MA 138 – Calculus 2 with Life Science Applications
Nonlinear Autonomous Systems: Theory
(Section 11.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Wednesday, April 26, 2017

Graphical Approach for 2×2 Systems

- We consider a system of two autonomous DEs of the form

$$\begin{cases} \dfrac{dx_1}{dt} &= f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} &= f_2(x_1, x_2) \end{cases}$$

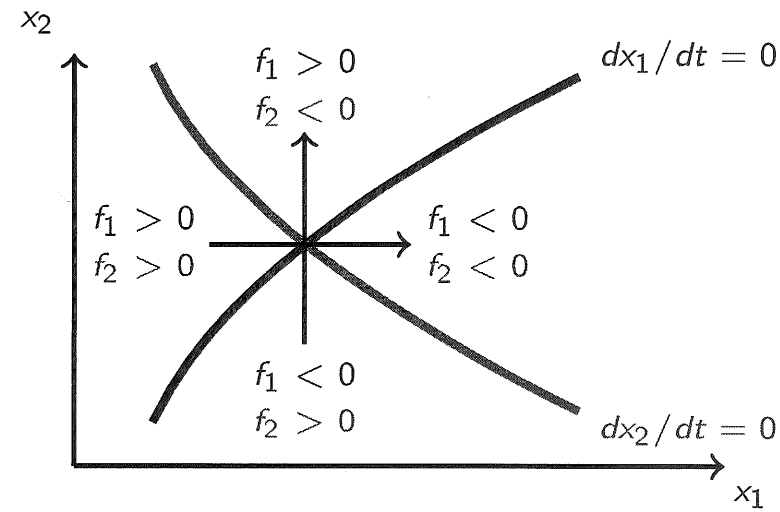
  i.e., we assume that the functions $f_i(\mathbf{x}) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ do not explicitly depend on t.

- Using vector notation, we can write the system as $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$.
- The curves

$$f_1(x_1, x_2) = 0 \qquad f_2(x_1, x_2) = 0.$$

  are called **zero isoclines** or **null clines**, and they represent the points in the x_1x_2-plane where the growth rates of the respective quantities

Let us assume that x_1 and x_2 are nonnegative; this restricts the discussion to the first quadrant of the x_1x_2-plane. The two curves in the picture on the right divide the first quadrant into four regions, and we label each region according to whether dx_i/dt (that is, f_i) is positive or negative.



The point where both null clines in the picture intersect is a point equilibrium or critical point, which we call $\widehat{\mathbf{x}}$. We can use the graph to determine the signs of the entries in the Jacobi matrix

$$D\mathbf{f}(\widehat{\mathbf{x}}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{where} \qquad a_{ij} = \frac{\partial f_i}{\partial x_j}(\widehat{\mathbf{x}}).$$

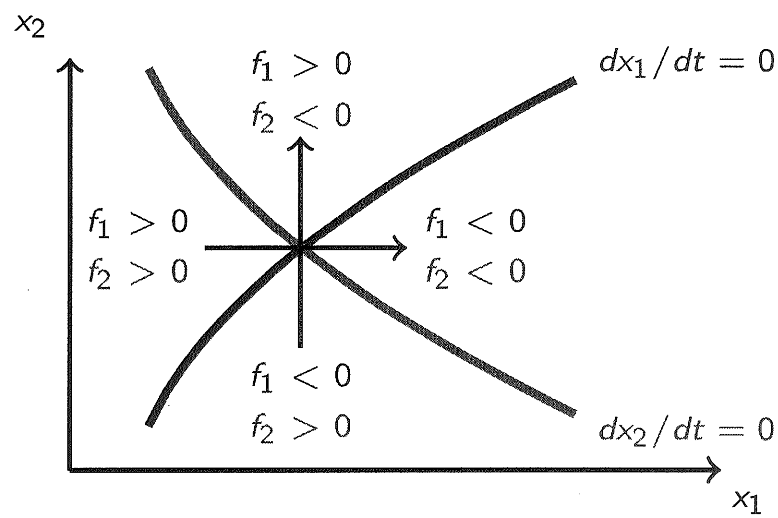
Clearly, the entry a_{11} is the effect of a change in f_1 in the x_1-direction when we keep x_2 fixed. To determine the sign of a_{11}, follow the horizontal arrow in the picture: The arrow goes from a region where f_1 is positive to a region where f_1 is negative, implying that f_1 is decreasing and hence $a_{11} < 0$.

The signs of the other three entries are found similarly, and we obtain

$$D\mathbf{f}(\widehat{\mathbf{x}}) = \begin{bmatrix} - & + \\ - & - \end{bmatrix}.$$

Thus, the trace of $D\mathbf{f}(\widehat{\mathbf{x}})$ is negative and the determinant of $D\mathbf{f}(\widehat{\mathbf{x}})$ is positive.

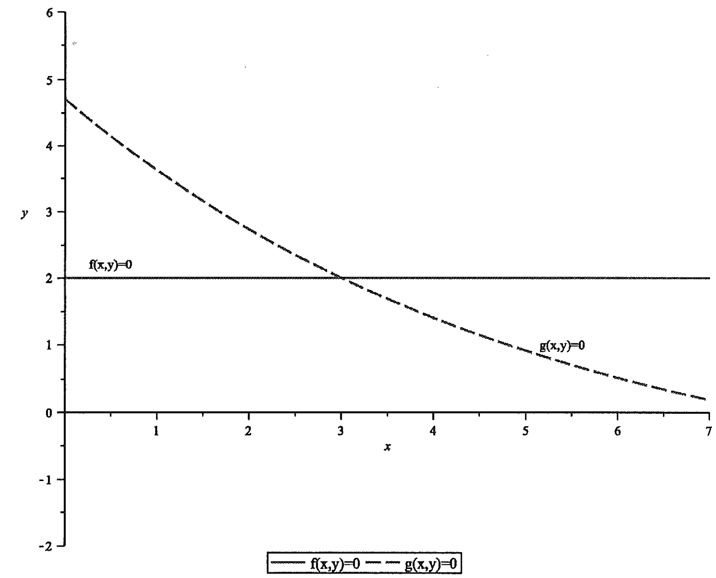
We conclude that both eigenvalues have negative real parts and, therefore, that the equilibrium is locally stable.



Example 1 (Problem #9, Exam 4, Spring 2012)

Consider the system of autonomous DEs

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

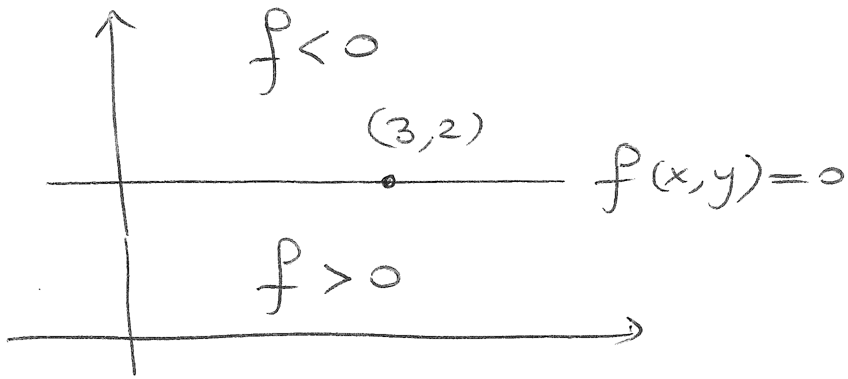


The zero isoclines (or nullclines) for this system of DEs are drawn in the picture on the side.

Assume that both functions f and g are positive in the region containing the origin and that f and g change sign when crossing their zero isoclines.

Use a graphical approach to find the sign structure of the Jacobi matrix at the equilibrium point $(3,2)$. Classify (if you can) the nature of the equilibrium point $(3,2)$. If not, say why.

Notice that the sign of $f(x,y)$ are

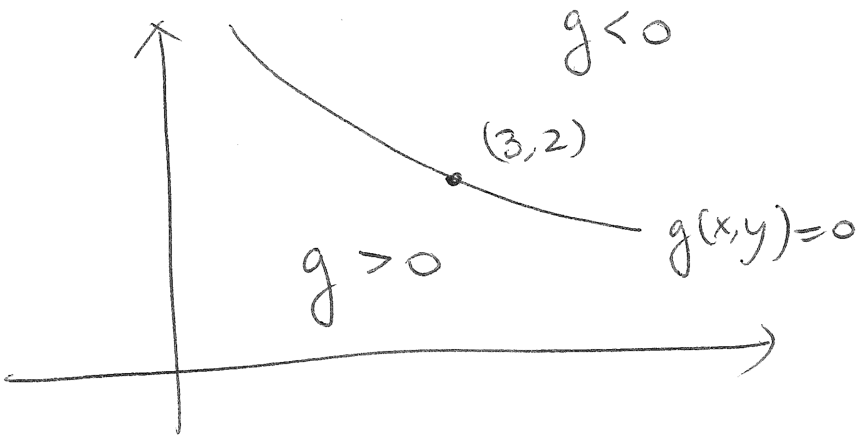


Thus:

$$\frac{\partial f}{\partial x}(3,2) = 0$$

$$\frac{\partial f}{\partial y}(3,2) < 0$$

Notice that the sign of $g(x,y)$ are



Thus

$$\frac{\partial g}{\partial x}(3,2) < 0$$

$$\frac{\partial g}{gy}(3,2) < 0$$

This means that the signs of the Jacobi matrix at the equilibrium (3,2):

$$D\begin{bmatrix} f \\ g \end{bmatrix}(3,2) = \begin{bmatrix} 0 & - \\ - & - \end{bmatrix}$$

thus

$$\boxed{\begin{array}{l} \text{trace} < 0 \\ \det < 0 \end{array}}$$

This means that the eigenvalues have different signs. Thus (3,2) is a saddle point (unstable equilibrium)

Example 2 (Example #3, Section 11.3, p. 628)

Use the graphical approach to analyze the equilibrium (3,2) of

$$\begin{cases} \dfrac{dx_1}{dt} &= 5 - x_1 - x_1x_2 + 2x_2 \\ \dfrac{dx_2}{dt} &= x_1x_2 - 3x_2 \end{cases}$$

Consider

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2) = 5 - x_1 - x_1 x_2 + 2x_2 \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2) = x_1 x_2 - 3x_2 \end{cases}$$

The equilibria are obtained by setting

$$f_1 = 5 - x_1 - x_1 x_2 + 2x_2 = 0$$

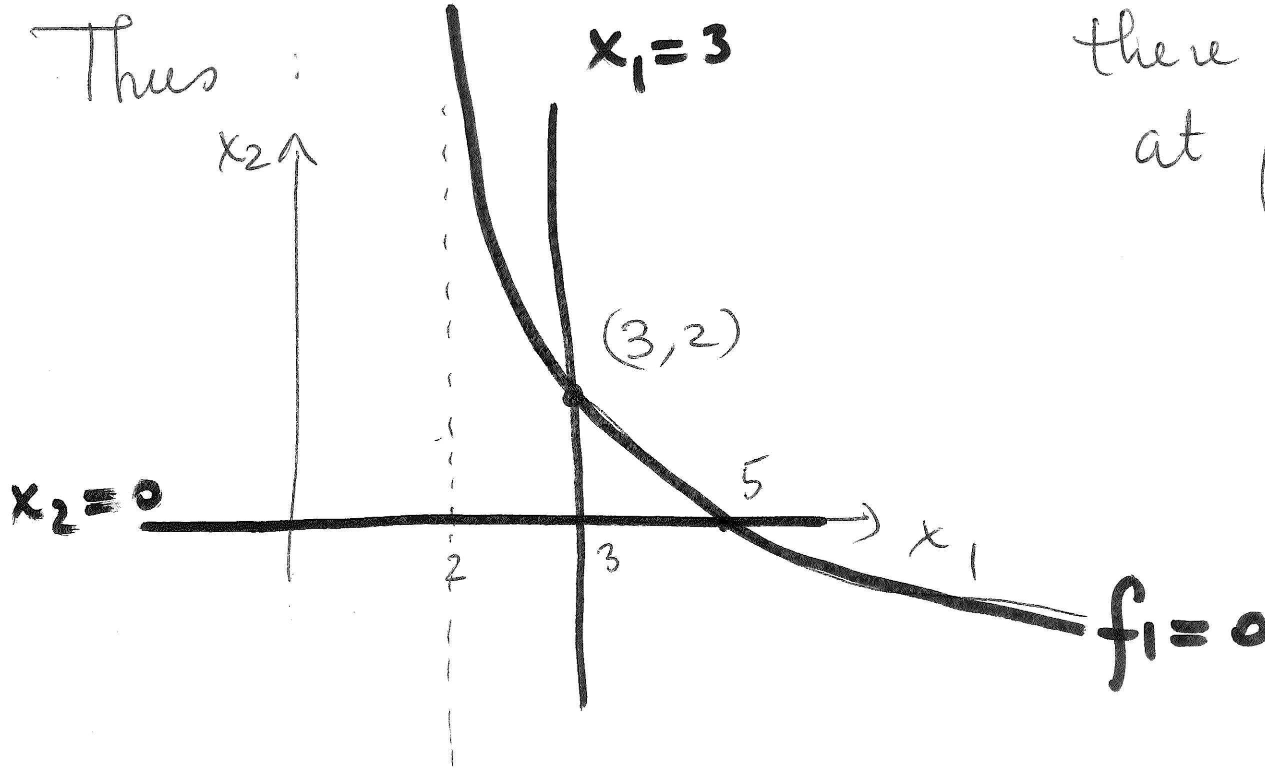
$$\iff x_2(2 - x_1) = x_1 - 5$$

$$\iff x_2 = \frac{x_1 - 5}{2 - x_1}$$

$$f_2 = x_1 x_2 - 3x_2 = 0$$

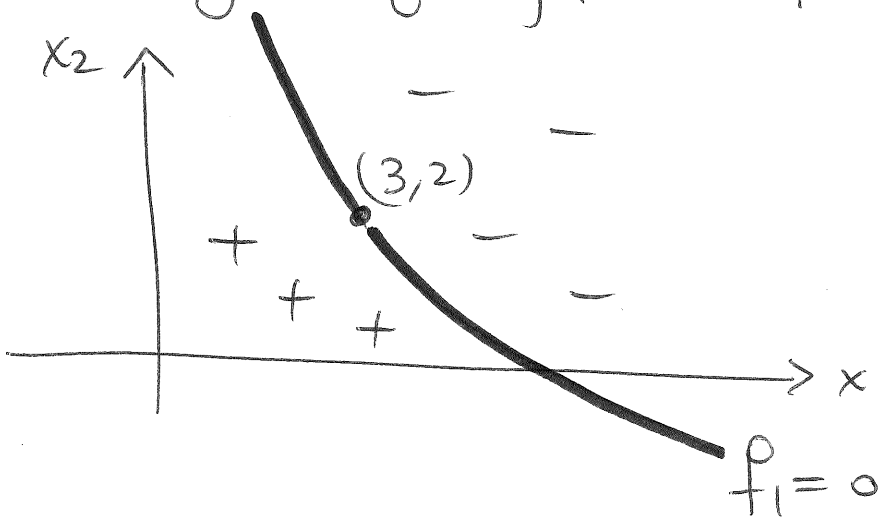
$$\iff x_2 = 0 \quad \text{or} \quad x_1 = 3$$

Thus:



there is an equilibrium at $\boxed{(3,2)}$

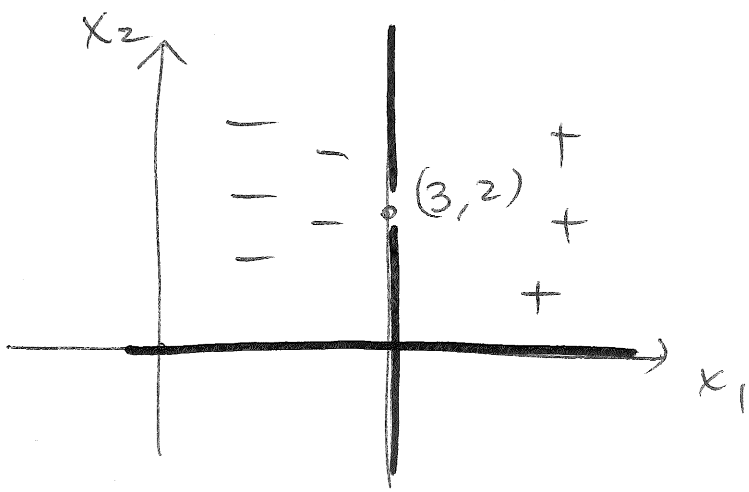
The sign of f_1 is:



Thus $\dfrac{\partial f_1}{\partial x_1} < 0$

$\dfrac{\partial f_1}{\partial x_2} < 0$

The sign of f_2 is:



Thus

$$\frac{\partial f_2}{\partial x_1} > 0$$

$$\frac{\partial f_2}{\partial x_2} = 0$$

Hence the Jacobi matrix of $\underline{f}$ at $(3,2)$ is

$$(D\underline{f})(3,2) = \begin{bmatrix} - & - \\ + & 0 \end{bmatrix}$$

trace < 0 and det > 0

Thus $(3,2)$ is a <u>stable equilibrium</u> (<u>sink</u>)

Remark

This simple graphical approach does not always give us the signs of the real parts of the eigenvalues, as illustrated in the following example: Suppose that we arrive at the Jacobi matrix in which the signs of the entries are

$$Df(\widehat{\mathbf{x}}) = \begin{bmatrix} + & - \\ - & - \end{bmatrix}.$$

The trace may now be positive or negative. Therefore, we cannot conclude anything about the eigenvalues. In this case, we would have to compute the eigenvalues or the trace and the determinant explicitly and cannot rely on the signs alone.