MA 138 – Calculus 2 with Life Science Applications
Iterated maps (Section 9.3)
Fibonacci's Numbers and a Population Model
(Handout)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

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Outline

The goal is to illustrate an application of large powers of matrices.

Our primary tools are the eigenvalues and eigenvectors of the matrix.

We then illustrate this approach with two familiar examples:

- Fibonacci's numbers;
- a simple population model.

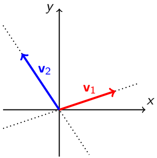
Iterated Maps

We restrict to the case in which A is a 2×2 matrix with real eigenvalues.

We saw that in this case the eigenvectors define lines through the origin that are invariant under the map A.

If the invariant lines are distinct, we say that the eigenvectors are **linearly independent**.

This notion can be formulated as follows in terms of eigenvectors: If we denote the two eigenvectors by $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if there does not exist a number c such that $\mathbf{v}_1 = c\mathbf{v}_2$.



Criterion for Linear Independence

Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2, and eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, resp. If $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

As a consequence of linear independence, we can write any vector $\mathbf{v}$ uniquely as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. That is,

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2,$$

where c_1 and c_2 are uniquely determined.

Apply now A to $\mathbf{v}$ (written as a linear combination of the two eigenvectors of A). Using the linearity of the map A, we find that

$$A\mathbf{v} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2$$

However, $\mathbf{v}_1$ and $\mathbf{v}_2$ are both eigenvectors corresponding to A. Hence, $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. We thus obtain

$$A\mathbf{v} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2.$$

This representation of $\mathbf{v}$ (namely, $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$) is particularly useful if we apply A repeatedly to $\mathbf{v}$. Applying A to $A\mathbf{v}$, we find that

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2) = c_1\lambda_1 A\mathbf{v}_1 + c_2\lambda_2 A\mathbf{v}_2 = c_1\lambda_1^2\mathbf{v}_1 + c_2\lambda_2^2\mathbf{v}_2,$$

(we used again the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of the matrix A). Continuing in this way we obtain $\boxed{A^n\mathbf{v} = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2}$.

Dominant Eigenvalue: Many biological processes correspond to matrices with a positive eigenvalue, say λ_1, strictly larger in magnitude than the other eigenvalue(s) and with the components of the associated eigenvector also positive. In this case the vector $A^n\mathbf{v}$ asymptotically approaches the line containing $\mathbf{v}_1$. In fact, if $|\lambda_2/\lambda_1| < 1$, we have

$$\lim_{n\longrightarrow\infty} \frac{A^n\mathbf{v}}{\lambda_1^n} = \lim_{n\longrightarrow\infty} \left\{ c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^n \mathbf{v}_2 \right\} = c_1\mathbf{v}_1.$$

Moreover, the relative sizes of the components of $A^n\mathbf{v}$ are proportional to the components of $\mathbf{v}_1$.

Example 1

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Find $A^{20}\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Example 2 Fibonacci's Numbers

We are all familiar with Fibonacci's sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$$

In the West, the Fibonacci sequence first appears in the book *Liber Abaci* (1202) by Leonardo of Pisa, known as Fibonacci. Fibonacci poses, and solves, a problem involving the growth of a population of rabbits based on idealized ($\equiv$ biologically unrealistic) assumptions.

What if we wanted to compute 'quickly' (this is the keyword!) the 1000$^{\text{th}}$ Fibonacci's number? Here is how matrices can help us.

Write $f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8,\ f_7 = 13, \ldots$

In other words, Fibonacci's numbers are given by the recursive relation $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, with $f_0 = 0$ and $f_1 = 1$.

Notice that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \cdots$$

That is, we can write the previous expressions as

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \quad \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \quad \begin{bmatrix} f_4 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} f_3 \\ f_2 \end{bmatrix} \quad \cdots$$

From this it also follows that

$$\begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2\begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \qquad \begin{bmatrix} f_4 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3\begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \qquad \cdots$$

In general, if we set $\mathbf{u}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ we have the *recursive relation*

$$\begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix} = \boxed{\mathbf{u}_{n+1} = A\mathbf{u}_n} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} \qquad \mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (1)$$

We can also solve (1) explicitly and produce the solution in terms of the powers of the matrix A. That is

$$\boxed{\mathbf{u}_n = A^n \mathbf{u}_0} \qquad \mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2)$$

- Notice that (1) gives us the '*transition*' (relation) between consecutive Fibonacci's numbers.
- More specifically, the equation $\mathbf{u}_{n+1} = A\mathbf{u}_n$ for $n \geq 0$ encodes the relation $f_{n+2} = f_{n+1} + f_n$ (the second relation encoded is the tautology $f_{n+1} = f_{n+1}$).

Notice that (2) gives us a way to calculate $\mathbf{u}_n$ (that is f_n and f_{n+1}) from $\mathbf{u}_0$ (that is f_0 and f_1) by means of the equation $\mathbf{u}_n = A^n \mathbf{u}_0$.

Let us compute the eigenvalues and eigenvectors of the matrix A introduced above. Despite the fact that A is rather simple, the eigenvalues and the eigenvectors of A are not 'nice' at all! The characteristic polynomial of A is

$$\det\left[\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

Hence we obtain the following eigenvectors

$$\lambda^2 - \lambda - 1 = 0 \qquad \Longleftrightarrow \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

$\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2}:}$ In order to find (one of) the eigenvector(s) $\mathbf{v}_1$ associated to λ_1 we need to solve the following system of equations

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \qquad \Longleftrightarrow \qquad (A - \lambda_1 I_2)\mathbf{v}_1 = \mathbf{0}.$$

That is we need to find the solutions of

$$\underbrace{\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}}_{A-\lambda_1 I_2} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbf{v}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since we are subtracting one of the two values that make the matrix A singular, we have that the system of two linear equations in a and b reduces to the single equation $a - \frac{1+\sqrt{5}}{2}\, b = 0$.

If we set $b = 1$ then $a = \frac{1+\sqrt{5}}{2}$. Hence we get the eigen pair

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \mathbf{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

$\boxed{\lambda_2 = \frac{1-\sqrt{5}}{2}:}$ In order to find (one of) the eigenvector(s) $\mathbf{v}_2$ associated to λ_2 we need to solve the following system of equations

$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \qquad \Longleftrightarrow \qquad (A - \lambda_2 I_2)\mathbf{v}_2 = \mathbf{0}.$$

That is we need to find the solutions of

$$\underbrace{\begin{bmatrix} 1 - \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix}}_{A-\lambda_2 I_2} \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{\mathbf{v}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since we are subtracting the other of the two values that make the matrix A singular, we have that the system of two linear equations in a and b reduces to the single equation $c - \frac{1-\sqrt{5}}{2}\, d = 0$.

If we set $d = 1$ then $c = \frac{1-\sqrt{5}}{2}$. Hence we get the eigen pair

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

Let us now rewrite the vector $\mathbf{u}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

That is we are seeking values c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{u}_0$

$$c_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff \begin{bmatrix} c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2} \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This system of linear equations leads to the solutions $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$. That is

$$\mathbf{u}_0 = \frac{1}{\sqrt{5}}\mathbf{v}_1 - \frac{1}{\sqrt{5}}\mathbf{v}_2.$$

Hence the relation $\mathbf{u}_n = A^n\mathbf{u}_0$ translates into the following

$$\mathbf{u}_n = A^n\left(\frac{1}{\sqrt{5}}\mathbf{v}_1 - \frac{1}{\sqrt{5}}\mathbf{v}_2\right) \iff \mathbf{u}_n = \frac{1}{\sqrt{5}}A^n\mathbf{v}_1 - \frac{1}{\sqrt{5}}A^n\mathbf{v}_2 \iff \mathbf{u}_n = \frac{1}{\sqrt{5}}\lambda_1^n\mathbf{v}_1 - \frac{1}{\sqrt{5}}\lambda_2^n\mathbf{v}_2$$

$$\iff \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

In other words we have

$$\mathbf{u}_n = A^n\mathbf{u}_0 = A^n(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A^n\mathbf{v}_1 + c_2A^n\mathbf{v}_2 = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2.$$

The above matrix equation translates into the following two (consistent) expressions

$$f_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad \text{and} \quad \boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}.$$

Observe that the eigenvalue $\lambda_1 = (1+\sqrt{5})/2 \approx 1.618$ (the largest, or *dominant*, eigenvalue) and the eigenvalue $\lambda_2 = (1-\sqrt{5})/2 \approx -0.618$. Hence $\lambda_2^n \to 0$ (in an oscillatory fashion) as $n \to \infty$. Thus we conclude that (for n sufficiently large)

$$\boxed{f_n = \text{closest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}.$$

The eigenvalue $\lambda_1 = (1+\sqrt{5})/2$ is also called the **golden ratio**.

Let us check the above result in the chart below

n	$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$	f_n
0	0.4472	0
1	0.7235	1
2	1.1707	1
3	1.8943	2
4	3.0649	3
5	4.9591	5
6	8.0239	8
7	12.9826	13
8	21.0059	21
9	33.9876	34
⋮	⋮	⋮

Example 3 A Population Model

If we analyze our description of the Fibonacci's numbers, we realize that we had a matrix A that was giving us a *transition* between a *set/state* $\mathbf{u}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ to the next set/state $\mathbf{u}_{n+1} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix}$. Thus, what we did for the Fibonacci's numbers can be applied to describe the dynamics of a population that have a finite number (not necessarily two) of stages in life.

Suppose that we consider an hypothetical animal that has *two life stages*:

juvenile **adult**.

Suppose that we count the number of members of this population on a weekly basis. Say

J_t = number of juveniles at week t $\qquad$ A_t = number of adults at week t.

The relation between the population during two consecutive weeks can reasonably be described as follows

$$J_{t+1} = J_t - mJ_t - gJ_t + fA_t, \tag{3}$$

where the term "$-mJ_t$" accounts for the fraction of juveniles that dies, the term "$-gJ_t$" accounts for the fraction of juveniles that becomes adult, and the term "$+fA_t$" accounts for the newborns;

$$A_{t+1} = A_t - \mu A_t + gJ_t, \tag{4}$$

where the term "$-\mu A_t$" accounts for the fraction of adults that dies and the term "$+gJ_t$" accounts for the fraction of juveniles that becomes adult.

Observe that m, g, f, μ are numbers that denote weekly rates, which we assume to be constant for each period. We can express express the relations described above in matrix form as

$$\underbrace{\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix}}_{\text{state } t+1} = \underbrace{\begin{bmatrix} 1-m-g & f \\ g & 1-\mu \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} J_t \\ A_t \end{bmatrix}}_{\text{state } t}. \tag{5}$$

If we define the vector $\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix}$ for any integer $t \geq 0$ then we can rewrite the above expression in the following recursive way

$$\mathbf{u}_{t+1} = A\mathbf{u}_t \qquad \text{with} \quad \mathbf{u}_0 = \begin{bmatrix} J_0 \\ A_0 \end{bmatrix};$$

in an explicit form we have

$$\mathbf{u}_t = A^t\mathbf{u}_0 \qquad \text{with} \quad \mathbf{u}_0 = \begin{bmatrix} J_0 \\ A_0 \end{bmatrix}.$$

Since we are interested in the dynamics of this population we have another example of large powers of a matrix.

(Numerical) Example

Suppose that $g = m = 0.5$, $f = 2$, and $\mu = 0.9$. Hence (5) becomes

$$\mathbf{u}_{t+1} = \begin{bmatrix} 0 & 2 \\ 0.5 & 0.1 \end{bmatrix} \mathbf{u}_t.$$

We can easily check that the eigen pairs are

$$\lambda_1 = 1.051 \quad \leftrightsquigarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1.9029 \\ 1 \end{bmatrix} \qquad \lambda_2 = -0.951 \quad \leftrightsquigarrow \quad \mathbf{v}_2 = \begin{bmatrix} -2.102 \\ 1 \end{bmatrix}.$$

As we discussed earlier, the general solution to our problem is

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} = A^t\mathbf{u}_0 = A^t(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A^t\mathbf{v}_1 + c_2A^t\mathbf{v}_2 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2,$$

where c_1 and c_2 are the values that allow us to rewrite the vector $\mathbf{u}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

As it happened in the case of the Fibonacci's numbers, one of the eigenvalues is dominant. Namely, $\lambda_1 = 1.051 > -0.951 = \lambda_2$.
Thus we can rewrite our solution as

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} = \lambda_1^t \left(c_1 \mathbf{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^t \mathbf{v}_2 \right) \underset{\text{as } t \to \infty}{\approx} \lambda_1^t c_1 \mathbf{v}_1.$$

It makes sense to call the **dominant** eigenvalue λ_1 the **growth rate** ($\lambda_1 = 1.051 \rightsquigarrow$ growth rate $= 5.1\%$) and the corresponding eigenvector $\mathbf{v}_1$ the **stable age structure**.
Also observe that the second term in the general solution leads to an oscillating (decaying) behavior caused by the factor $(-0.951)^t$.

As we observed earlier in the long run we have

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} \approx c_1 (1.051)^t \begin{bmatrix} 1.9029 \\ 1 \end{bmatrix}.$$

This implies that the ratio $\dfrac{J_t}{A_t} = \dfrac{c_1(1.051)^t 1.9029}{c_1(1.051)^t} = 1.9029$ is constant.
This means that in the long run the population will consist of 65.6% of juveniles and 34.4% of adults[1]. In other words there will be about 1.9 juveniles for every adult.

Remark. The above population model is an example of a *Leslie matrix*.
You can read more about Leslie matrices (even for populations with more than two life stages!) on pages 459-464 and 483-486 of our textbook *Calculus for Biology and Medicine* by Claudia Neuhauser. The example is taken from the book *Mathematical Methods in Biology* by J.D. Logan and W. Wolensky (pages 103-105).

[1] If x represents the percentage of juvenile then $100 - x$ represents the percentage of adults. Hence the equation (ratio) $x/(100 - x) = 1.9029$ gives the solution $x = 190.29/2.9029 \approx 65.6$ and $100 - x \approx 34.4$.