MA 138 – Calculus 2 with Life Science Applications
Vector Valued Functions
(Section 10.4)

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Vector-valued functions

- So far, we have considered only real-valued functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.
- We now extend our discussion to functions whose the range is a subset of $\mathbb{R}^m$ — that is, $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.
- Such functions are vector-valued functions, since they take on values that are represented by vectors:

$$\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad (x_1, x_2, \ldots, x_n) \mapsto \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

- Here, each function $f_i(x_1, \ldots, x_n)$ is a real-valued function:

$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad (x_1, x_2, \ldots, x_n) \mapsto f_i(x_1, x_2, \ldots, x_n).$$

We will encounter vector-valued functions where $n = m = 2$ in Chapter 11.

Example

As an example, consider a community consisting of two species.
Let u and v denote the respective densities of the species and $f(u, v)$ and $g(u, v)$ the per capita growth rates of the species as functions of the densities u and v.
We can then write this relationship as a map

$$\mathbf{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (u, v) \mapsto \begin{bmatrix} f(u, v) \\ g(u, v) \end{bmatrix}.$$

E.g., in the Lotka-Volterra predator-prey model: $(u, v) \mapsto \begin{bmatrix} \alpha - \beta v \\ \gamma u - \delta \end{bmatrix}$,

where α, β, γ, and δ are constants.

Review

- We have defined earlier the linearization at a point (x_0, y_0) of a real-valued function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$; namely,

$$L_f(x, y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

- We can write the above equation in matrix notation as

$$L_f(x, y) = f(x_0, y_0) + \underbrace{\begin{bmatrix} \dfrac{\partial f(x_0, y_0)}{\partial x} & \dfrac{\partial f(x_0, y_0)}{\partial y} \end{bmatrix}}_{1\times 2 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{2\times 1 \text{ matrix}}.$$

Our Goal

- Our task is to define the linearization at a point (x_0, y_0) of vector-valued functions whose domain and range are $\mathbb{R}^2$; that is,

$$\mathbf{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (x,y) \mapsto \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

- To do so, we linearize at the point (x_0, y_0) each component of $\mathbf{h}(x,y)$

$$L_f(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)$$

$$L_g(x,y) = g(x_0,y_0) + \frac{\partial g(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial g(x_0,y_0)}{\partial y}(y-y_0).$$

- We define the linearization of $\mathbf{h}(x,y)$ at the point (x_0,y_0) to be the vector-valued function $\mathbf{L}(x,y)$

$$\mathbf{L}(x,y) = \begin{bmatrix} L_f(x,y) \\ L_g(x,y) \end{bmatrix}.$$

The Jacobi (or Derivative) Matrix

We can rewrite the linearization $\mathbf{L}(x,y)$ at a point (x_0,y_0) of the vector-valued functions $\mathbf{h}(x,y)$ in the following matrix form

$$\begin{aligned}\mathbf{h}(x,y) \approx \mathbf{L}(x,y) &= \begin{bmatrix} L_f(x,y) \\ L_g(x,y) \end{bmatrix} \\ &= \begin{bmatrix} f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0) \\ g(x_0,y_0) + \frac{\partial g(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial g(x_0,y_0)}{\partial y}(y-y_0) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix}}_{\mathbf{h}(x_0,y_0)} + \underbrace{\begin{bmatrix} \frac{\partial f(x_0,y_0)}{\partial x} & \frac{\partial f(x_0,y_0)}{\partial y} \\ \frac{\partial g(x_0,y_0)}{\partial x} & \frac{\partial g(x_0,y_0)}{\partial y} \end{bmatrix}}_{(D\mathbf{h})(x_0,y_0)} \cdot \begin{bmatrix} (x-x_0) \\ (y-y_0) \end{bmatrix}\end{aligned}$$

$(D\mathbf{h})(x_0,y_0)$ is a 2×2 matrix called the **Jacobi matrix** of $\mathbf{h}$ at (x_0,y_0).

Example 1 (Problem #10, Exam 3, Spring 2012)

Consider the vector valued function $\mathbf{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$\mathbf{h}(x,y) = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}.$$

(a) Compute the **Jacobi matrix** $(D\mathbf{h})(x,y)$ and evaluate it at the point $(1,2)$.

(b) Find the linear approximation of $\mathbf{h}(x,y)$ at the point $(1,2)$.

(a) $$D\underline{h}(x,y) = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 - 3y^2 \\ 6x^2y^2 & 4x^3 + 1 \end{bmatrix}$$

where $\underline{h}(x,y) = \begin{bmatrix} h_1(x,y) \\ h_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}$

hence at $(x=1, y=2)$ we have $D\underline{h}(1,2) = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}$

(b) $$\begin{aligned}\underline{L}(x,y) &= \begin{bmatrix} h_1(1,2) \\ h_2(1,2) \end{bmatrix} + D\underline{h}(1,2) \cdot \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 10 \end{bmatrix} + \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= \begin{bmatrix} -6 + 4(x-1) - 11(y-2) \\ 10 + 24(x-1) + 9(y-2) \end{bmatrix} = \begin{bmatrix} 4x - 11y + 12 \\ 24x + 9y - 32 \end{bmatrix}\end{aligned}$$

Example 2 (Problem #46, Section 10.4, p. 536)

Find a linear approximation to

$$\mathbf{f}(x,y) = \begin{bmatrix} \sqrt{2x+y} \\ x - y^2 \end{bmatrix}$$

at $(1,2)$. Use your result to find an approximation for $\mathbf{f}(1.05, 2.05)$.

$$\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} \sqrt{2x+y} \\ x - y^2 \end{bmatrix}$$

$$(D\underline{f})(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\cancel{2}}{\cancel{2}\sqrt{2x+y}} & \frac{1}{2\sqrt{2x+y}} \\ 1 & -2y \end{bmatrix}$$

$$(D\underline{f})(1,2) = \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix}$$

$$\underline{L}(x,y) = \underbrace{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\underline{f}(1,2)} + \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{y}{4} + 1 \\ x - 4y + 4 \end{bmatrix}$$

$$\underline{f}(1.05, 2.05) \approx \underline{L}(1.05, 2.05) = \begin{bmatrix} 2.0375 \\ -3.15 \end{bmatrix}$$

exact value $= \begin{bmatrix} 2.03715 \\ -3.1525 \end{bmatrix}$

Example 3 (Example # 9, Section 10.4, p. 534)

Consider the function $\mathbf{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (x,y) \mapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$, with

$$u(x,y) = y\,e^{-x} \qquad \text{and} \qquad v(x,y) = \sin x + \cos y.$$

Find the linear approximation to $\mathbf{f}(x,y)$ at $(0,0)$.
Compare $\mathbf{f}(0.1, -0.1)$ with its linear approximation.

$$\underline{f}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} y e^{-x} \\ \sin x + \cos y \end{bmatrix}$$

$$(D\underline{f})(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos x & -\sin y \end{bmatrix}$$

$$(D\underline{f})(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Linear approximation of $\underline{f}$ at $(0,0)$

$$\underline{L}(x,y) = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\underline{f}(0,0)} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} = \begin{bmatrix} y \\ 1+x \end{bmatrix}$$

$$\underline{f}(0.1, -0.1) = \begin{bmatrix} -0.09048 \\ 1.0948 \end{bmatrix} \approx \underline{L}(0.1, -0.1) = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}$$

The General Case

- Consider the function $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, say $\mathbf{f}(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

  where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, are m real-valued functions of n variables.
- The Jacobi matrix of $\mathbf{f}$ is an $m \times n$ matrix of the form

$$(D\mathbf{f})(x_1, \ldots, x_n) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

- The linearization of $\mathbf{f}$ about the point $(x_1^*, \ldots, x_n^*)$ is then

$$\mathbf{L}(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1^*, \ldots, x_n^*) \\ \vdots \\ f_m(x_1^*, \ldots, x_n^*) \end{bmatrix} + (D\mathbf{f})(x_1^*, \ldots, x_n^*) \cdot \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$