MA 138 – Calculus 2 with Life Science Applications
Functions of Two or More Independent Variables
(Section 10.1)

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Coordinate Systems (in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Any point P in the plane can be represented as an ordered pair of real numbers. To locate a point in space, three numbers are required.

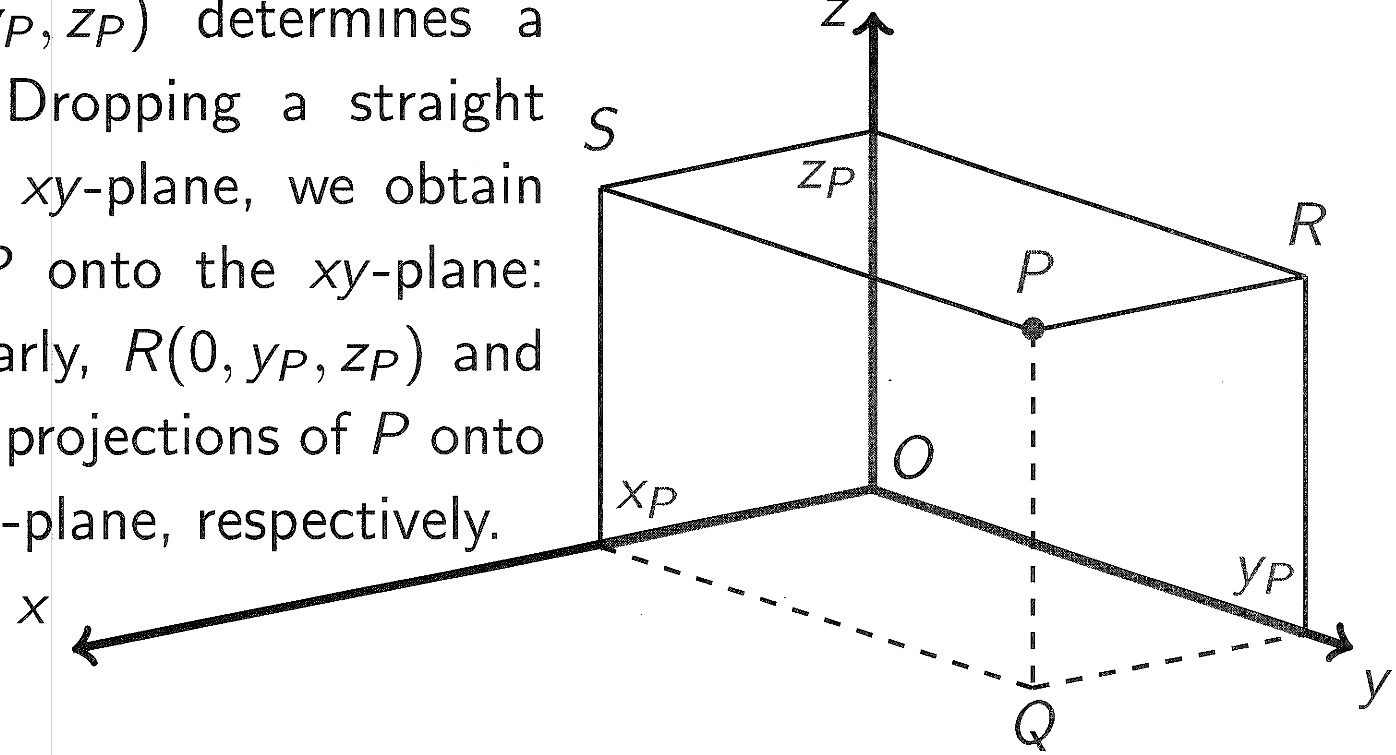
We first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the coordinate axes and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical. The direction of the z-axis is determined by the right-hand rule: If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine three coordinate planes: The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes.

These three coordinate planes divide space into eight parts, called octants. The first octant is determined by the positive axes.

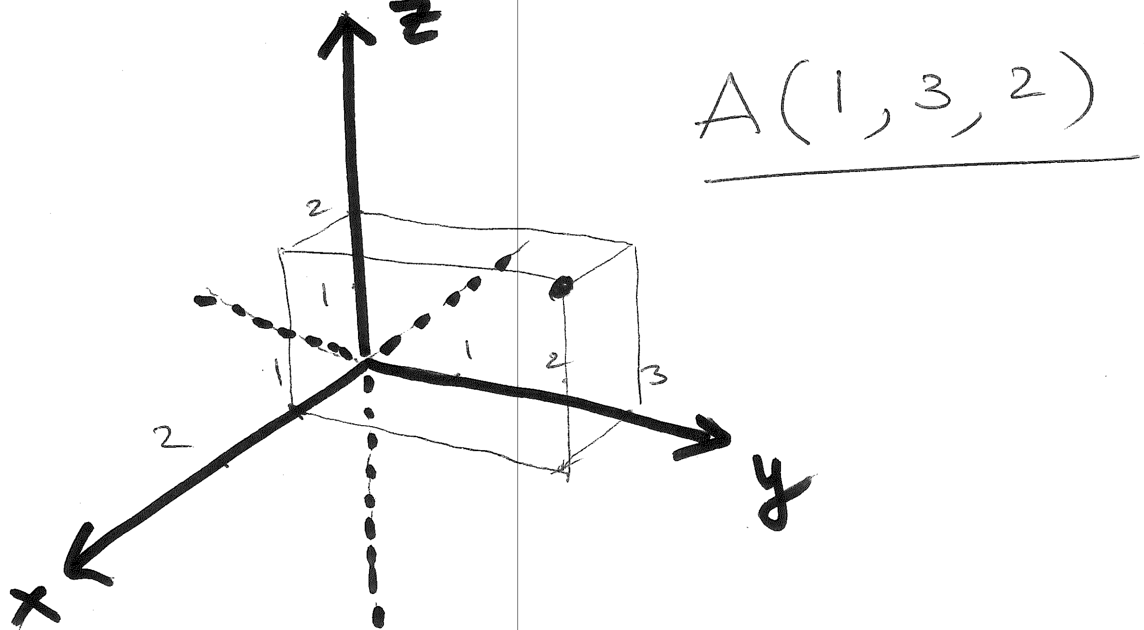
Now if P is any point in space, let x_P be the distance from P to the yz-plane, let y_P be the distance from P to the xz-plane, and let z_P be the distance from P to the xy-plane. We represent the point P by the ordered triple (x_P, y_P, z_P) of real numbers and we call them the coordinates of P.

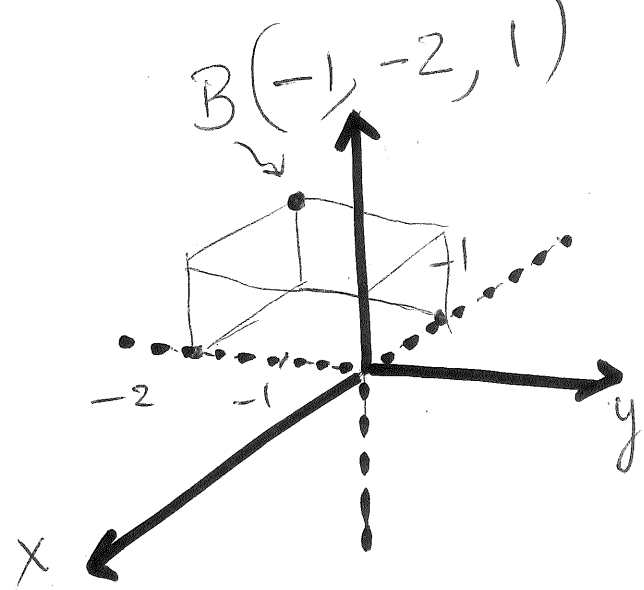
The point $P(x_P, y_P, z_P)$ determines a rectangular box. Dropping a straight line from P to the xy-plane, we obtain the projection of P onto the xy-plane: $Q(x_P, y_P, 0)$. Similarly, $R(0, y_P, z_P)$ and $S(x_P, 0, z_P)$ are the projections of P onto the yz-plane and xz-plane, respectively.

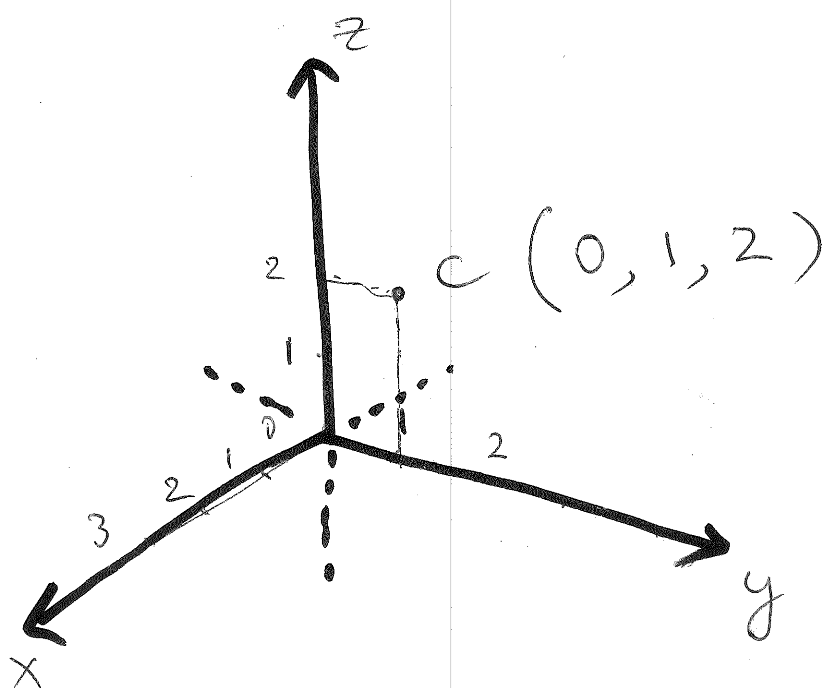


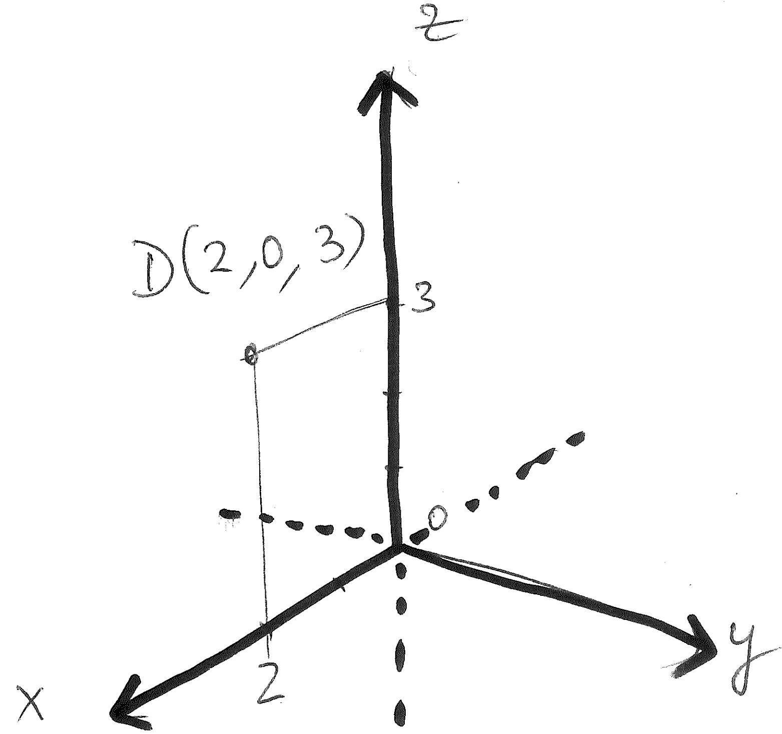
Example 1 (Problems # 3, 4, Section 10.1, p. 511)

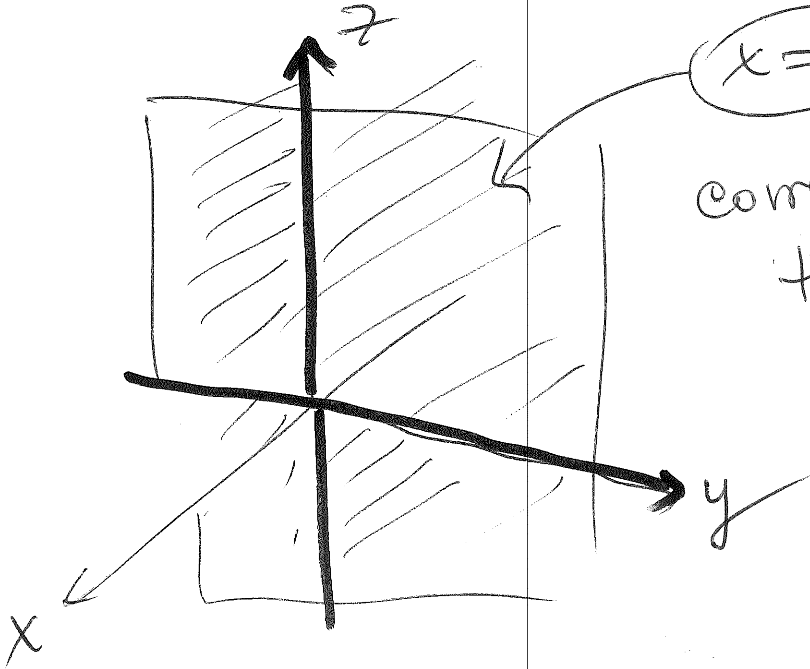
- Locate the following points in a three-dimensional Cartesian coordinate system:
  $A(1,3,2)$ $B(-1,-2,1)$ $C(0,1,2)$ $D(2,0,3)$
- Describe the set of all points in $\mathbb{R}^3$ that satisfy the following expressions:
  (a) $x=0$ (b) $y=0$ (c) $z=0$ (d) $z\geq 0$ (e) $y\leq 0$


A(1, 3, 2)
z
y
x


B(-1, -2, 1)
y
x

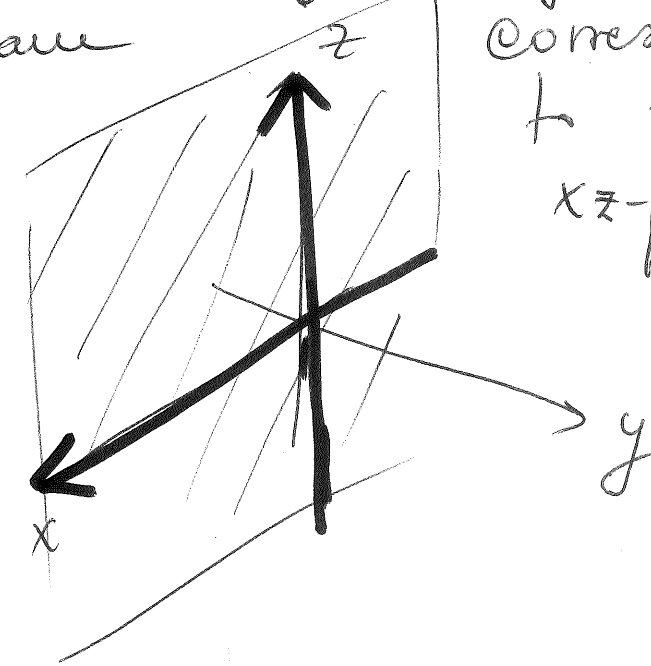

C (0, 1, 2)
z
y
x


D(2, 0, 3)
z
y
x



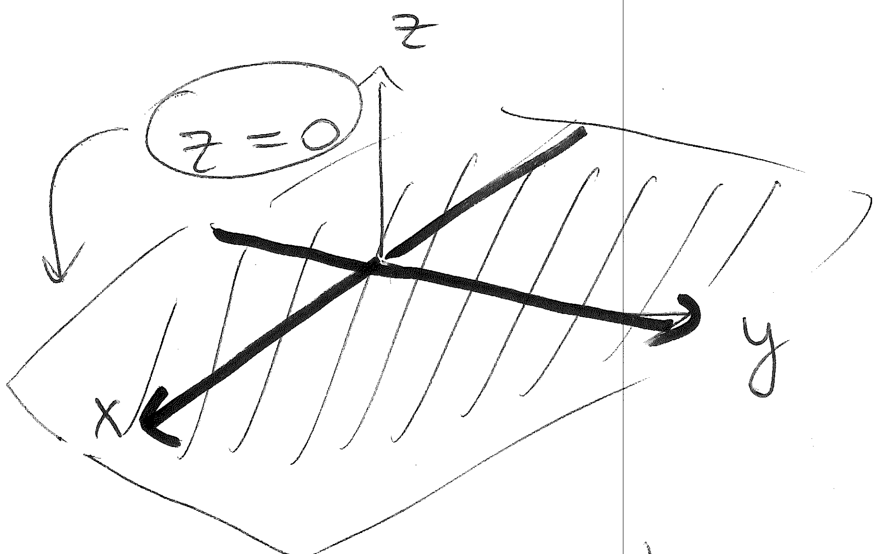
$x = 0$

corresponds to the yz-plane



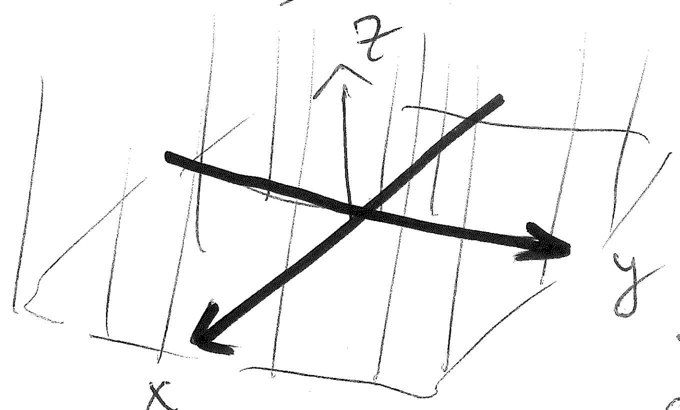
$y = 0$

corresponds to the xz-plane



$z = 0$

it corresponds to the xy-plane



$z \geq 0$

everything above the xy-plane

$y \leq 0$ everything below the xy plane

Functions of Two or More Independent Variables

We consider functions for which

- the **domain** consists of pairs of real numbers (x, y) with $x, y \in \mathbb{R}$ or, more generally, of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$ with $x_1, x_2, \ldots, x_n \in \mathbb{R}$. We write $\mathbb{R}^n$ to denote the set of all n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$.
- the **range** consists of subsets of the real numbers.

Real-Valued Functions

Suppose $D \subset \mathbb{R}^n$. Then a real-valued function f on D assigns a real number to each element in D, and we write

$$f : D \longrightarrow \mathbb{R}, \qquad (x_1, x_2, \ldots, x_n) \mapsto f(x_1, x_2, \ldots, x_n)$$

The set D is the **domain** of the function f, and the set

$$\{w \in \mathbb{R} \mid f(x_1, x_2, \ldots, x_n) = w \text{ for some } (x_1, x_2, \ldots, x_n) \in D\}$$

is the **range** of the function f.

Graph of a Function of Two Variables

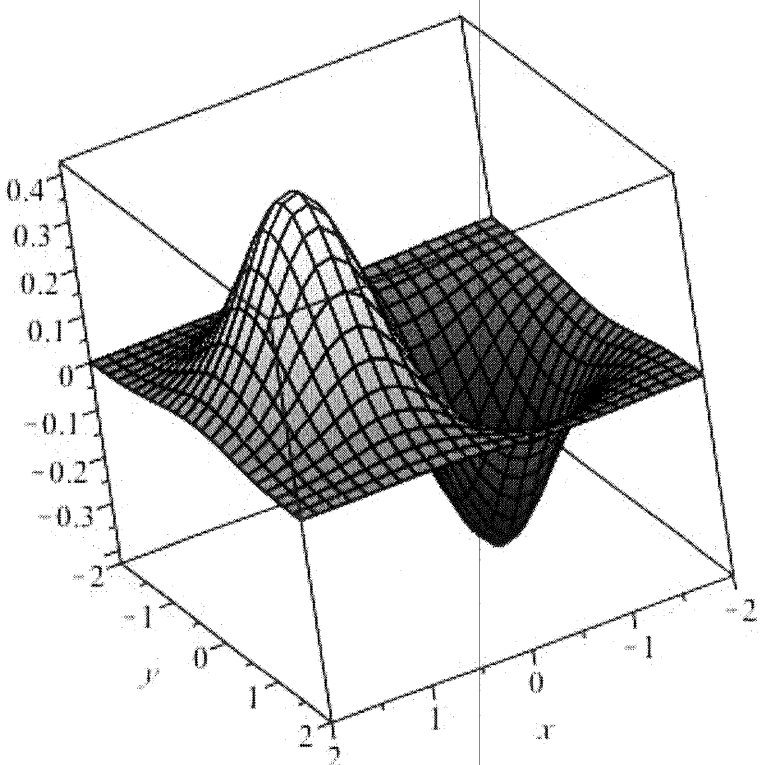
- If f is a function of two independent variables, we usually denote the independent variables by x and y, and write $f(x, y)$.
- We also write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variable z is the dependent variable.
- If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is well-defined.
- To visualize a function of two variables we often consider its graph.

Graph of a Function of Two Variables

The graph of a function f of two independent variables with domain D is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for $(x, y) \in D$. That is, the graph of f is the set

$$\text{Graph}(f) = \{(x, y, z) \,|\, z = f(x, y) \text{ with } (x, y) \in D\}.$$

The graph of $f(x, y)$ is therefore a surface in three-dimensional space, as illustrated, for example, by the following picture



which shows the graph of the function

$$f(x, y) = x\, e^{-x^2-y^2}$$

over the square $[-2, 2] \times [-2, 2]$.

Graphing a surface in three-dimensional space is difficult. Fortunately, good computer software is now available that facilitates this task.

Example 2 (Online Homework # 2)

Suppose $f(x,y) = xy^2 + 7$. Compute the following values

- $f(4,-2)$
- $f(-2,4)$
- $f(t,4t)$
- $\dfrac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$

$$f(x,y) = xy^2 + 7$$

* $f(4,-2) = 4(-2)^2 + 7 = 23$
* $f(-2,4) = (-2)(4)^2 + 7 = -25$

* $f(t,4t) = (t)(4t)^2 + 7 = \underline{16t^3 + 7}$

* $$\frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} =$$

* $$\frac{[x_0(y_0+h)^2 + 7] - [x_0 y_0^2 + 7]}{h} = \frac{x_0(y_0^2 + 2y_0h + h^2) + 7 - x_0 y_0^2 - 7}{h}$$

$$= \frac{\cancel{x_0 y_0^2} + 2x_0 y_0 h + x_0 h^2 \cancel{+7} \cancel{- x_0 y_0^2} \cancel{-7}}{h}$$

$$= \frac{2x_0 y_0 h + x_0 h^2}{h} = \frac{\cancel{h}(2x_0 y_0 + x_0 h)}{\cancel{h}}$$

$$= 2x_0 y_0 + x_0 h$$

Hence $\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = \boxed{2x_0 y_0}$

Example 3 (Online Homework # 3)

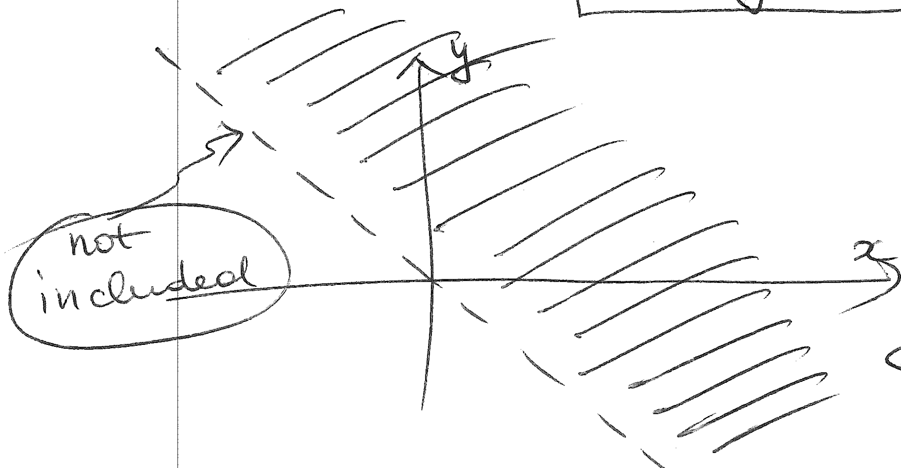
Find the domain of the following functions

- $f(x, y) = \ln(x + y)$
- $g(x, y) = \sqrt{x^2 y^3}$
- $h(x, y) = e^{-\frac{1}{x+y}}$
- $k(x, y) = x^2 + y^3$

(1) $\boxed{f(x,y) = \ln(x+y)}$

all (x,y) such that $\boxed{x+y>0}$ so

$\underline{\underline{y > -x}}$

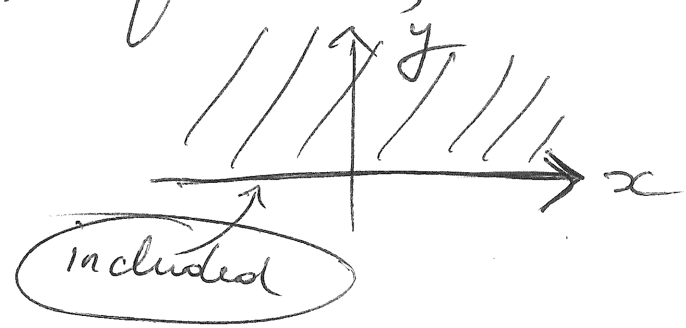


the upper half of that plane

(2) $g(x,y) = \sqrt{x^2 y^3}$

we need all (x,y) such that $x^2 y^3 \geq 0$

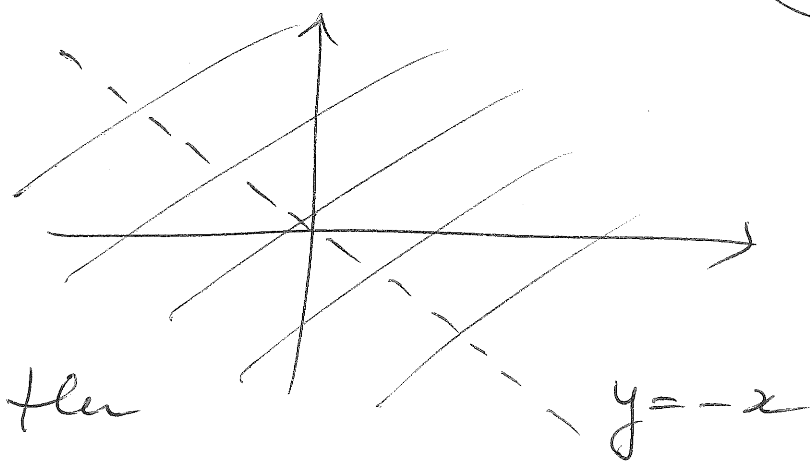
x can be anything since it is squared, but y needs to be positive



(3) $h(x,y) = e^{-\frac{1}{x+y}}$

we need all (x,y) such that $x+y \neq 0$

so $y \neq -x$



all points in the xy plane except the line $y = -x$

(4) $k(x,y) = x^2 + y^3$

there is no restriction on x or y.

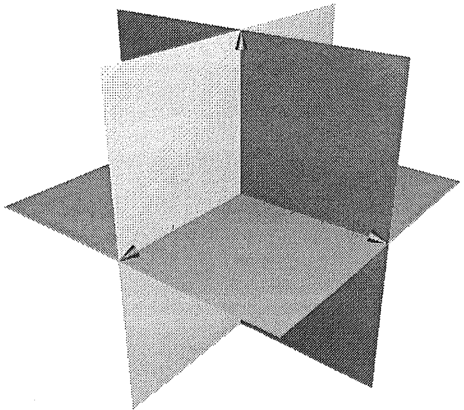
domain: all $\mathbb{R}^2$.

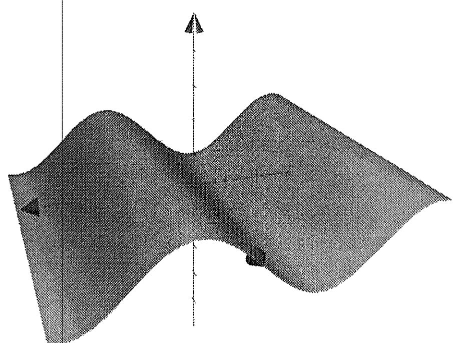
Example 4 (Online Homework # 4)

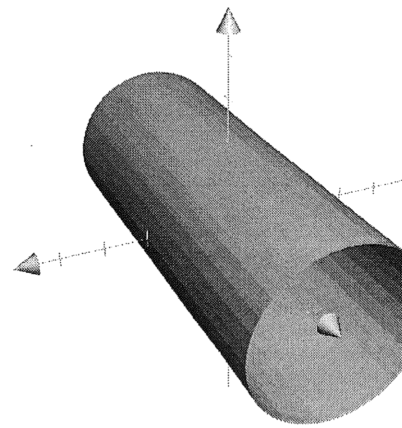
Match the equation of the surface

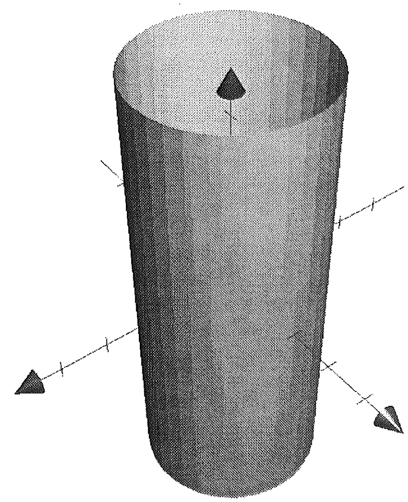
$$z = \sin x \qquad x^2 + y^2 = 4 \qquad xyz = 0 \qquad x^2 + z^2 = 4$$

with one of the graphs below

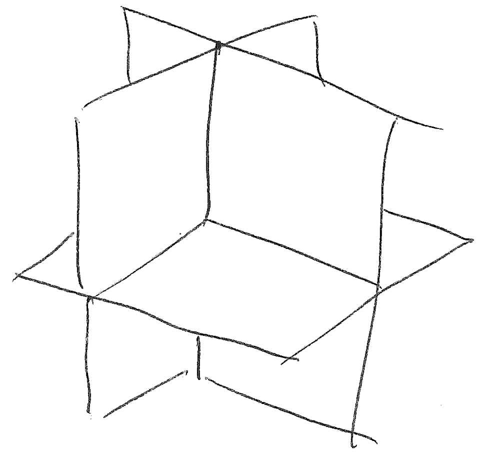




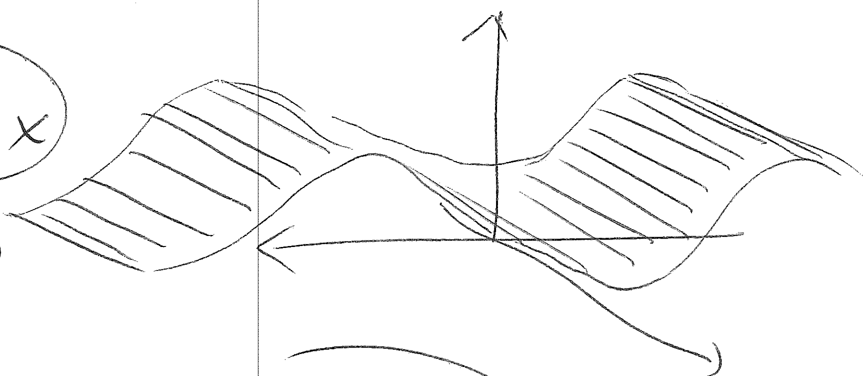




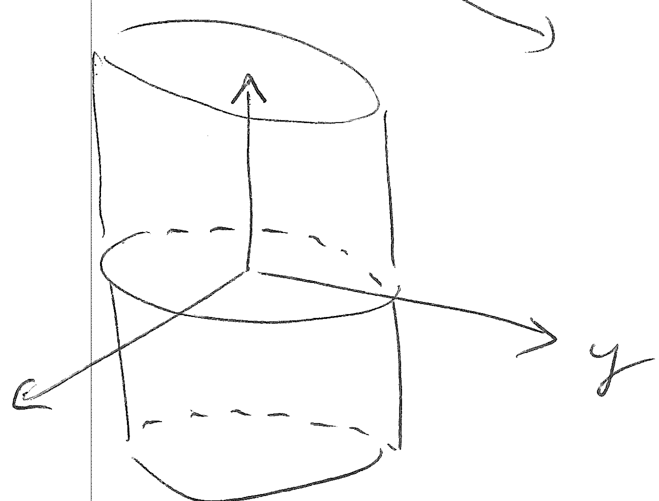
$xyz = 0 \longleftrightarrow$



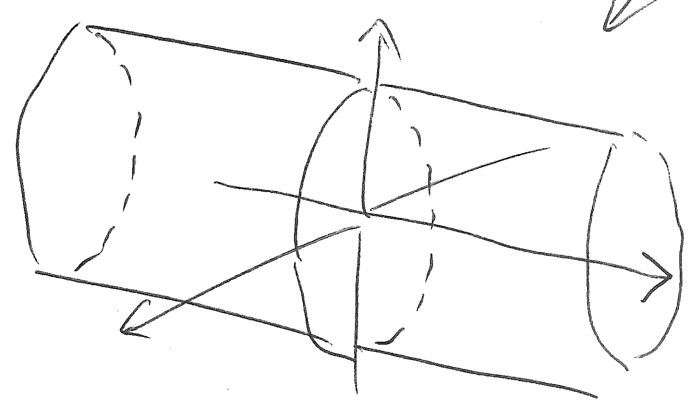
$z = \sin x$



$x^2 + y^2 = 4$



$x^2 + z^2 = 4$



Level Curves (or Contour Lines)

Another way to visualize functions is with **level curves** or **contour lines**. This approach is used, for instance, in topographical maps.

There is a *subtle distinction* between level curves and contour lines, in that level curves are drawn in the function domain whereas contour lines are drawn on the surface.

This distinction is not always made, and often the two terms are used interchangeably. Our text almost exclusively uses level curves, for which we now give the precise definition:

Level curves

Suppose that $f : D \longrightarrow R$, $D \subset \mathbb{R}^2$. Then the level curves of f comprise the set of points (x, y) in the xy-plane where the function f has a constant value; that is, $f(x, y) = c$.

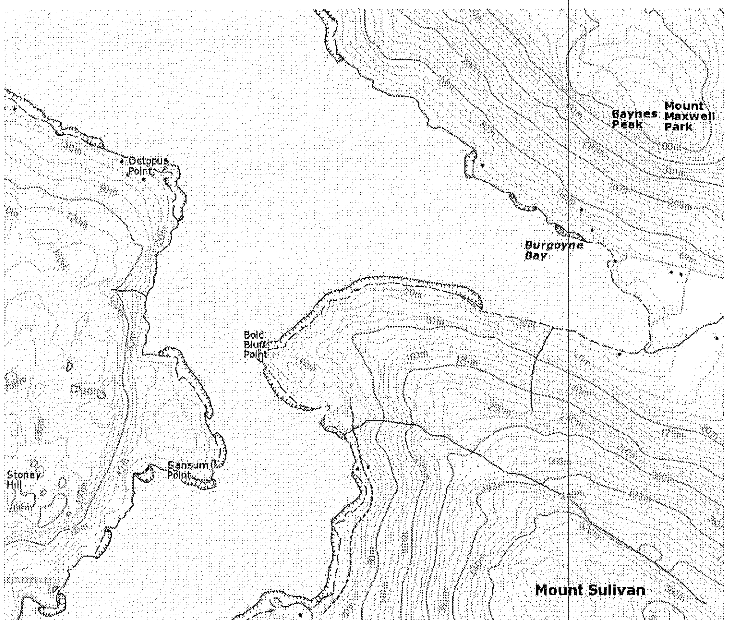
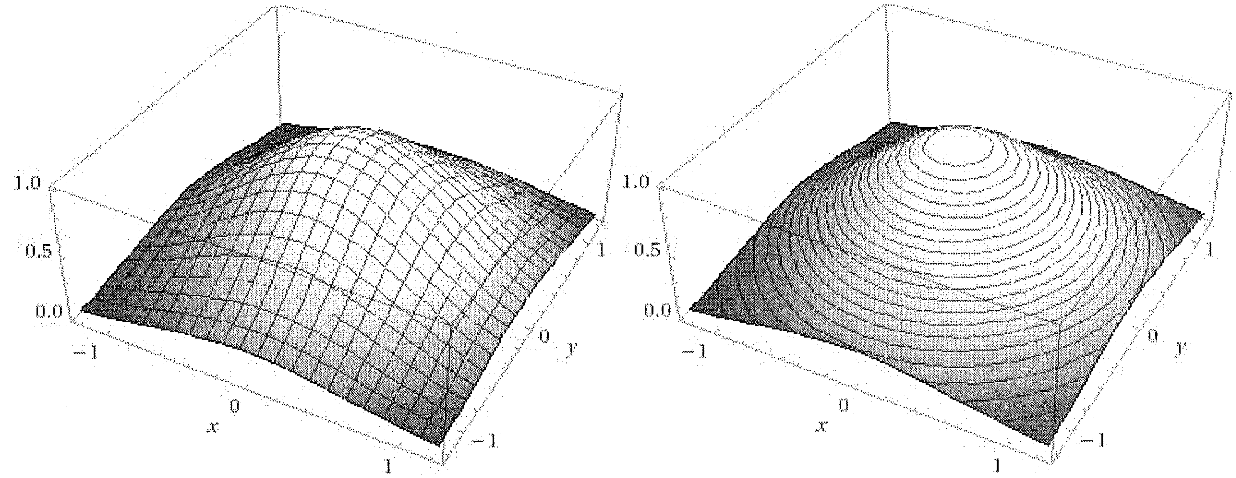
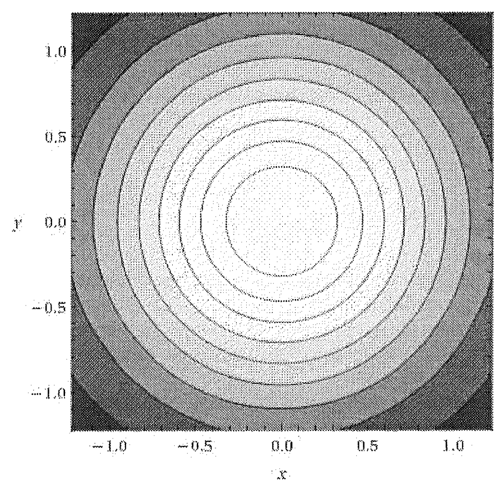


FIGURE: topographical map

Graph of $z = e^{-x^2-y^2}$



The picture on the **left** shows the mesh plot on the graph of the function; the picture on the **right** shows the contour lines on the graph.



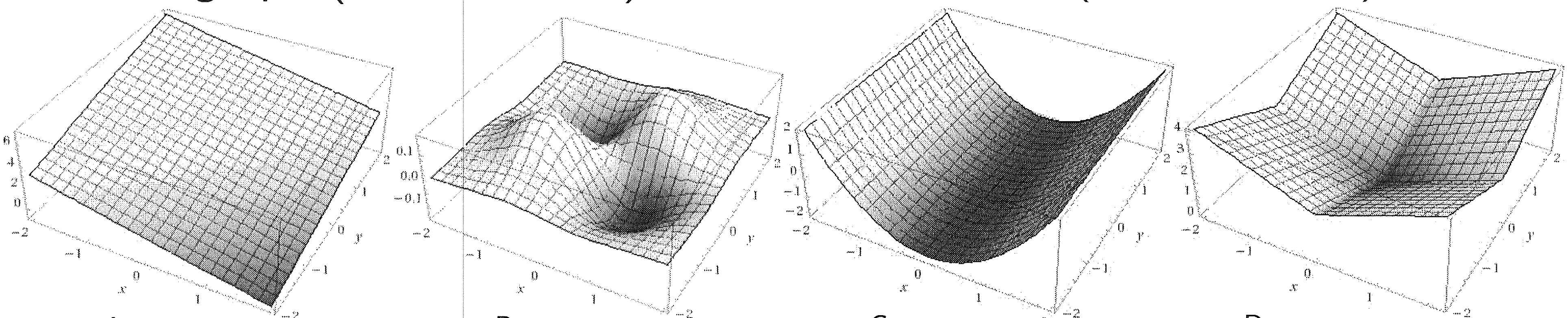
The picture shows the level curves of the function $z = e^{-x^2-y^2}$ in the xy-plane

Example 5 (Online Homework # 5, 6, 7)

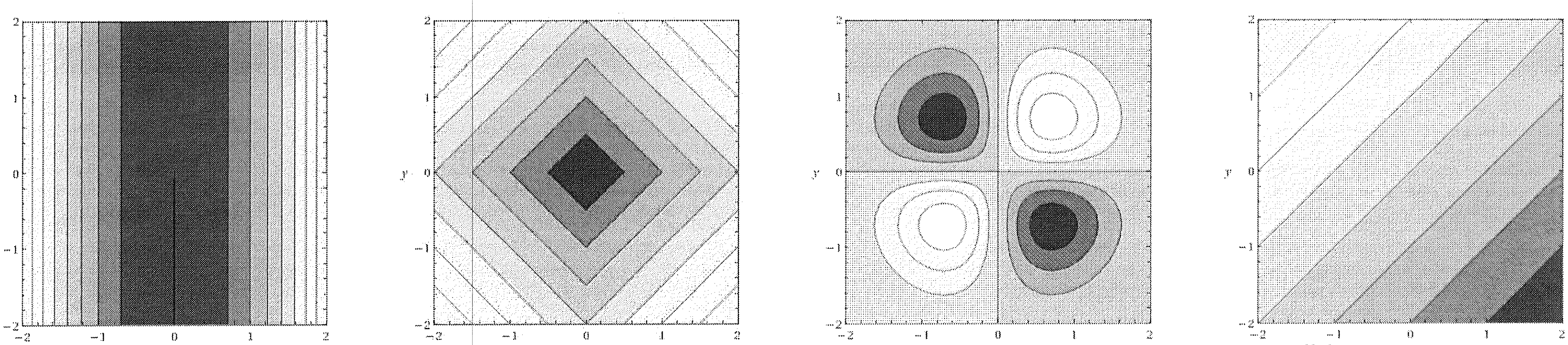
Match each of the following functions of two variables x and y

$f(x,y) = x^2 - 2 \quad g(x,y) = 3 - x + y \quad h(x,y) = |x| + |y| \quad k(x,y) = xye^{-x^2-y^2}$

with its graph (labeled A.-D.) and its level curves (labeled I.-IV.).



A. B. C. D.



I. II. III. IV.

$f(x,y) = x^2 - 2 \longleftrightarrow$ C. $\longleftrightarrow$ I.

$g(x,y) = 3 - x + y \longleftrightarrow$ A. $\longleftrightarrow$ IV.

$h(x,y) = |x| + |y| \longleftrightarrow$ D. $\longleftrightarrow$ II.

$k(x,y) = xy\, e^{-x^2 - y^2} \longleftrightarrow$ B. $\longleftrightarrow$ III.

Example 6 (Problem #4, Exam 3, Spring 2012)

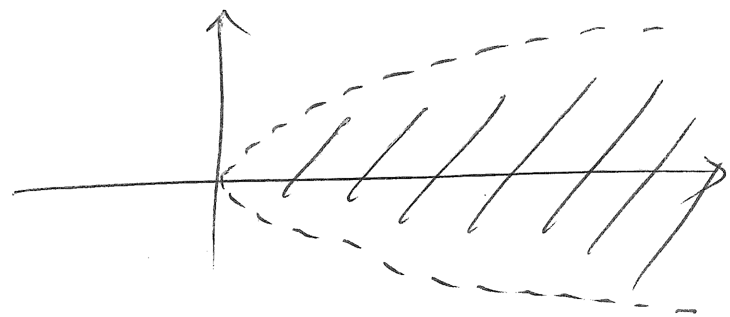
Find the largest possible domain for $f(x, y) = \ln(x - 2y^2)$.

Determine explicitly the equations of the level curves $f(x, y) = c$ and graph them in the domain of f.

$f(x,y) = \ln(x - 2y^2)$

domain: all $(x,y) \in \mathbb{R}^2$ such that

$$\boxed{x - 2y^2 > 0} \iff x > 2y^2$$



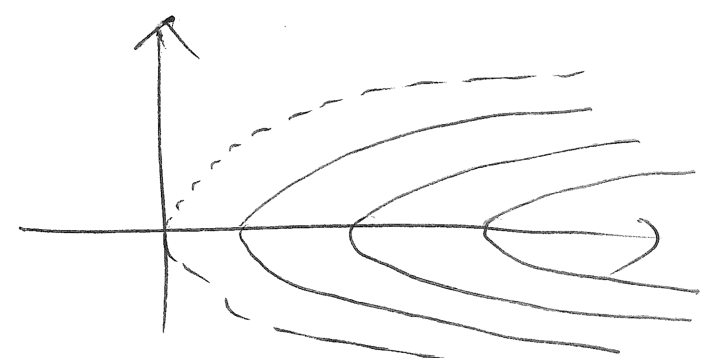
parabola not included

level curves:

$$\ln(x - 2y^2) = c \iff x - 2y^2 = e^c$$

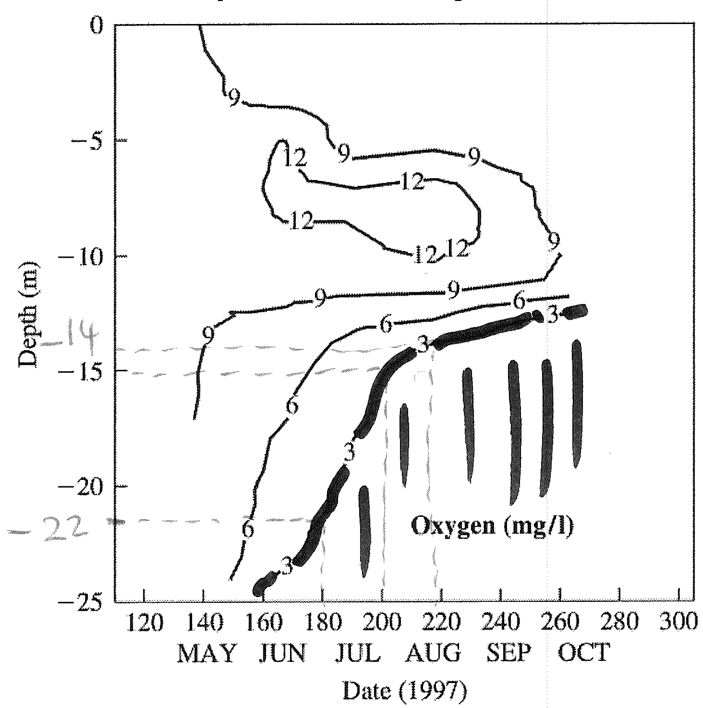
$$\boxed{x = 2y^2 + C}$$

translates of parabola



Example 7 (Problem # 25, Section 10.1, p. 512)

The picture below shows the oxygen concentration for Long Lake, Clear Water County (Minnesota). The water flea *Daphnia* can survive only if the oxygen concentration is higher than 3 mg/l. Suppose that you wanted to sample the Daphnia population in 1997 on days 180, 200, and 220. Below which depths can you be fairly sure not to find any Daphnia?



day 180: below ≈ 22

day 200: below ≈ 15

day 220: below ≈ 14

↑ approximations