  $\frac{dy}{dx} = y^2 - 4$ .