MA 138 – Calculus 2 with Life Science Applications
Linear Systems
(Section 9.1)

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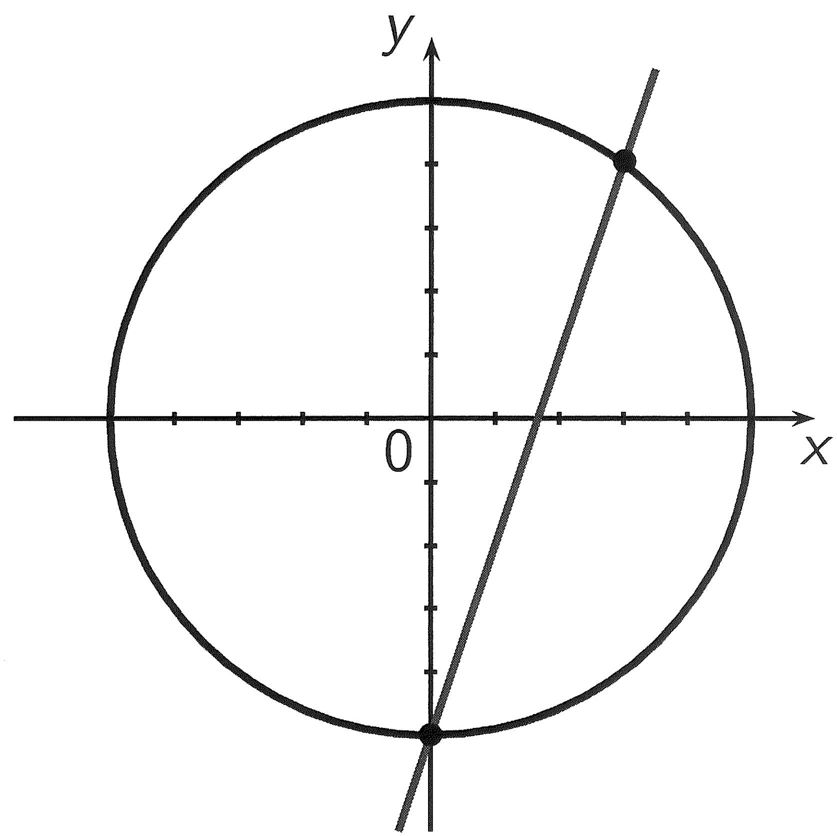
Systems of Equations and Their Solutions

A **system of equations** is a set of equations that involve the same variables. A **solution** of a system is an assignment of values for the variables that makes each equation in the system true. To **solve** a system means to find all solutions of the system.



Example 1

Show that $(0, -10)$ and $(6, 8)$ are solutions of the system

$$\begin{cases} x^2 + y^2 = 100 \\ 3x - y = 10 \end{cases}$$

$$\begin{cases} x^2 + y^2 = 100 \\ 3x - y = 10 \end{cases}$$

Observe that $(0,-10)$ satisfies both equations as:

$$\begin{cases} 0^2 + (-10)^2 \stackrel{?}{=} 100 \quad \checkmark \\ 3(0) - (-10) \stackrel{?}{=} 10 \quad \checkmark \end{cases}$$

Similarly $(6,8)$ satisfies both equations

$$\begin{cases} 6^2 + 8^2 \stackrel{?}{=} 100 \quad \checkmark \\ \underbrace{3 \cdot 6}_{18} - (8) \stackrel{?}{=} 10 \quad \checkmark \end{cases}$$

A Flashback

We have already encountered systems of linear equations when we were discussing **partial fraction decomposition** in Section 7.3.

E.g., suppose that we have to find the integral of $\dfrac{6x+4}{(3x+1)(2x+2)}$.

The method says that we need to find constants A and B such that

$$\begin{aligned} \frac{6x+4}{(3x+1)(2x+2)} &= \frac{A}{3x+1} + \frac{B}{2x+2} \\ &= \frac{A(2x+2)+B(3x+1)}{(3x+1)(2x+2)} \\ &= \frac{(2A+3B)x+2A+B}{(3x+1)(2x+2)}. \end{aligned}$$

Thus, we need to solve the following equations

$$2A+3B=6 \qquad 2A+B=4.$$

Systems of Linear Equations

A **system of two linear equations in two variables** has the form

$$\begin{cases} a_1\,x + b_1\,y &= c_1 \\ a_2\,x + b_2\,y &= c_2 \end{cases}$$

where a_1, a_2, b_1, b_2, c_1, and c_2 are numbers. We can use two methods

- the substitution method
- the elimination method

to solve such systems algebraically. For linear systems, though, the elimination method is usually easier.

Geometrically: The graph of a linear system in two variables is a pair of lines. Thus, from a graphic point of view, to solve the system means that we must find the intersection point(s) of the lines.

Substitution Method

In the substitution method we start with one equation in the system and solve for one variable in terms of the other variable.

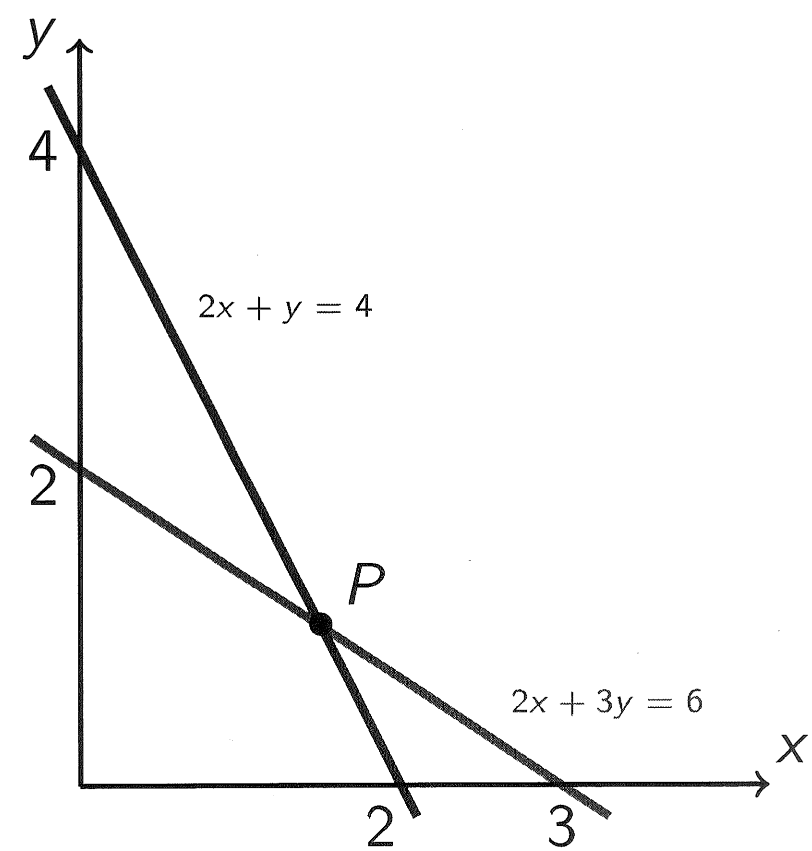
The following describes the procedure.

1. **Solve for One Variable:** Choose one equation and solve for one variable in terms of the other variable.
2. **Substitute:** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute:** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

Example 2 (Example 1, Section 9.1, p. 433)

Find the solution of the system of linear equations

$$\begin{cases} 2x + 3y = 6 \\ 2x + y = 4 \end{cases}$$



$$\begin{cases} 2x + 3y = 6 \\ 2x + y = 4 \end{cases}$$

We use substitution. We solve for y in the second equation

$$\begin{cases} 2x + 3y = 6 \\ y = 4 - 2x \end{cases}$$

substitute in the first equation

We obtain: $2x + 3(4 - 2x) = 6$

hence $2x - 6x + 12 = 6 \iff$

$-4x = 6 - 12 \iff -4x = -6 \iff$

$x = 6/4 = \boxed{3/2}$

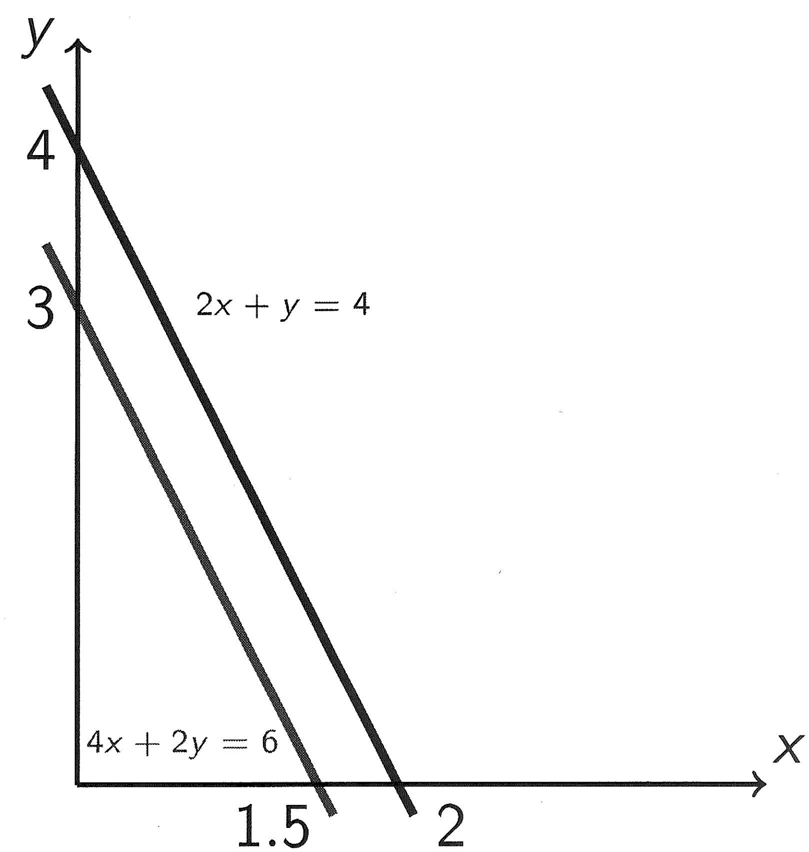
now back substitute

$y = 4 - 2(3/2) = \boxed{1}$

Example 3 (Example 2, Section 9.1, p. 434)

Find the solution of the system of linear equations

$$\begin{cases} 4x + 2y = 6 \\ 2x + y = 4 \end{cases}$$



$$\begin{cases} 4x + 2y = 6 \\ 2x + y = 4 \end{cases}$$

We can do what we did in the first problem:

$$\begin{cases} 4x + 2y = 6 \\ y = 4 - 2x \end{cases}$$ substitute

$$4x + 2(4 - 2x) = 6$$

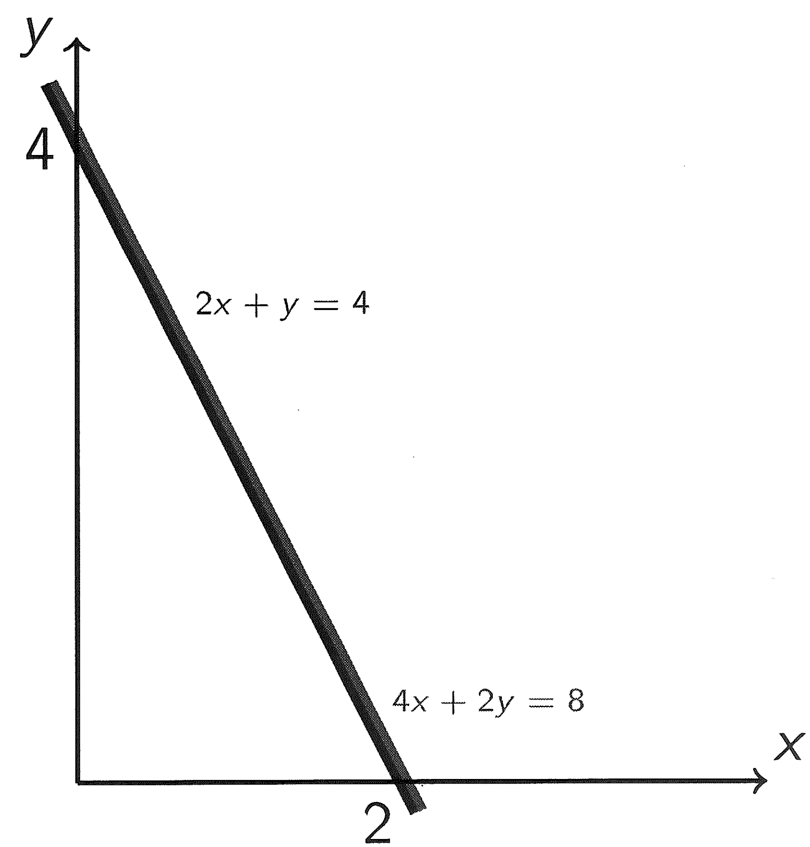
$$\cancel{4x} + 8 - \cancel{4x} = 6$$

We obtain $\underline{\underline{8 = 6 \quad \text{impossible!!}}}$

Example 4 (Example 3, Section 9.1, p. 434)

Find the solution of the system of linear equations

$$\begin{cases} 4x + 2y = 8 \\ 2x + y = 4 \end{cases}$$



$$\begin{cases} 4x + 2y = 8 \\ 2x + y = 4 \end{cases}$$

We can do what we did in the previous problems:

$$\begin{cases} 4x + 2y = 8 \\ y = 4 - 2x \end{cases}$$ substitute

We obtain that the first equation becomes

$$4x + 2(4 - 2x) = 8$$

$$\Updownarrow$$

$$4x + 8 - 4x = 8$$

$$\Updownarrow$$

$$8 = 8 \quad \text{true!}$$

But no condition on x !

This is because the first equation is twice the second one.

Thus we have just one equation

$$\boxed{2x + y = 4}$$

the points on the line all satisfy both equation. If we set $x = t$ when t is any real number then

$y = 4 - 2x = 4 - 2t$. Thus we can write the line of solutions as

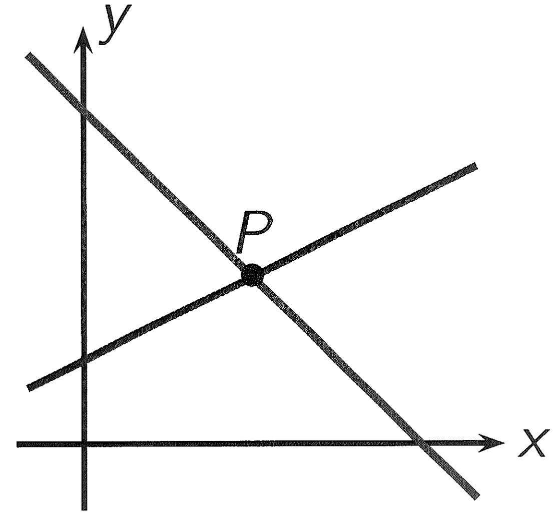
$$\text{Line} = \{ (t, 4-2t) \mid t \in \mathbb{R} \}$$

This is called parametrization of the line

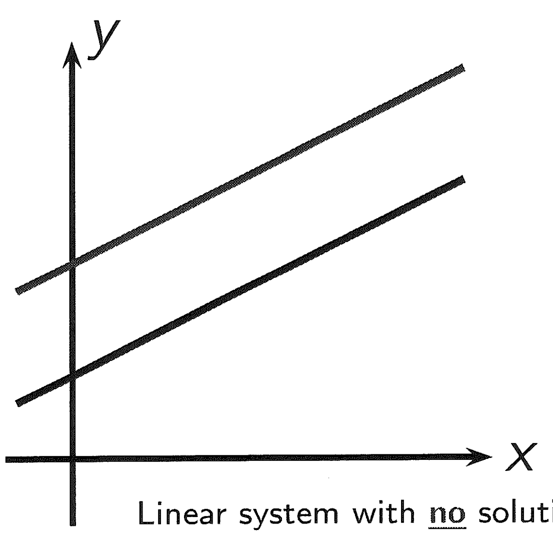
Number of Solutions of a Linear System in Two Variables

For a system of linear equations in two variables, exactly one of the following is true:

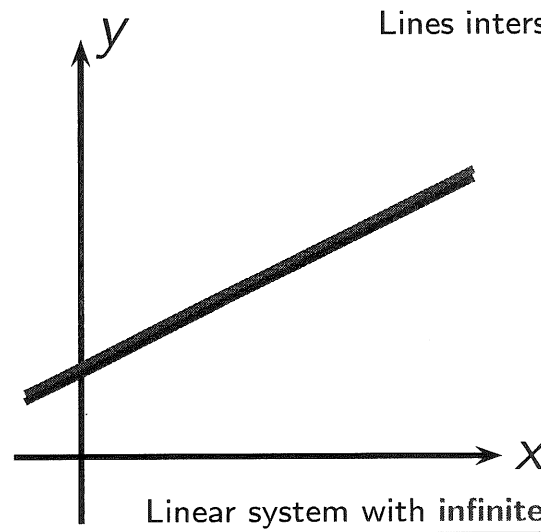
1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.



Linear system with one solution.
Lines intersect at a single point.



Linear system with no solution.
Lines are parallel, so they do not intersect.



Linear system with infinitely many solutions.
Lines coincide.

Towards the Elimination Method

- The substitution method of solving systems of linear equations we have employed so far works well only for systems in few variables. To solve large systems of m linear equations in n variables, we need to develop a more efficient method.

- The basic strategy is to transform the system of linear equations into an **equivalent system** of equations that has the same solutions as the original, but a much simpler form.

- In a nutshell, in the elimination method we try to combine the equations using sums or differences to eliminate one of the variables.

- We illustrate this approach in the next example. We tag all equations with labels of the form (R_i); R_i stands for "ith row. The labels will allow us to keep track of our computations.

Example 2 (Revisited)

- We multiply the coefficients of the first equation(≡row) by $\frac{1}{2}$

$$\begin{cases} 2x + 3y = 6 \\ 2x + y = 4 \end{cases} \overset{\frac{1}{2}R_1}{\rightsquigarrow} \begin{cases} x + \frac{3}{2}y = 3 \\ 2x + y = 4 \end{cases}$$

- We add -2 times the first equation to the second one

$$\begin{cases} x + \frac{3}{2}y = 3 \\ 2x + y = 4 \end{cases} \underset{R_2 - 2R_1}{\rightsquigarrow} \begin{cases} x + \frac{3}{2}y = 3 \\ -2y = -2 \end{cases}$$

- We multiply the coefficients of the second equation by $-\frac{1}{2}$

$$\begin{cases} x + \frac{3}{2}y = 3 \\ -2y = -2 \end{cases} \underset{-\frac{1}{2}R_2}{\rightsquigarrow} \begin{cases} x + \frac{3}{2}y = 3 \\ y = 1 \end{cases}$$

- We add $-\frac{3}{2}$ the second equation to the first one

$$\begin{cases} x + \frac{3}{2}y = 3 \\ y = 1 \end{cases} \overset{R_1 - \frac{3}{2}R_2}{\rightsquigarrow} \begin{cases} x = \frac{3}{2} \\ y = 1 \end{cases}$$

A Shorthand Notation— **The Augmented Matrix**

When we transform a system of linear equations, we make changes only to the coefficients of the variables. It is therefore **convenient to introduce a notation that simply keeps track of all the coefficients**.

Example 2 (Revisited)

$$\begin{cases} 2x + 3y = 6 \\ 2x + y = 4 \end{cases} \quad \leftrightsquigarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow \tfrac{1}{2}R_1 \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow R_2 - 2R_1 \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & -2 & -2 \end{array}\right]$$

$$\rightsquigarrow -\tfrac{1}{2}R_2 \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 1 & 1 \end{array}\right] \rightsquigarrow R_1 - \tfrac{3}{2}R_2 \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Remark about Examples 3 and 4

Example 3

$$\begin{cases} 4x + 2y = 6 \\ 2x + y = 4 \end{cases} \quad \leftrightsquigarrow \quad \left[\begin{array}{cc|c} 4 & 2 & 6 \\ 2 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 2 & 6 \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow_{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow_{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 0 & 1 \end{array}\right]$$

Example 4

$$\begin{cases} 4x + 2y = 8 \\ 2x + y = 4 \end{cases} \quad \leftrightsquigarrow \quad \left[\begin{array}{cc|c} 4 & 2 & 6 \\ 2 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 2 & 8 \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow_{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 2 \\ 2 & 1 & 4 \end{array}\right] \rightsquigarrow_{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Example 5 (Online Homework # 3)

Determine the value of k for which the following system

$$\begin{cases} -6x & - & 6y & = & -2 \\ -21x & - & 21y & = & k \\ -15x & - & 15y & = & -5 \end{cases}$$

is consistent ($\equiv$ the system has solution).

$$\left[\begin{array}{cc|c} -6 & -6 & -2 \\ -21 & -21 & k \\ -15 & -15 & -5 \end{array}\right] \rightsquigarrow \;-\tfrac{1}{6}R_1 \left[\begin{array}{cc|c} 1 & 1 & \tfrac{1}{3} \\ -21 & -21 & k \\ -15 & -15 & -5 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{r} \\ R_2+21R_1 \\ R_3+15R_1 \end{array} \left[\begin{array}{cc|c} 1 & 1 & \tfrac{1}{3} \\ 0 & 0 & k+\tfrac{21}{3} \\ 0 & 0 & 0 \end{array}\right]$$

To be consistent we need

$$k + \frac{21}{3} = 0 \iff \boxed{k = -7}$$

Example 6 (Online Homework # 4)

Solve the system

$$\begin{cases} 2x + 5y = a \\ -3x - 7y = b \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 5 & a \\ -3 & -7 & b \end{array}\right] \rightsquigarrow \begin{matrix} \tfrac{1}{2}R_1 \\ \end{matrix} \left[\begin{array}{cc|c} 1 & 5/2 & a/2 \\ -3 & -7 & b \end{array}\right]$$

$$\begin{matrix} \\ R_2+3R_1 \end{matrix} \left[\begin{array}{cc|c} 1 & 5/2 & a/2 \\ 0 & -7+\frac{15}{2} & b+\frac{3}{2}a \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 5/2 & a/2 \\ 0 & 1/2 & \frac{2b+3a}{2} \end{array}\right]$$

$$\rightsquigarrow \begin{matrix} \\ 2R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 5/2 & a/2 \\ 0 & 1 & 2b+3a \end{array}\right] \rightsquigarrow \begin{matrix} R_1-\tfrac{5}{2}R_2 \\ \end{matrix} \left[\begin{array}{cc|c} 1 & 0 & -7a-5b \\ 0 & 1 & 2b+3a \end{array}\right]$$

$$\frac{a}{2} - \frac{5}{2}(2b+3a) = \frac{a-10b-15a}{2}$$

$$= \boxed{-7a-5b}$$

Therefore the solution is

$$x = -7a - 5b$$

$$y = 3a + 2b$$

Example 7 (Online Homework # 6)

Determine the values of h and k so that the following system has an infinite number of solutions

$$\begin{cases} x - 8y = h \\ 2x + ky = -10 \end{cases}$$

Determine h an k so that the system

$$\begin{cases} x - 8y = h \\ 2x + ky = -10 \end{cases}$$

has infinitely many solutions.

i.e. the 2 equations must be the same

$$x - 8y = h \iff y = \frac{1}{8}x - \frac{h}{8}$$

$$2x + ky = -10 \iff y = -\frac{2}{k}x - \frac{10}{k}$$

$$\iff \frac{1}{8} = -\frac{2}{k} \quad \text{and} \quad -\frac{h}{8} = -\frac{10}{k}$$

Thus $\underline{k = -16}$ and $\underline{h = \frac{80}{-16} = -5}$

Otherwise, set up the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -8 & h \\ 2 & k & -10 \end{array}\right] \xrightarrow[R_2 - 2R_1]{} \left[\begin{array}{cc|c} 1 & -8 & h \\ 0 & k+16 & -10-2h \end{array}\right]$$

To have infinite solutions the last row has to be $\left[\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right]$

$\therefore$ $k + 16 = 0$ $\qquad$ $-10 - 2h = 0$

$\boxed{k = -16}$ $\qquad$ $\boxed{h = -5}$

Example 8

A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second solution gives a mixture that is 15% acid, whereas 100 mL of the first mixed with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

Let $x/100$ be the concentration of the first solution

and $y/100$ be the concentration of the second solution.

When we blend 300 mL of the first solution to 600 mL of the second solution we obtain 900 mL of a solution 15% acid. Thus the amount of acid is

$$300 \cdot \frac{x}{100} + 600 \cdot \frac{y}{100} = 900 \cdot \frac{15}{100}$$

$$\iff 3x + 6y = 135 \iff \boxed{x + 2y = 45}$$

When we blend 100 mL of the first solution to 500 mL of the second solution we obtain 600 mL of a solution with 12.5% acid.

Thus

$$100\,\frac{x}{100} + 500\,\frac{y}{100} = 600 \cdot \frac{12.5}{100}$$

$$\Longleftrightarrow \boxed{x + 5y = 75}$$

We need to solve

$$\begin{cases} x + 2y = 45 \\ x + 5y = 75 \end{cases}$$

Check that $(x = 25,\ y = 10)$ is the solution.

i.e. 1st solution is 25% acid

2nd solution is 10% acid.