MA162: Finite mathematics

Matrix Inverses

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SCHEDULE: Solutions

Last Time

- Last time we saw that matrices and vector possess an interesting algebraic structure, which allows us to add, subtract, and multiply matrices (provided certain size conditions are satisfied)

The Identity Matrix

- Given a positive integer n, the $n \times n$ identity matrix is the $n \times n$ matrix with ones along the main diagonal and zeros everywhere else.
- The 2×2 identity

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The 4×4 identity

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The $n \times n$ identity is usually denoted by I_n. Sometimes, we write I, when the size is clear from context.

Properties of the identity

- I is the matrix analogue of the number 1 in real number algebra
- If A is an $n \times m$ matrix, then

$$AI_m = A$$

and

$$I_nA = A$$

In the same way that that

number $\times$ 1 = number

i.e

$3 \cdot 1 = 3$

$1 \cdot 5 = 5$ etc.

Matrix Inverses

- The multiplicative inverse of a real number a is the real number a^{-1} satisfying

$$aa^{-1} = 1$$

- For real numbers, the multiplicative inverse is just the reciprocal. For example, $4^{-1} = \frac{1}{4}$ since $4 \cdot \frac{1}{4} = 1$.
- The inverse of matrix A is the matrix A^{-1} with the property that

$$AA^{-1} = I$$

- There is no simple "reciprocal operation" for computing A^{-1}
- Only square matrices can have inverses, and not all square matrices have inverses.

Computing Inverse of a Diagonal Matrix

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Need a matrix B so that

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Try $B = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

It's usually not this easy!!

$$AB = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5\cdot\frac{1}{5}+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+(-4)(-\frac{1}{4})+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+\frac{1}{3}\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Computing Inverse of a Triangular Matrix

Augmented matrix with Full identity on Right side.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix} = A$$

We find inverses use a variant of Gauss-Jordan.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \mapsto -\frac{1}{4}R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

Aleady a pivot so start at bottom right

$$\xrightarrow[R_2 \mapsto R_2 - R_3]{R_1 \mapsto R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 1/2 \\ 0 & \cancel{2} & 0 & 0 & 1/2 & 1/8 \\ 0 & 0 & 1 & 0 & 0 & -1/4 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cancel{1} & -\frac{3}{2} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{8} \\ 0 & 1/2 & 1/8 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$

Matrix Inverses

Find the inverse to $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. We need to find a matrix so that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying out, we get

$$\begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the first column we find $a+c=1$ and $a+2c=0$. Putting these in the top row, Replacing the second with the second minus the first, we obtain $c=-1$ and $a=2$.
From the second column we find $b+d=0$ and $b+2d=1$. Replacing the second with the second minus the first, we obtain $d=1$ and $b=-1$.
Therefore, the inverse matrix must be

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Matrix Inverses

A different point of view:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

can be viewed as

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which correspond to solving the two systems

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right]$$

We could use Gauss-Jordan on both of these augmented matrices. But since the have the same left hand sides, we can combine and do Gauss-Jordan a single time on this matrix

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

Using Gauss-Jordan to find inverse

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

General Process: To find the matrix inverse of an $n \times n$ matrix, form the augmented matrix

$$[A|I_n]$$

Then apply the Gauss-Jordan process, attempting to get

$$[I_n|B]$$

If you can reduce the left side to the identity, then A has an inverse and the inverse is $A^{-1} = B$.
If the left hand side cannot be reduced to the identity, then A does not have an inverse.

Solving systems of equations via inverses

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & + & 2y & + & 3z & = & 4 \\ x & + & 4y & + & 9z & = & 6 \end{array}$$

can be expressed as $AX = B$ and so we can solve for X by apply A^{-1} to each side: $X = A^{-1}B$. Lets find the inverse:

To Find A^{-1}

New Pivot

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \xrightarrow{R_3 \mapsto \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_2 \mapsto R_2 - 2R_3]{R_1 \mapsto R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

Solving systems of equations via inverses

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \Longrightarrow A^{-1} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

So x, y, and z can be found by finishing the above matrix multiplication.

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}}_{3\times 3}\underbrace{\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}}_{3\times 1}.$$

Result will be 3×1.

$$= \begin{bmatrix} 3\cdot 2 + (-\frac{5}{2})4 + \frac{1}{2}\cdot 6 \\ -3\cdot 2 + 4\cdot 4 + (-1)6 \\ 1\cdot 2 + (-\frac{3}{2})4 + (\frac{1}{2})6 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

One advantage to solving systems using matrix inverses is that we can now easily solve lots of systems, provided the left side (coefficient matrix) is fixed.

For example,

old left side

New right.

$$\begin{aligned} x + y + z &= -3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 7 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$$

Already did the hard work of computing inverse!

$$\Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3(-3) + (-\frac{5}{2})(4) + \frac{1}{2}\cdot 7 \\ (-3)(-3) + 4\cdot 4 + (-1)7 \\ 1(-3) + (-\frac{3}{2})4 + \frac{1}{2}\cdot 7 \end{bmatrix}$$

If the left side changed, then we'd need to recompute the inverse.

$$= \begin{bmatrix} -14.5 \\ 18 \\ -5.5 \end{bmatrix}.$$