MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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A Comparison Result for Improper Integrals

In many cases, it is difficult (if not impossible) to evaluate an integral exactly. For example, it takes some work to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad\qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi.$$

In dealing with improper integrals, it frequently suffices to know whether the integral converges.

Instead of computing the value of the improper integral exactly, we can then resort to simpler integrals that either dominate or are dominated by the improper integral of interest.

We will explain this idea graphically.

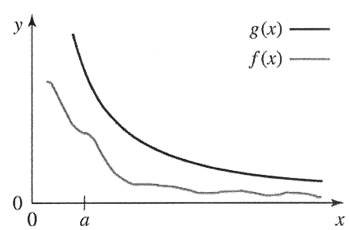
Convergence Test

Test for Convergence

We assume that $f(x) \geq 0$ for $x \geq a$.

To show that $\int_a^{\infty} f(x)\,dx$ is convergent it is enough to find a function $g(x)$ such that

- $g(x) \geq f(x)$ for all $x \geq a$;
- $\int_a^{\infty} g(x)\,dx$ is convergent.

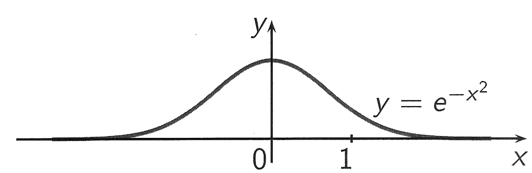


It is clear from the graph that $\quad 0 \leq \int_a^{\infty} f(x)\,dx \leq \int_a^{\infty} g(x)\,dx.$

If $\int_a^{\infty} g(x)dx < \infty$, it follows that $\int_a^{\infty} f(x)\,dx$ is convergent, since $\int_a^{\infty} f(x)\,dx$ must take on a value between 0 and $\int_a^{\infty} g(x)dx$.

$\boxed{\text{Example 1}}$ (Example #9, Section 7.4, p. 361)

Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

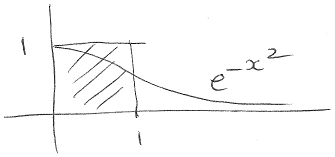


Note: It is an hard fact to show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

By symmetry

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = 2\int_0^{+\infty} e^{-x^2}\,dx$$

Now $$\int_0^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{} + \int_1^{\infty} e^{-x^2}\,dx$$



this number is finite and less than the area of the unit square

We need to show that $\int_1^{\infty} e^{-x^2}\,dx$ is finite.

But for $x \geq 1$ we have that

$$x^2 \geq x \implies -x^2 \leq -x \implies \boxed{e^{-x^2} \leq e^{-x}}$$

Thus $e^{-x^2} \leq e^{-x}$ for $x \geqslant 1$

Hence $$\int_1^{+\infty} e^{-x^2}\,dx \leq \int_1^{+\infty} e^{-x}\,dx =$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left[e^{-1} - e^{-b}\right]$$

$$= \frac{1}{e} - \underbrace{\lim_{b\to\infty}\frac{1}{e^b}}_{=0} = 1/e$$

<u>Converges</u>

Thus:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = 2\left[\int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx\right]$$

$$\leq 2\left(1 + \frac{1}{e}\right) \cong \underline{\underline{2.7358}}$$

It is a hard fact to show that actually

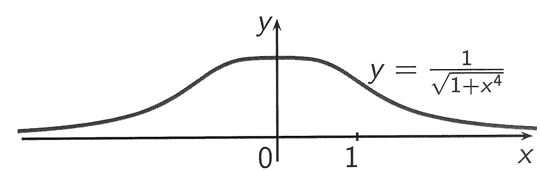
$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi} \cong \underline{\underline{1.772}}$$

We could easily show with the Comparison theorem that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx \leq 2\left(\frac{e+1}{e}\right) \cong 2.7358$$

Example 2 (Problem #36, Section 7.4, p. 363)

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx$ converges.



$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{1+x^4}}\,dx = \text{by symmetry} = 2\int_0^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx$$

$$= 2\left\{ \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx + \int_1^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx \right\}$$

Notice that $\boxed{\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \le 1}$ = area of the unit square.

For $1 \le x < \infty$ we have that $x^4 < 1+x^4$

and also $x^2 = \sqrt{x^4} < \sqrt{1+x^4}$ because $\sqrt{\cdot}$ is an increasing function.

Thus $\frac{1}{\sqrt{1+x^4}} < \frac{1}{x^2}$

and so $0 \le \int_1^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx \le \int_1^{\infty} \frac{1}{x^2}\,dx = 1$

Thus $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{1+x^4}}\,dx \le 2(1+1) = \underline{\underline{4}}$

and so it converges.

Example 3 (Online Homework # 8)

Let f(x) be a continuous function defined on the interval $[2,\infty)$ such that

$$f(4) = 7 \qquad |f(x)| < x^3 + 3 \qquad \int_4^{\infty} f(x)e^{-x/8}\,dx = -6.$$

Determine the value of

$$\int_4^{\infty} f'(x)e^{-x/8}\,dx.$$

data: $f(4)=7$; $|f(x)| < x^3+3$; $\int_4^{\infty} f(x)e^{-x/8}\,dx = -6$

Question : $\int_4^{\infty} f'(x)e^{-x/8}\,dx = ?$

Integration by parts says:

$$\boxed{\int f(x)e^{-x/8}\,dx = -8f(x)e^{-x/8} - \int -8f'(x)e^{-x/8}\,dx}$$

$$\therefore \int f'(x)e^{-x/8}\,dx = \frac{1}{8}\left[8f(x)e^{-x/8} + \int f(x)e^{-x/8}\,dx \right]$$

with the limits of integration :

$$\int_4^{\infty} f'(x)e^{-x/8}\,dx = \Big[f(x)e^{-x/8}\Big]_4^{\infty} + \frac{1}{8}\int_4^{\infty} f(x)e^{-x/8}\,dx$$

Thus:

$$\int_4^\infty f'(x)\,e^{-x/8}\,dx = \left[\underbrace{\lim_{b\to\infty} f(b)\,e^{-b/8}}\right] - f(4)\,e^{-1/2} + \frac{1}{8}\int_4^\infty f(x)\,e^{-x/8}\,dx$$

$$= 0 - (7)\cdot e^{-0,5} + \frac{1}{8}(-6)$$

$$\approx -4,9957$$

$$|f(x)| < x^3+3 \implies -(x^3+3) \le f(x) \le x^3+3$$

$$\implies -(x^3+3)\,e^{-x/8} \le f(x)\,e^{-x/8} \le (x^3+3)\,e^{-x/8}$$

Hence $\lim_{x\to\infty} f(x)\,e^{-x/8} = 0$ by the Sandwich Theorem.

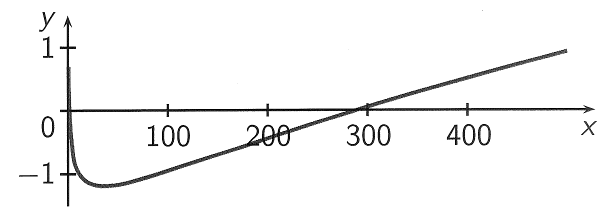
In fact $\lim_{x\to\infty}(x^3+3)\,e^{-x/8} = \lim_{x\to\infty}\dfrac{x^3+3}{e^{x/8}} = 0$ by L'Hôpital

Example 4 (Problem #8(*b*), Exam 1, Spring 14)

It is given that for $x \ge 300$ the inequality $3\ln x \le \sqrt{x}$ holds.
Use the above inequality and the Comparison Theorem for improper integrals to conclude that

$$\int_{300}^\infty e^{-\sqrt{x}}\,dx$$

coverges.



graph of $y = \sqrt{x} - 3\ln(x)$

For $x \ge 300$ we have $\boxed{3\ln x \le \sqrt{x}}$.

But: $3\ln x \le \sqrt{x} \iff -3\ln x \ge -\sqrt{x}$

$$\iff -\sqrt{x} \le -3\ln x = \ln(x^{-3})$$

Since the exponential function is an increasing function:

$$\boxed{e^{-\sqrt{x}} \le e^{\ln(x^{-3})} = x^{-3} = \frac{1}{x^3}}$$

Thus by the comparison Theorem:

$$\int_{300}^\infty e^{-\sqrt{x}}\,dx \le \int_{300}^\infty \frac{1}{x^3}\,dx \quad \text{which converges}$$

$\boxed{\text{Note}}$ that:

$$\int_{300}^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_{300}^b x^{-3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\right]_{300}^b =$$

$$= +\frac{1}{2(300)^2} - \underbrace{\lim_{b\to\infty}\frac{1}{2b^2}}_{=0} = \frac{1}{2(300)^2} < +\infty$$

$$\approx \boxed{5.5\cdot 10^{-6}}$$

$\boxed{\text{Note}}$ we can actually compute

$$\int e^{-\sqrt{x}}\,dx = \text{set } \boxed{u=\sqrt{x}} \qquad du = \frac{1}{2\sqrt{x}}\,dx$$

so $2u\,du = dx$

$$= \int 2u\,e^{-u}\,du = \text{by parts} = -2u\,e^{-u} - \int(-2e^{-u})\,du$$

$$= -2u\,e^{-u} - 2e^{-u} + C = -2e^{-u}(1+u) + C$$

$$= \boxed{-2\,e^{-\sqrt{x}}(1+\sqrt{x}) + C}$$

Thus:

$$\int_{300}^{+\infty} e^{-\sqrt{x}}\,dx = \lim_{b\to\infty} \int_{300}^{b} e^{-\sqrt{x}}\,dx =$$

$$= \lim_{b\to\infty} \left[-2\,e^{-\sqrt{x}}\left(1+\sqrt{x}\right)\right]_{300}^{b} =$$

$$= 2\,e^{-\sqrt{300}}\left(1+\sqrt{300}\right) - \underbrace{2\lim_{b\to\infty} e^{-\sqrt{b}}\left(1+\sqrt{b}\right)}_{=0 \text{ by L'Hôpital}}$$

$$= 2\,e^{-\sqrt{300}}\left(1+\sqrt{300}\right)$$

$$\simeq \underline{1.10095\cdot 10^{-6}}$$

Example 5 (Problem #9(b), Exam 1, Spring 13)

If x is really big, say bigger than 100, one has

$$-\sqrt{x} \leq -2\ln x = \ln\left(\frac{1}{x^2}\right).$$

Use this inequality together with the fact that the exponential is an increasing function to determine if

$$\int_{100}^{\infty} e^{-\sqrt{x}}\,dx$$

converges or diverges.

For $x \geq 100$ one has $-\sqrt{x} \leq -2\ln x = \ln\left(\frac{1}{x^2}\right)$

Thus $e^{-\sqrt{x}} \leq e^{\ln(1/x^2)} = \frac{1}{x^2}$ since the exponential is an increasing function.

So
$$0 \leq \int_{100}^{\infty} e^{-\sqrt{x}}\,dx \leq \int_{100}^{\infty} \frac{1}{x^2}\,dx < +\infty$$

<u>Converges!</u>

In fact
$$\int_{100}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_{100}^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_{100}^{b}$$

$$= \lim_{b\to\infty}\left[\frac{1}{100} - \frac{1}{b}\right] = \frac{1}{100}.$$

Example 6 (Problem #42, Section 7.4, p. 363)

Determine whether $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx$ is convergent or not.

By symmetry

$$\int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x}}\, dx = 2\int_0^{+\infty} \frac{1}{e^x+e^{-x}}\, dx$$

Now observe that for $x>0$ we have:

$$e^x \le e^x + e^{-x} \quad (\text{as } e^{-x} \text{ is always positive})$$

Thus $\dfrac{1}{e^x+e^{-x}} \le \dfrac{1}{e^x} = e^{-x}$, Thus

$$\int_0^{\infty} \frac{1}{e^x+e^{-x}}\, dx \le \int_0^{\infty} e^{-x}\, dx = 1 \quad \text{as seen earlier}$$

$$\therefore \boxed{\int_{-\infty}^{+\infty} \frac{1}{e^x+e^{-x}}\, dx \le 2} \quad \underline{\text{Converges}}$$

Example 7

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx \le 4.$

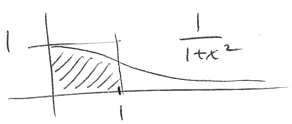
Note: We will show at the end of the lecture that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$

By symmetry $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = 2\int_0^{+\infty} \frac{1}{1+x^2}\, dx$

Now we can also consider

$$\int_0^{+\infty} \frac{1}{1+x^2}\, dx = \underbrace{\int_0^1 \frac{1}{1+x^2}\, dx}_{\le 1} + \int_1^{\infty} \frac{1}{1+x^2}\, dx$$

$\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$ is less than the area of the unit square



For $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\, dx$ observe that $x^2 < 1+x^2$

$$\Longrightarrow \frac{1}{1+x^2} \le \frac{1}{x^2}$$

Thus $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\, dx \le \int_1^{\infty} \frac{1}{x^2}\, dx = 1$ By an earlier example

Thus: $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = 2\left[\int_0^1 \frac{1}{1+x^2}\, dx + \int_1^{\infty} \frac{1}{1+x^2}\, dx\right]$

$$\le 2(1+1) = 4$$

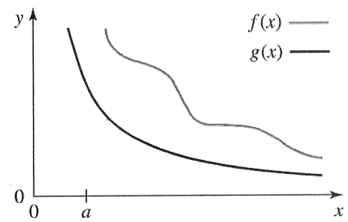
Divergence Test

Test for Divergence

We assume that $f(x) \geq 0$ for $x \geq a$.

To show that $\int_a^\infty f(x)\,dx$ is divergent it is enough to find a function $g(x)$ such that

- $g(x) \leq f(x)$ for all $x \geq a$;
- $\int_a^\infty g(x)\,dx$ is divergent.



It is clear from the graph that $\int_a^\infty f(x)\,dx \geq \int_a^\infty g(x)\,dx \geq 0.$

If $\int_a^\infty g(x)dx$ is divergent, it follows that $\int_a^\infty f(x)\,dx$ is divergent.

$\boxed{\text{Example 8}}$ (Example #10, Section 7.4, p. 361)

Show that $\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$ is divergent.

Consider $\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$. The integrand looks very messy. However notice that for $x \geq 1$

$\sqrt{x} \leq x$ so that $x + \sqrt{x} \leq x + x$.

Since the square root function is increasing

$$\sqrt{x+\sqrt{x}} \leq \sqrt{2x} \rightsquigarrow \frac{1}{\sqrt{2x}} \leq \frac{1}{\sqrt{x+\sqrt{x}}}$$

Hence:

$$0 \leq \underbrace{\frac{1}{\sqrt{2}}\int_1^\infty \frac{1}{\sqrt{x}}\,dx}_{\text{diverges as seen earlier}} \leq \int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$$

$\therefore$ our original integral diverges

The Inverse Tangent Function $\tan^{-1}(x)$

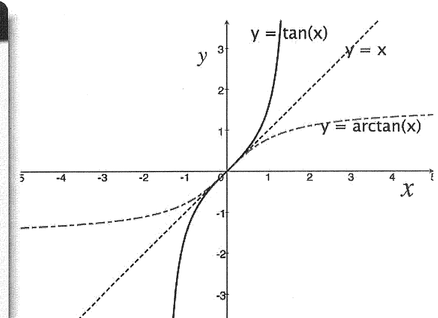
The only functions that have an inverse are one-to-one functions.
The tangent function is not one-to-one.
We can make it one-to-one by restricting its domain to the interval $(-\pi/2, \pi/2)$. Its inverse is denoted by $\tan^{-1}$ or arctan.

Inverse Tangent Function

The inverse tangent function, $\tan^{-1}$, has

- domain $\mathbb{R}$
- range $(-\pi/2, \pi/2)$.

$$\tan^{-1}(x) = y$$
$$\Longleftrightarrow$$
$$x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$



Derivative of $\tan^{-1}(x)$ (Example 4, Section 4.7, p. 187)

We want to compute $\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{dy}{dx}$.
Notice that $\tan^{-1}(x) = y$ is equivalent to $x = \tan(y)$. If we differentiate with respect to x the latter equation and apply the chain rule, we obtain

$$\frac{d}{dx}(x) = 1 = \frac{d}{dx}(\tan(y)) = \frac{d}{dy}(\tan(y)) \cdot \frac{dy}{dx} = \sec^2(y) \cdot \frac{dy}{dx}.$$

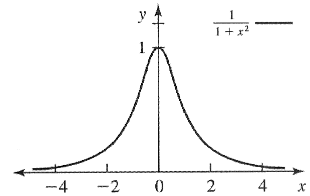
Thus

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)}.$$

We used the trigonometric identity $\sec^2(y) = 1 + \tan^2(y)$ to get the denominator in the rightmost term. Since $x = \tan(y)$, it follows that $x^2 = \tan^2(y)$, and, hence,

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}.$$

Integral of $\dfrac{1}{1+x^2}$ (Example 4, Section 7.4, p. 356)



From the previous discussion we have

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C.$$

Moreover

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= 2\int_0^{\infty} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{b\longrightarrow\infty} \int_0^{b} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{b\longrightarrow\infty} \left[\tan^{-1}(x)\right]_0^b \\
&= 2\lim_{b\longrightarrow\infty} \left[\tan^{-1}(b) - \tan^{-1}(0)\right] \\
&= 2\,(\pi/2 - 0) \quad = \quad \pi.
\end{aligned}$$