

# MA 138 – Calculus 2 with Life Science Applications

## Solving Differential Equations

(Section 8.1)

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## Differential Equations ( $\equiv$ DEs)

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

For example

$$\blacksquare \frac{dy}{dx} + 6y = 7;$$

$$\blacksquare \frac{dy}{dt} + 0.2ty = 6t;$$

$$\blacksquare \frac{dP}{dt} = \sqrt{Pt};$$

$$\blacksquare xy' + y = y^2.$$

Differential equations can contain derivatives of any order; for example,

$$\blacksquare \frac{d^2y}{dx^2} + 6\frac{dy}{dx} = xy \quad \text{or} \quad y'' + 6y' - xy = 0$$

is a DE containing the first and second derivative of the function  $y = y(x)$ .

If a differential equation contains only the first derivative,

it is called a **first-order differential equation**:  $\frac{dy}{dx} = h(x, y)$ .

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DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

**exponential growth model:**  $\frac{dN}{dt} = rN \quad N(0) = N_0;$

**logistic growth model:**  $\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) \quad N(0) = N_0;$

**von Bertalanffy models:**  $\frac{dL}{dt} = k(L_\infty - L) \quad L(0) = L_0.$

$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \quad W(0) = W_0;$$

**Solow's economic growth model:**  $\frac{dk}{dt} = sk^\alpha - \delta k \quad k(0) = k_0.$

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### Example 1

Consider the differential equation  $(t+1)\frac{dy}{dt} - y + 6 = 0$ .

Which of the following functions

$$y_1(t) = t + 7 \quad y_2(t) = 3t + 21 \quad y_3(t) = 3t + 9$$

are solutions for all  $t$ ?

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$$(t+1) \frac{dy}{dt} - y + 6 = 0$$

(1)  $y_1(t) = t+7$  so  $\frac{dy_1}{dt} = 1$

Hence  $(t+1) \cdot \frac{dy_1}{dt} - y_1 + 6 \stackrel{?}{=} 0$

becomes  $(t+1)(1) - (t+7) + 6 \stackrel{?}{=} 0$   
 $t+1 - t - 7 + 6 \stackrel{?}{=} 0$   
 $-0 = 0$  Yes

(2)  $y_2(t) = 3t+21$  so  $\frac{dy_2}{dt} = 3$

Hence  $(t+1) \cdot \frac{dy_2}{dt} - y_2 + 6 \stackrel{?}{=} 0$

becomes  $(t+1)(3) - (3t+21) + 6 \stackrel{?}{=} 0$   
 $3t+3 - 3t - 21 + 6 \stackrel{?}{=} 0$   
 $-12 = 0$  No

(3)  $y_3(t) = 3t+9$  so  $\frac{dy_3}{dt} = 3$

Hence  $(t+1) \frac{dy_3}{dt} - y_3 + 6 \stackrel{?}{=} 0$

becomes  $(t+1)(3) - (3t+9) + 6 \stackrel{?}{=} 0$   
 $3t+3 - 3t - 9 + 6 \stackrel{?}{=} 0$   
 $0 = 0$  Yes

Hence  $y_1(t)$  and  $y_3(t)$  are solutions

but  $y_2(t)$  is not a solution

## Separable Differential Equations

We will restrict ourselves to first-order differential equations

$$\frac{dy}{dx} = h(x, y) \quad \text{of the form} \quad \frac{dy}{dx} = f(x)g(y).$$

That is, the right-hand side of the equation is the product of two functions, one depending only on  $x$ ,  $f(x)$ , the other only on  $y$ ,  $g(y)$ .

Such equations are called **separable differential equations**.

This type of differential equations includes two special cases:

■ **pure-time differential equations:**  $\frac{dy}{dx} = f(x)$  [i.e.,  $g(y) \equiv 1$ ]

■ **autonomous differential equations:**  $\frac{dy}{dx} = g(y)$  [i.e.,  $f(x) \equiv 1$ ]

(DEs of this form are frequently used in biological models.)

In order to solve the separable differential equation

$$\frac{dy}{dx} = f(x)g(y), \quad (*)$$

we divide both sides of (\*) by  $g(y)$  [assuming that  $g(y) \neq 0$ ]:

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

Now, if  $y = u(x)$  is a solution of (\*), then  $u(x)$  satisfies

$$\frac{1}{g[u(x)]} u'(x) = f(x).$$

If we integrate both sides with respect to  $x$ , we find that

$$\int \frac{1}{g[u(x)]} u'(x) dx = \int f(x) dx \quad \text{or} \quad \int \frac{1}{g(y)} dy = \int f(x) dx$$

since  $g[u(x)] = g(y)$  and  $u'(x)dx = dy$ .

### Example 1 (again)

Solve the differential equation  $(t+1)\frac{dy}{dt} - y + 6 = 0$ .

$$(t+1)\frac{dy}{dt} - y + 6 = 0 \iff (t+1)\frac{dy}{dt} = y - 6$$

Separate variables and integrate

$$\frac{dy}{y-6} = \frac{dt}{t+1} \iff \int \frac{1}{y-6} dy = \int \frac{1}{t+1} dt$$

$$\implies \ln(y-6) = \ln(t+1) + C$$

take the exponential of both sides

$$e^{\ln(y-6)} = e^{\ln(t+1)+C} = e^{\ln(t+1)} \cdot e^C$$

$$\therefore y-6 = (t+1)A \quad \text{where } A = e^C$$

$$\therefore \boxed{y = A(t+1) + 6}$$

If you prefer we can write

$$y = A \cdot t + A + 6$$

$$\text{Thus for } A=1 \iff y_1(t) = t + 7$$

$$\text{for } A=3 \iff y_3(t) = 3t + 9$$

But we cannot get  $y_2(t)$ .

### Example 2 (Online Homework # 2)

Solve the following initial value problem

$$\frac{dy}{dt} + 0.2ty = 6t$$

with  $y(0) = 4$ .

$$\frac{dy}{dt} + 0.2ty = 6t \iff \frac{dy}{dt} = 6t - 0.2ty$$

$$\frac{dy}{dt} = 0.2t(30 - y) = -0.2t(y - 30)$$

Separate and integrate:

$$\frac{dy}{y-30} = -0.2t dt \iff \int \frac{1}{y-30} dt = \int -0.2t dt$$

$$\Rightarrow \ln(y-30) = -0.1t^2 + C \quad \text{take } \underline{\underline{\text{exp.}}}$$

$$e^{\ln(y-30)} = e^{-0.1t^2 + C}$$

$$\Rightarrow \boxed{y - 30 = A \cdot e^{-0.1t^2}}$$

$$\therefore y = 30 + A e^{-0.1t^2}$$

Use the initial condition  $y(0) = 4$

$$\therefore 4 = 30 + A \underbrace{e^{-0.1(0)^2}}_1 \quad \therefore \boxed{A = -26}$$

$$\therefore \boxed{y = 30 - 26 e^{-0.1t^2}}$$

### Example 3 (Online Homework # 3)

Find the solution of the differential equation

$$\frac{dP}{dt} = \sqrt{Pt}$$

that satisfies the initial condition  $P(1) = 7$ .

$$\frac{dP}{dt} = \sqrt{Pt} \quad \text{with } P(1) = 7$$

$$\frac{dP}{dt} = \sqrt{P} \cdot \sqrt{t} \implies \frac{1}{\sqrt{P}} dP = \sqrt{t} dt$$

(separate variables)

$$\int P^{-1/2} dP = \int t^{1/2} dt \implies \boxed{2P^{1/2} = \frac{2}{3}t^{3/2} + C}$$

(integrate)

(simplify by 2 on both sides)

$$\boxed{P^{1/2} = \frac{1}{3}t^{3/2} + \tilde{C}}$$

$$\tilde{C} = \frac{C}{2}$$

Constant

Now use the initial condition.

$$P^{1/2} = \frac{1}{3} t^{3/2} + \tilde{C}$$

$$P(1) = 7$$

$$\therefore \sqrt{7} = \frac{1}{3} (1)^{3/2} + \tilde{C} \quad \therefore \tilde{C} = \sqrt{7} - \frac{1}{3}$$

Hence :

$$P(t) = \left( \frac{1}{3} t^{3/2} + \sqrt{7} - \frac{1}{3} \right)^2 \quad \left| \ln \right.$$

$$= \frac{1}{9} \left( t\sqrt{t} + 3\sqrt{7} - 1 \right)^2$$

### Example 4 (Online Homework # 5)

Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies the initial condition  $y(1) = -1$ .

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$$x y' + y = y^2$$

$$y(1) = -1$$

$$x \left( \frac{dy}{dx} \right) = y^2 - y \quad \longleftrightarrow \quad \boxed{\frac{1}{y^2 - y} dy = \frac{1}{x} dx}$$

(separate variables)

now integrate

$$\int \frac{1}{y(y-1)} dy = \int \frac{1}{x} dx$$

partial fractions... check that

$$\boxed{\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}}$$

Thus :

$$\int \left( \frac{1}{y-1} - \frac{1}{y} \right) dy = \int \frac{1}{x} dx$$

$$\therefore \ln(y-1) - \ln y = \ln x + C$$

$$\therefore \ln \left( \frac{y-1}{y} \right) = \ln x + C$$

Take exponential of both sides

$$\frac{y-1}{y} = A \cdot x \quad \text{where } A = e^C$$

$$\therefore y-1 = Axy \quad ; \quad \text{where } \boxed{x=1 \quad y=-1} \quad \text{initial condition}$$

$$\text{So } -2 = A(1)(-1) \quad \therefore A = 2$$

$$\text{Thus } y-1 = 2xy$$

$$\Rightarrow y - 2xy = 1$$

$$y(1-2x) = 1$$

$$y = \frac{1}{1-2x}$$

Solution to  
our IVP

## Pure-Time Differential Equations

In many applications, the independent variable represents time. If the rate of change of a function depends only on time, we call the resulting differential equation a **pure-time differential equation**. Such a differential equation is of the form

$$\frac{dy}{dx} = f(x), \quad x \in I, \quad y(x_0) = y_0,$$

where  $I$  is an interval and  $x$  represents time; the number  $x_0$  is in the interval  $I$ .

The solution can then be written as

$$y(x) = y_0 + \int_{x_0}^x f(u) du.$$

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### Example 5 (Example # 1, Section 8.1, p. 392)

Suppose that the volume  $V(t)$  of a cell at time  $t$  changes according to

$$\frac{dV}{dt} = \sin t \quad \text{with} \quad V(0) = 3.$$

Find  $V(t)$ .

$$\frac{dV}{dt} = \sin t \quad \text{with} \quad V(0) = 3$$

$$V(t) = V(0) + \int_0^t \frac{dV}{du} du$$

$$= 3 + \int_0^t \sin(u) du$$

$$= 3 + \left[ -\cos(u) \right]_0^t =$$

$$= 3 + \left[ -\cos(t) + \cos(0) \right]$$

$$= \boxed{4 - \cos(t)}$$

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## Autonomous Differential Equations

Many of the differential equations that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not explicitly depend on  $x$ . These equations are called **autonomous differential equations**.

Formally, we can solve this autonomous differential equation by separation of variables. We begin by dividing both sides of the equation by  $g(y)$  and multiplying both sides by  $dx$ , to obtain

$$\frac{1}{g(y)} dy = dx.$$

Integrating both sides then gives  $\int \frac{1}{g(y)} dy = \int dx$ .

## Example 6 (Online Homework # 1)

Find the particular solution of the differential equation

$$\frac{dy}{dx} + 6y = 7$$

satisfying the initial condition  $y(0) = 0$ .

$$\frac{dy}{dx} + 6y = 7$$

$$y(0) = 0$$

$$\frac{dy}{dx} = 7 - 6y \iff \frac{1}{7-6y} dy = dx$$

$$\int \frac{6}{6y-7} dy = \int (-6) dx \quad \text{Thus}$$

$$\ln(6y-7) = -6x + C \quad \text{Take exp}$$

$$6y-7 = A \cdot e^{-6x} \quad \text{when } A = e^C$$

$$\boxed{6y = 7 + A e^{-6x}} \quad \text{lu}$$

when  $x=0$  then  $y=0$  is our initial condition.

So from  $6y = 7 + A e^{-6x}$  we get

$$0 = 7 + A \underbrace{e^0}_1 \quad \therefore \boxed{A = -7}$$

$$\therefore 6y = 7 - 7 e^{-6x} \quad \text{or}$$

$$\boxed{y = \frac{7}{6} (1 - e^{-6x})}$$

**Example 7** (Problem # 35, Section 8.1, p. 405)Find the general solution of the differential equation  $\frac{dy}{dx} = y^2 - 4$ .

$$\frac{dy}{dx} = y^2 - 4 \implies \frac{dy}{y^2 - 4} = dx$$

Hence after we integrate:

$$\int \frac{1}{y^2 - 4} dy = \int dx$$

Note that  $\frac{1}{y^2 - 4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$

$$= \frac{A(y+2) + B(y-2)}{(y-2)(y+2)}$$

So that we want A and B such that

$$1 = A(y+2) + B(y-2)$$

if  $y=2$  then  $1 = 4 \cdot A$ ; if  $y=-2$  then  $1 = B(-4)$ 

Thus  $\frac{1}{y^2 - 4} = \frac{1}{4} \left[ \frac{1}{y-2} - \frac{1}{y+2} \right]$

"  $\int \frac{1}{4} \left[ \frac{1}{y-2} - \frac{1}{y+2} \right] dy = \int dx$  or

$$\int \left( \frac{1}{y-2} - \frac{1}{y+2} \right) dy = \int 4 dx$$

$$\ln(y-2) - \ln(y+2) = 4x + C$$

or  $\ln\left(\frac{y-2}{y+2}\right) = 4x + C$  Take exp.

$$\frac{y-2}{y+2} = e^{4x} \cdot A \quad \text{where } A = e^C$$

$$y-2 = A e^{4x} \cdot (y+2)$$

$$y - y A e^{4x} = 2 A e^{4x} + 2$$

$$y(1 - A e^{4x}) = 2(1 + A e^{4x})$$

$$\therefore y = 2 \frac{1 + A e^{4x}}{1 - A e^{4x}}$$

Multiply top and bottom by  $e^{-4x}$ 

$$y = 2 \frac{e^{-4x} + A}{e^{-4x} - A}$$

$$\lim_{x \rightarrow \infty} y(x) = 2 \frac{0+A}{0-A} = -2$$

So,