

MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
 (Section 7.3)

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Lecture 6

Example 1

Evaluate the following indefinite integrals

- $\int \frac{5}{(3x+2)^4} dx;$
- $\int \frac{2x-2}{(x^2-2x+5)^3} dx.$

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(1) $\int \frac{5}{(3x+2)^4} dx$ set $\boxed{u = 3x+2}$ so that $\frac{du}{dx} = 3$
 $\boxed{dx = \frac{1}{3} du}$

hence if we substitute back

$$= \int \frac{5}{u^4} \cdot \frac{1}{3} du = \int \frac{5}{3} u^{-4} du = \frac{5}{3} \cdot \frac{1}{-3} u^{-3} + C$$

$$= -\frac{5}{9} \cdot \frac{1}{u^3} + C = \boxed{-\frac{5}{9} \cdot \frac{1}{(3x+2)^3} + C}$$

(2) $\int \frac{2x-2}{(x^2-2x+5)^3} dx$ set $\boxed{u = x^2-2x+5}$ so
 $\frac{du}{dx} = 2x-2$ $\boxed{(2x-2)dx = du}$

hence

$$= \int \frac{du}{u^3} = \int u^{-3} du = -\frac{1}{2} u^{-2} + C = -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{2(x^2-2x+5)^2} + C}$$

Section 7.3: Rational Functions and Partial Fractions

- A rational function f is the quotient of two polynomials. That is,

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

- To integrate such a function, we write $f(x)$ as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$

where $A, B, C, a, b,$ and c are constants and n is a positive integer.

- In this form, the quadratic polynomial $ax^2 + bx + c$ can no longer be factored into a product of two linear functions with real coefficients.

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Proper Rational Functions

- The rational function $f(x) = P(x)/Q(x)$ is said to be **proper** if the degree of the polynomial in the numerator, $P(x)$, is strictly less than the degree of the polynomial in the denominator, $Q(x)$,

$$f(x) = \frac{P(x)}{Q(x)} \text{ proper} \iff \deg P(x) < \deg Q(x).$$

- Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \quad f_2(x) = \frac{x}{x+2} \quad f_3(x) = \frac{2x-3}{x^2+x}$$

is proper? Only $f_3(x)$ is proper.

- The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write $f(x)$ as a sum of a polynomial and a **proper** rational function.

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Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

Long Division Algorithm

If $A(x)$ and $B(x)$ are polynomials, with $B(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree strictly less than the degree of $B(x)$, such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials $A(x)$ and $B(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

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Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4 \text{ by } B(x) = x - 3.$$

$$\begin{array}{r} x-3 \overline{) 2x^2 - x - 4} \\ \underline{2x^2 - 6x} \\ + 5x - 4 \end{array}$$

(Complete the above table and check your work!)

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$$\begin{array}{r} \overline{) 2x^2 - x - 4} \\ \underline{2x^2 - 6x} \text{ subtract} \\ 5x - 4 \\ \underline{5x - 15} \text{ subtract} \\ \underline{} 11 \text{ remainder} \end{array}$$

This means that

$$2x^2 - x - 4 = (x-3) \cdot (2x+5) + 11$$

$$\begin{aligned} \text{check: } & (2x^2 + 5x - 6x - 15) + 11 \\ & = 2x^2 - x - 4 \quad \checkmark \end{aligned}$$

■ **Synthetic division** is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$, where c is a number. In synthetic division we write only the essential part of the long division table.

■ In synthetic division we abbreviate the polynomial $A(x)$ by writing only its coefficients.

Moreover, instead of $B(x) = x - c$, we simply write 'c.'

Writing c instead of $-c$ allows us to add instead of subtract!

Example 2 (revisited): Divide

$$A(x) = 2x^2 - x - 4 \text{ by } B(x) = x - 3.$$

$$\begin{array}{r|rrr} 3 & 2 & -1 & -4 \\ & & 6 & 15 \\ \hline & 2 & 5 & 11 \end{array}$$

We obtain $Q(x) = 2x + 5$ and $R(x) = 11$. That is,

$$2x^2 - x - 4 = (2x + 5)(x - 3) + 11.$$

Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write $f(x)$ as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

$$\begin{array}{r} x^2 + 4x + 3 \overline{) x^3} \\ \underline{x^3 + 4x^2 + 3x} \quad \text{subtract} \\ -4x^2 - 3x \\ \underline{-4x^2 - 16x - 12} \quad \text{subtract} \\ 13x + 12 \quad \text{remainder} \end{array}$$

This means that

$$x^3 = (x-4)(x^2+4x+3) + 13x+12$$

Thus
$$\frac{x^3}{x^2+4x+3} = \frac{(x-4)(x^2+4x+3) + 13x+12}{x^2+4x+3}$$

Now split the fraction on the right hand side as the sum of 2 fractions:

$$\frac{x^3}{x^2+4x+3} = \frac{(x-4)(x^2+4x+3)}{x^2+4x+3} + \frac{13x+12}{x^2+4x+3}$$

simplify

$$= x-4 + \frac{13x+12}{x^2+4x+3}$$

↑
this is a proper
rational fraction

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

$Q(x)$ is a product of m distinct linear factors. $Q(x)$ is thus of the form

$$Q(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_m)$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are the m distinct roots of $Q(x)$.

The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left[\frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_m}{x - \alpha_m} \right]$$

We will see in the next examples how the constants A_1, A_2, \dots, A_m are determined.

Example 3 (cont.d)

Evaluate the indefinite integral: $\int \frac{x^3}{x^2 + 4x + 3} dx$.

Note: from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x - 4) dx + \int \frac{13x + 12}{(x + 3)(x + 1)} dx.$$

We have shown before that

$$\frac{x^3}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{x^2 + 4x + 3}$$

Thus:

$$\int \frac{x^3}{x^2 + 4x + 3} dx = \int (x - 4) dx + \int \frac{13x + 12}{x^2 + 4x + 3} dx$$

\downarrow easy
 \downarrow let's look at this

$$= \frac{1}{2}x^2 - 4x + C$$

Notice: $\frac{13x + 12}{x^2 + 4x + 3} = \frac{13x + 12}{(x + 1)(x + 3)} = \frac{A}{x + 3} + \frac{B}{x + 1}$

want

for some constants A and B

$$\frac{13x + 12}{(x + 1)(x + 3)} = \frac{A}{x + 3} + \frac{B}{x + 1} = \frac{A(x + 1) + B(x + 3)}{(x + 3)(x + 1)}$$

give common denominator

$$= \frac{Ax + A + Bx + 3B}{(x + 3)(x + 1)} = \frac{(A + B)x + (A + 3B)}{(x + 1)(x + 3)}$$

This means that

$$\textcircled{13}x + \textcircled{12} = \textcircled{(A + B)}x + \textcircled{(A + 3B)} \quad \underline{\underline{\text{So}}}$$

$$\begin{cases} A + B = 13 \\ A + 3B = 12 \end{cases} \iff \begin{cases} A = 13 - B \\ A = 12 - 3B \end{cases}$$

$$\text{so } 13 - B = 12 - 3B \iff 2B = -1 \implies \boxed{B = -\frac{1}{2}}$$

$$\text{and } A = 13 - B = 13 - \left(-\frac{1}{2}\right) = \frac{27}{2}$$

This means that
$$\frac{13x+12}{x^2+4x+3} = \frac{27}{2} \cdot \frac{1}{x+3} - \frac{1}{2} \cdot \frac{1}{x+1}$$

Thus

$$\int \frac{13x+12}{x^2+4x+3} dx = \frac{27}{2} \int \frac{1}{x+3} dx - \frac{1}{2} \int \frac{1}{x+1} dx$$

$$= \frac{27}{2} \ln|x+3| - \frac{1}{2} \ln|x+1| + C$$

Thus:

$$\int \frac{x^3}{x^2+4x+3} = \frac{1}{2}x^2 - 4x + \frac{27}{2} \ln|x+3| - \frac{1}{2} \ln|x+1| + C$$

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

\Leftrightarrow

$$A(x+1) + B(x+3) = 13x + 12 \quad (*)$$

Set $x = -1$ in (*). We obtain

$$A \cdot 0 + B \cdot (-1+3) = 13(-1) + 12$$

\Leftrightarrow

$$B \cdot (2) = -1$$

\Leftrightarrow

$$B = -1/2$$

Set $x = -3$ in (*). We obtain

$$A \cdot (-3+1) + 0 = 13(-3) + 12$$

\Leftrightarrow

$$A \cdot (-2) = -27$$

\Leftrightarrow

$$A = 27/2$$

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Example 4 (Online Homework # 8)

Find the integral: $\int_2^5 \frac{2}{x^2-1} dx.$

Consider the fraction $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$.

We want to decompose as:

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \text{give common denominator}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

Thus $2 = A(x+1) + B(x-1)$

evaluate at $x=1$: $2 = 2 \cdot A + B \cdot 0 \quad \therefore A=1$

evaluate at $x=-1$: $2 = A \cdot (0) + B(-2) \quad \therefore B=-1$

Thus:
$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

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$$\begin{aligned} \text{Thus: } \int \frac{2}{x^2-1} dx &= \int \frac{1}{x-1} dx - \int \frac{1}{x+1} dx \\ &= \ln|x-1| - \ln|x+1| + C \\ &= \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$\begin{aligned} \text{Finally: } \int_2^5 \frac{2}{x^2-1} dx &= \ln \left| \frac{x-1}{x+1} \right| \Big|_2^5 = \\ &= \ln\left(\frac{4}{6}\right) - \ln\left(\frac{1}{3}\right) \\ &= \ln\left(\frac{4}{6} \cdot 3\right) = \underline{\underline{\ln(2)}} \approx \underline{\underline{0.6931}} \end{aligned}$$

Example 5 (Online Homework # 6)

Evaluate the indefinite integral: $\int \frac{1}{x(x+1)} dx$.

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We need to decompose the fraction $\frac{1}{x(x+1)}$ as

$$\frac{A}{x} + \frac{B}{x+1} \quad \text{Thus:}$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + B \cdot x}{x(x+1)}$$

$$\text{Thus } 1 = A(x+1) + Bx$$

• evaluate at $x=0$

$$1 = A \cdot 1 + B \cdot 0$$

$$\therefore \boxed{A=1}$$

• evaluate at $x=-1$

$$1 = A \cdot 0 + B(-1)$$

$$\therefore \boxed{B=-1}$$

$$(A+B)x + A = 1$$

implies

$$\begin{cases} A+B=0 \\ A=1 \end{cases}$$

$$\therefore \boxed{A=1} \quad B=-1$$

No matter which method we choose, we obtain

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

Thus

$$\begin{aligned} \int \frac{1}{x(x+1)} dx &= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx \\ &= \ln|x| - \ln|x+1| + C \\ &= \underline{\underline{\ln \left| \frac{x}{x+1} \right| + C}} \end{aligned}$$