MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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Improper Integrals

We discuss definite integrals of two types with the following characteristics:

(1) **one or both limits of integration are infinite**; that is, the integration interval is unbounded. For example

$$\int_1^{\infty} e^{-x}\,dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx;$$

(These integrals are very important in Probability and Statistics!)

(2) **the integrand becomes infinite at one or more points of the interval of integration**. For example

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \qquad \text{or} \qquad \int_0^1 \frac{1}{2\sqrt{x}}\,dx.$$

We call such integrals **improper integrals**.

Type 2: Unbounded Integrand

What if the integrand becomes infinite at one or both endpoints of the interval of integration?

- If f is continuous on $(a,b]$ and $\lim\limits_{x\longrightarrow a^+} f(x) = \pm\infty$, we define

$$\int_a^b f(x)\,dx := \lim_{c\longrightarrow a^+} \int_c^b f(x)\,dx$$

provided that this limit exists.

- If f is continuous on $[a,b)$ and $\lim\limits_{x\longrightarrow b^-} f(x) = \pm\infty$, we define

$$\int_a^b f(x)\,dx := \lim_{c\longrightarrow b^-} \int_a^c f(x)\,dx$$

provided that this limit exists.

If the limit does not exist, we say that the integral diverges.

Example 1 (Problem #12, Section 7.4, page 362)

Determine whether the improper integral

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$$

is convergent. If the integral is convergent, compute its value.

Example 2 (Problem #26, Section 7.4, page 362)

Determine whether the improper integral

$$\int_1^e \frac{1}{x \ln x}\, dx$$

is convergent. If the integral is convergent, compute its value.

Example 3 (Online Homework #7)

Determine whether the improper integral

$$\int_0^9 \frac{4}{(x-6)^2}\, dx$$

is convergent. If the integral is convergent, compute its value.

Example 4 (Problem #34, Section 7.4, page 363)

Let p be a positive real number. Show that

$$\int_0^1 \frac{1}{x^p}\, dx = \begin{cases} \dfrac{1}{1-p} & \text{for } 0 < p < 1 \\ \infty & \text{for } p \geq 1. \end{cases}$$

E.g.: $\int_0^1 \frac{1}{x}\, dx$ and $\int_0^1 \frac{1}{x^2}\, dx$ both diverge (as $p = 1, 2$, respectively).

E.g.: $\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2$ and $\int_0^1 \frac{1}{\sqrt[3]{x}}\, dx = \frac{3}{2}$ (as $p = 1/2, 1/3$, respectively).

Example 5 (Problem #15, Section 7.4, page 362)

Determine whether the improper integral

$$\int_{-1}^1 \ln|x|\, dx.$$

is convergent. If the integral is convergent, compute its value.