MA162: Finite mathematics
Linear Programming: Introducing the Simplex Algorithm

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SCHEDULE:

Standard Linear Programming Problem

A standard Linear Programming Problem is one in which

- The objective function is to be **maximized**

- All of the variables in the problem are **non-negative**

- All other linear constraints can be written so that the expression involving the variables is less than or equal to a nonnegative constant.

Slack Variables

- Basic idea: *equalities* are easier to deal with than *inequalities*

- We will convert *inequalities* into *equalities*.

- HOW? Consider the inequality $x + y \leq 100$, with $x \geq 0$ and $y \geq 0$.

- Lets introduce another variable, say z, with $z \geq 0$. z measures the extent to which $x + y$ is below 100. Then

$$x + y + z = 100$$

and $x \geq 0$, $y \geq 0$, $z \geq 0$ is equivalent to the original inequality involving x and y.

- z is called a **slack variable.**

More on Slack Variables

- Slack variables are not just abstract mathematical creations. They measure useful things!

- Farmer has 100 acres to plant corn and wheat.

- x is number of acres of corn, y number of acres of wheat.

- $x + y \leq 100$ says total of corn and wheat cannot exceed the total amount of land.

- $x + y + z = 100$? In this case, z measures the amount of unused land.

Example of Simplex Algorithm

Winston Furniture Company manufactures tables and chairs. Each table requires 40 board feet of wood and 3 labor-hours. Each chair requires 16 board feet of wood and 4 labor-hours. Profit for each table is $45 and profit for each chair is $20. In a certain week, the company has 3200 board feet available and 520 labor-hours available. How many tables and how many chairs should they produce to maximize their profit?

Inequality version

- Let x denote number of tables, y number of chairs.

- Board feet: $40x + 16y \leq 3200$

- Labor-hours: $3x + 4y \leq 520$

- Reality check: $x \geq 0, y \geq 0$

- Objective: Maximimize $P = 45x + 20y$

Slack version

- Let u denote unused board feet, v unused labor.

- Board feet: $40x + 16y + u = 3200$

- Labor-hours: $3x + 4y + v = 520$

- Reality check: $x \geq 0, y \geq 0, u \geq 0, v \geq 0$

- Objective: Maximimize $P = 45x + 20y$

Building a simplex-tableaux

- We put the two linear constraints (Board-feet and Labor-hours) and profit into an augmented matrix

- $$\begin{bmatrix} 40 & 16 & 1 & 0 & 0 & 3200 \\ 3 & 4 & 0 & 1 & 0 & 520 \\ -45 & -20 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- The bottom row corresponds to $P = 45x + 20y$, but rewritten as $-45x - 20y + P = 0$.

- Currently, x and y are Non-basic variables and u, v and P are Basic variables.

- The corner points correspond to setting the Non-basic variables to 0.

- Currently, set $x = 0$ and $y = 0$.

- Above matrix tells us $u = 3200$, $v = 520$ and $P = 0$.

- This tells us that if we make NO tables and NO chairs, then we will have 3200 board feet of wood left over, 520 unused labor, and we earn 0 profit.

Example of Simplex Algorithm

- KEY IDEA of the Simplex Algorithm: Perform row operations to change which variables are basic and which variables are non-basic.

- Do this in a smart way, so that profit *increases*, and all of the variables remain nonnegative.

- How to choose row operation: Look at profit row and identify entry with most negative value. (If all values in profit row are nonnegative, then you are done.)

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$$\begin{bmatrix} 40 & 16 & 1 & 0 & 0 & 3200 \\ 3 & 4 & 0 & 1 & 0 & 520 \\ -45 & -20 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- In this case, we look at column 1. (One additional table brings in \$45 of profit compared to \$20 of profit from one additional chair.)

Example of Simplex Algorithm

- Now choose which row. How to decide?

- Look at the rows, excluding the bottom row. From these, select the rows with positive entries in the pivot column.

- For each of the selected rows, divide the entry on the right hand side by the corresponding entry in the pivot column.

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$$\left[\begin{array}{ccccc|c} 40 & 16 & 1 & 0 & 0 & 3200 \\ 3 & 4 & 0 & 1 & 0 & 520 \\ -45 & -20 & 0 & 0 & 1 & 0 \end{array}\right]$$

- In this case, $3200/40 = 80$ and $520/3 = 173.33$. We use the first row to pivot, since the ratio is smaller. (Reasoning: "If we increase number of tables, we will run out of wood before we run out of labor")

Example of Simplex Algorithm

So, first row, first column will be our pivot entry.

$$
\begin{bmatrix}
40 & 16 & 1 & 0 & 0 & 3200 \\
3 & 4 & 0 & 1 & 0 & 520 \\
-45 & -20 & 0 & 0 & 1 & 0
\end{bmatrix}
\xrightarrow{R_1 \mapsto R_1/40}
\begin{bmatrix}
1 & 0.4 & 0.025 & 0 & 0 & 80 \\
3 & 4 & 0 & 1 & 0 & 520 \\
-45 & -20 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
\xrightarrow[R_2 \mapsto R_2 - 3R_1]{R_3 \mapsto R_3 + 45R_1}
\begin{bmatrix}
1 & 0.4 & 0.025 & 0 & 0 & 80 \\
0 & 2.8 & -0.075 & 1 & 0 & 280 \\
0 & -2 & 1.125 & 0 & 1 & 3600
\end{bmatrix}
$$

- Now x, v and P are the basic variables, and y and u are non-basic variables.

- Set non-basic to zero and solve for non-basic: $x = 80$, $v = 280$, $P = 3600$.

- In plain english: "Make 80 tables, 0 chairs, earn \$3600 in profit. We have 0 unused board feet of wood and 280 unused labor hours."

Example of Simplex Algorithm

- Is this best possible? NO, there is still a negative entry in the profit row.
- The next pivot entry will correspond to the most negative entry in the profit row, in this case, column 2.
- How to decide which row? Select each row with positive entry in new pivot column, and divide the right hand side of these rows by the entries in the pivot column.
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$$\begin{bmatrix} 1 & 0.4 & 0.025 & 0 & 0 & 80 \\ 0 & 2.8 & -0.075 & 1 & 0 & 280 \\ 0 & -2 & 1.125 & 0 & 1 & 3600 \end{bmatrix}$$

- In this case, $80/0.4 = 200$ and $280/2.8 = 100$. We use the second row to pivot, since the ratio is smaller. (Reasoning: "If we increase number of chairs, we will run out labor before we run out of tables")

So, second row, second column will be our pivot entry.

$$\begin{bmatrix} 1 & 0.4 & 0.025 & 0 & 0 & | & 80 \\ 0 & 2.8 & -0.075 & 1 & 0 & | & 280 \\ 0 & -2 & 1.125 & 0 & 1 & | & 3600 \end{bmatrix}$$

$$\xrightarrow{R_2 \mapsto R_2/2.8} \begin{bmatrix} 1 & 0.4 & 0.025 & 0 & 0 & | & 80 \\ 0 & 1 & -0.0268 & 0.3571 & 0 & | & 100 \\ 0 & -2 & 1.125 & 0 & 1 & | & 3600 \end{bmatrix}$$

$$\xrightarrow[R_1 \mapsto R_1 - 0.4R_2]{R_3 \mapsto R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 0.0357 & -0.1429 & 0 & | & 40 \\ 0 & 1 & -0.0268 & 0.3571 & 0 & | & 100 \\ 0 & 0 & 1.0714 & 0.7143 & 1 & | & 3800 \end{bmatrix}$$

- Now x, y and P are the basic variables, and v and u are non-basic variables.
- Set non-basic to zero and solve for basic: $x = 40$, $y = 100$, $P = 3800$.
- In plain english: "Make 40 tables, 100 chairs, earn \$3800 in profit. We have 0 unused board feet of wood and 0 unused labor hours."

Example of Simplex Algorithm

- Is this best possible?

- YES.

- WHY?

- Bottom row can be interpreted as

$$P = 3800 - 1.0714u - 0.7143v$$

  P can only be increased if u or v are decreased. But u and v are currently 0, so they cannot be decreased any further.

- Therefore, 40 tables, 100 chairs, $3800 profit, no unused wood or labor gives the best solution.

Why use simplex algorithm?

- The ease of the method of corners relies on having few variables, which results in low dimensional geometry.

- Simplex algorithm doesn't directly use the geometry, so simplex algorithm can be used regardless of the number of variables.

- Method of Corners requires us to look at ALL corners. Simplex Algorithm can find the optimal corner without visiting every corner.

- Simplex Algorithm returns more information: it tells us which resources were not fully used. This extra information can be very useful in management decisions.