MA 138 – Calculus 2 with Life Science Applications
Handout

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Direction fields of differential equations

- Many differential equations cannot be solved conveniently by analytical methods, so it is important to consider what qualitative information can be obtained about their solutions without actually solving the equations.
- A **direction field** (or **slope field**) is a graphical representation of the solutions of a first-order differential equation of the form

$$\frac{dy}{dx} = f(x, y).$$

- Imagine that you are standing at a point $P(\alpha, \beta)$ in the xy-plane and that the above differential equation determines your future location. Where should you go next? You move along a curve whose tangent line at the point $P(\alpha, \beta)$ has slope $dy/dx|_P = f(\alpha, \beta)$.

- We (or, better, a computer) can construct a direction field (or slope field) by evaluating the function $f(x, y)$ at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point.
- Thus each line segment is tangent to the graph of the solution passing through that point.
- A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a given differential equation.
- The graph of a solution to the given differential equation is a curve in the xy-plane. It is often useful to regard this curve as the path, or **trajectory** traversed by a moving particle.
- The xy-plane is called the **phase plane** and a representative set of trajectories is referred to as a **phase portrait**.

Direction fields of differential equations...with SAGE

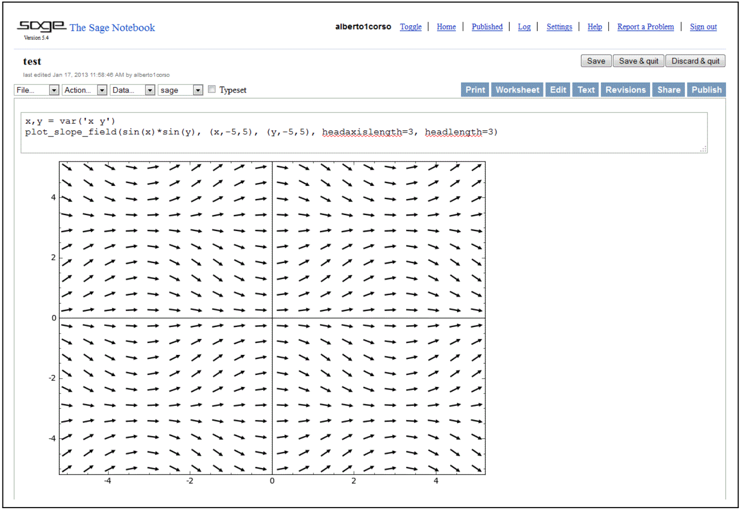
- SAGE is a free open-source mathematics software system.

  `www.sagemath.org/`

- To try sage online follow the appropriate links at the above address and select one of the OpenID providers (say, for example, Google or Yahoo).
- It is easy to plot direction (slope) fields of a differential equation using SAGE. For this we use the command

  `plot_slope_field`

The picture below shows you a snapshot of a session in SAGE with the direction field of the differential equation $dy/dx = \sin(x)\sin(y)$.



Example 1

Consider the differential equation

$$\frac{dy}{dx} = x^2y^2 \qquad \rightsquigarrow \qquad \int \frac{dy}{y^2} = \int x^2\,dx.$$

The general solution is

$$y = \frac{-3}{x^3 + C},$$

where C is a constant. If we make the constant equal to $6, -3$, and 0.3, respectively, we obtain the three solutions below

$$y_1 = \frac{-3}{x^3+6} \qquad y_2 = \frac{-3}{x^3-3} \qquad y_3 = \frac{-3}{x^3+0.3}$$

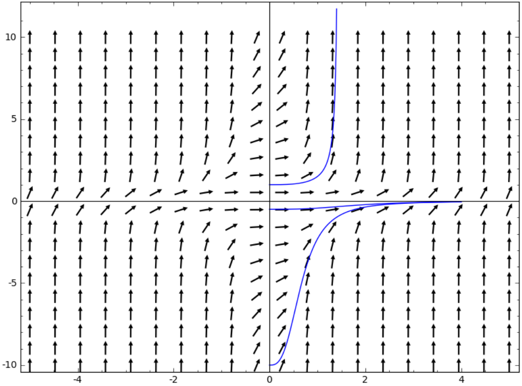
which correspond to the initial conditions

$$y_1(0) = -0.5 \qquad y_2(0) = 1 \qquad y_3(0) = -10$$

respectively.

Below are the commands to plot the direction field of the given differential equation as well as the graphs of those three particular solutions.

```
x,y=var('x,y')
v=plot_slope_field(x^2*y^2,(x,-5,5),(y,-10,10), headaxislength=3, headlength=3)
a=6
b=-3
c=0.3
d1=plot(-3/(x^3+a),(x,0,4))
d2=plot(-3/(x^3+b),(x,0,1.4))
d3=plot(-3/(x^3+c),(x,0,4))
show(v+d1+d2+d3)
```



Phase Portrait 1: direction field of $dy/dy = x^2y^2$ and some particular solutions.

Remark (about Example 1)

If you compute the limit as x tends to infinity of

$$y = \frac{-3}{x^3 + C}$$

you see that for any choice of C the limit is 0.

This seems inconsistent with the behavior of y_2 in the previous phase portrait. (It seems very different from the behavior of y_1 and y_3.) This difference is due to the fact that

$$\lim_{x \to (\sqrt[3]{3})^-} \frac{-3}{x^3 - 3} = +\infty,$$

that is, the solution y_2 has a discontinuity at $x = \sqrt[3]{3}$.

Example 2

Consider the differential equation

$$\frac{dy}{dx} = y^2 - 4 \qquad \rightsquigarrow \qquad \int \frac{dy}{(y-2)(y+2)} = \int dx.$$

Using the method of partial fractions, we saw that the general solution is

$$y = 2 \cdot \frac{1 + Ce^{4x}}{1 - Ce^{4x}} = 2 \cdot \frac{e^{-4x} + C}{e^{-4x} - C},$$

where C is a constant. If we make the constant equal to $2, -1$, and 0.1, respectively, we obtain the three solutions

$$y_1 = 2 \cdot \frac{1 + 2e^{4x}}{1 - 2e^{4x}} \qquad y_2 = 2 \cdot \frac{1 - e^{4x}}{1 + e^{4x}} \qquad y_3 = 2 \cdot \frac{1 + 0.1e^{4x}}{1 - 0.1e^{4x}},$$

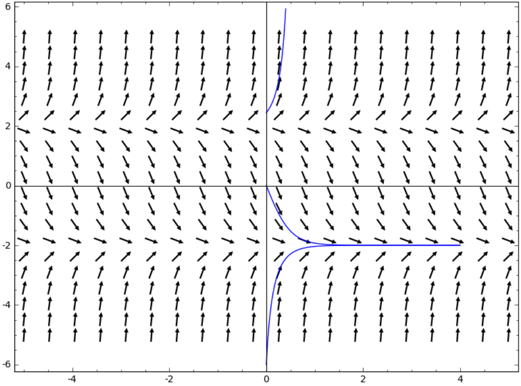
which correspond to the initial conditions

$$y_1(0) = -6 \qquad y_2(0) = 0 \qquad y_3(0) = \frac{22}{9} \approx 2.\overline{4},$$

respectively.

Below are the commands to plot the direction field of the given differential equation as well as the graphs of those three particular solutions.

```
x,y=var('x,y')
v=plot_slope_field((y^2-4),(x,-5,5),(y,-5,5),headaxislength=3, headlength=3)
a=2
b=-1
c=0.1
d1=plot(2*(1+a*e^(4*x))/(1-a*e^(4*x)),(x,0,4)
d2=plot(2*(1+b*e^(4*x))/(1-b*e^(4*x)),(x,0,4)
d3=plot(2*(1+c*e^(4*x))/(1-c*e^(4*x)),(x,0,0.
show(v+d1+d2+d3)
```



Phase Portrait 2: direction field of $dy/dy = y^2 - 4$ and some particular solutions.

Remark (about Example 2)

If you compute the limit as x tends to infinity of

$$y = 2 \cdot \frac{1 + Ce^{4x}}{1 - Ce^{4x}} = 2 \cdot \frac{e^{-4x} + C}{e^{-4x} - C}$$

you see that for any choice of C the limit is -2.

This seems inconsistent with the behavior of y_3 in the previous phase portrait. (It seems very different from the behavior of y_1 and y_2.) This difference is due to the fact that

$$\lim_{x \to (\ln(10)/4)^-} 2 \cdot \frac{1 + 0.1e^{4x}}{1 - 0.1e^{4x}} = +\infty,$$

that is, the solution y_3 has a discontinuity at $x = \ln(10)/4$.

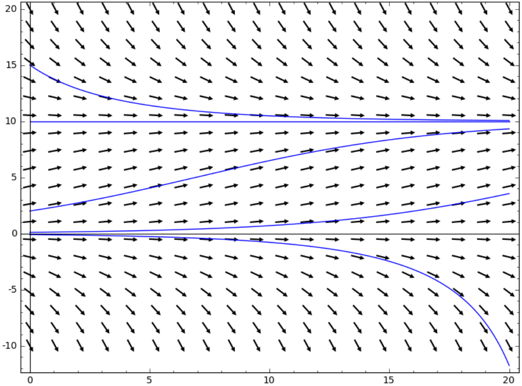
Example 3 (Logistic Growth Model)

This is an example with $r = 0.2$ ($\equiv$ 20% growth rate) and $K = 10$.

$$\frac{dN}{dt} = 0.2\,N\,(1 - N/10)$$

$$\rightsquigarrow$$

$$N(t) = \frac{10}{1 + (10/N_0 - 1)e^{-0.2t}}$$



Phase Portrait 3: direction field of $dN/dt = 0.2\,N\,(1 - N/10)$ and some particular solutions.