MA 138 – Calculus 2 with Life Science Applications
Linear Maps
(Section 9.3)

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Outline

- We mostly focus on 2×2 matrices, but point out that we can generalize our discussion to arbitrary $n \times n$ matrices.
- Consider a map of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or, in short,} \qquad \mathbf{v} \mapsto A\mathbf{v}$$

  where A is a 2×2 matrix and $\mathbf{v}$ is a 2×1 (column) vector.
- Since $A\mathbf{v}$ is a 2×1 vector, this map takes a 2×1 vector and maps it into a 2×1 vector. This enables us to apply A repeatedly: We can compute $A(A\mathbf{v}) = A^2\mathbf{v}$, which is again a 2×1 vector, and so on.
- We will **first** look at vectors $\mathbf{v}$, **then** at maps $\mathbf{v} \mapsto A\mathbf{v}$, and **finally** at iterates of the map A (i.e., $A^2\mathbf{v}$, $A^3\mathbf{v}$, and so on).

Graphical Representation of (Column) Vectors

We assume that $\mathbf{v} = \begin{bmatrix} x_{\mathbf{v}} \\ y_{\mathbf{v}} \end{bmatrix}$ is a 2×1 matrix.

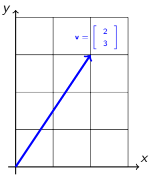
We call $\mathbf{v}$ a column vector or simply a **vector**.

Since a 2×1 matrix has just two components, we can represent a vector in the plane.

For instance, to represent the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

in the x-y plane, we draw an arrow from the origin $(0,0)$ to the point $(2,3)$.



Addition of Vectors

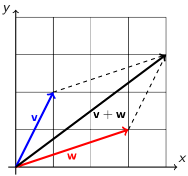
Because vectors are matrices, we can add vectors using matrix addition.

For instance,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

This vector sum has a simple geometric representation. The sum $\mathbf{v} + \mathbf{w}$ is the diagonal in the parallelogram that is formed by the two vectors $\mathbf{v}$ and $\mathbf{w}$.

The rule for vector addition is therefore referred to as the **parallelogram law**.



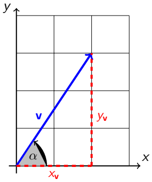
Length of Vectors

The length of the vector $\mathbf{v} = \begin{bmatrix} x_\mathbf{v} \\ y_\mathbf{v} \end{bmatrix}$, denoted by $|\mathbf{v}|$, is the distance from the origin $(0,0)$ to the point $(x_\mathbf{v}, y_\mathbf{v})$.

By Pythagoras Theorem we have

$$\text{length of } \mathbf{v} = \|\mathbf{v}\| = \sqrt{x_\mathbf{v}^2 + y_\mathbf{v}^2}$$

We define the direction of $\mathbf{v}$ as the angle α between the positive x-axis and the vector $\mathbf{v}$. The angle α is in the interval $[0, 2\pi)$ and satisfies $\tan\alpha = y_\mathbf{v}/x_\mathbf{v}$.



We thus have two distinct ways of representing vectors in the plane: We can use

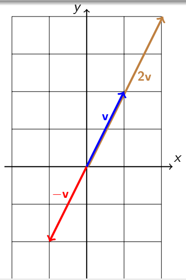
- either the endpoint $(x_\mathbf{v}, y_\mathbf{v})$
- or the length and direction $(\|\mathbf{v}\|, \alpha)$.

Scalar Multiplication of Vectors

Multiplication of a vector by a scalar is carried out componentwise.

If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 2, we get $2\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This operation corresponds to changing the length of the vector by the factor 2.

If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by -1, then the resulting vector is $-\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, which has the same length as the original vector, but points in the opposite direction.



Linear Maps (also called Linear Transformations)

We start with a graphical approach to study maps of the form

$$\mathbf{v} \mapsto A\mathbf{v}$$

where A is a 2×2 matrix and $\mathbf{v}$ is a 2×1 vector.

Since $A\mathbf{v}$ is a 2×1 vector as well, the map A takes the 2×1 vector $\mathbf{v}$ and maps it to the 2×1 vector $A\mathbf{v}$ can be thought of as a map from the plane $\mathbb{R}^2$ to the plane $\mathbb{R}^2$.

We will discuss simple examples of maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $\mathbf{v} \mapsto A\mathbf{v}$, that take the vector $\mathbf{v}$ and rotate, stretch, or contract it.

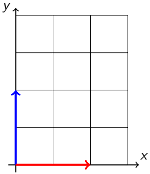
For an arbitrary matrix A, vectors may be moved in a way that has no simple geometric interpretation.

Example 1 (Reflections)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

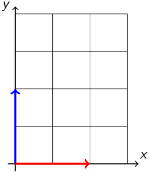


Example 2 (Contractions or Expansions)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

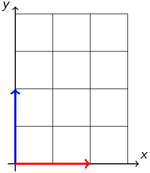


Example 3 (Shears)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

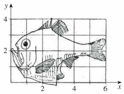
$$A_3 = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$



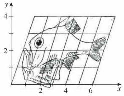
Example 4

Sir D'Arcy Wentworth Thompson (May 2, 1860 - June 21, 1948) was a Scottish biologist, mathematician, and classics scholar. He was a pioneer of mathematical biology. Thompson is remembered as the author of the distinctive 1917 book *On Growth and Form*. The book led the way for the scientific explanation of morphogenesis, the process by which patterns are formed in plants and animals.

For example, Thompson illustrated the transformation of *Argyropelecus offers* into *Sternoptyx diaphana* by applying a 20° shear mapping (≡ transvection). What is the form of the matrix that describes this change?



Argyropelecus offers



Sternoptyx diaphana

(**source**: WIKIPEDIA)

Rotations

The following matrix rotates a vector in the x-y plane by an angle α:

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If $\alpha > 0$ the rotation is counterclockwise; if $\alpha < 0$ it is clockwise.

Properties of Rotations:

- $\det(R_\alpha) = \cos^2\alpha + \sin^2\alpha = 1$.
- A rotation by an angle α followed by a rotation by an angle β should be equivalent to a single rotation by a total angle $\alpha + \beta$. In fact, using the usual trigonometric identities, we have

$$\begin{aligned} R_\alpha R_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R_{\alpha+\beta} \end{aligned}$$

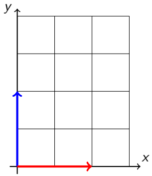
- The previous identity shows that the product of rotations is commutative: $R_\alpha R_\beta = R_\beta R_\alpha$.

Example 5 (Rotations)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



Properties of Linear Maps

According to the properties of matrix multiplication, the map $\mathbf{v} \mapsto A\mathbf{v}$ satisfies the following conditions:

- $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$, and
- $A(\lambda \mathbf{v}) = \lambda(A\mathbf{v})$, where λ is a scalar.

Because of these two properties, we say that the map $\mathbf{v} \mapsto A\mathbf{v}$ is **linear**.

Example 6 (Problem # 2, Section 9.3, p 486)

Show by direct calculation that $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$ and $A(\lambda \mathbf{v}) = \lambda(A\mathbf{v})$.

$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x + x' \\ y + y' \end{bmatrix} \\
&= \begin{bmatrix} a(x + x') + b(y + y') \\ c(x + x') + d(y + y') \end{bmatrix} \\
&= \begin{bmatrix} (ax + by) + (ax' + by') \\ (cx + dy) + (cx' + dy') \end{bmatrix} \\
&= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} + \begin{bmatrix} ax' + by' \\ cx' + dy' \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}
\end{aligned}$$

$A(\lambda \mathbf{v}) = \lambda(A\mathbf{v})$

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\lambda \begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} a(\lambda x) + b(\lambda y) \\ c(\lambda x) + d(\lambda y) \end{bmatrix} \\
&= \begin{bmatrix} \lambda(ax + by) \\ \lambda(cx + dy) \end{bmatrix} = \lambda \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \\
&= \lambda\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)
\end{aligned}$$

Example 7

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find $A\mathbf{u}$ and $A\mathbf{v}$.

Composition of Linear Maps $\equiv$ Product of Matrices

Consider two linear maps $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$ given by the matrices A_f and A_g

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

That is the coordinates are transformed according to the rules

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad \begin{cases} x'' = \alpha x' + \beta y' \\ y'' = \gamma x' + \delta y' \end{cases}$$

If we compose the two maps we obtain the transformation

$$\begin{cases} x'' = \alpha(ax + by) + \beta(cx + dy) = (\alpha a + \beta c)x + (\alpha b + \beta d)y \\ y'' = \gamma(ax + by) + \delta(cx + dy) = (\gamma a + \delta c)x + (\gamma b + \delta d)y \end{cases}$$

whose matrix representation corresponds to the product $A_g A_f$ of the two matrices

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{bmatrix}}_{A_{g \circ f}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f} \begin{bmatrix} x \\ y \end{bmatrix}$$