MA162: Finite mathematics

Financial Mathematics

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February 3, 2014

SCHEDULE:

Loans

- An amount \$P is borrowed. (P stands for principal, or present value)
- The loan is to be repaid by making *regular* payments of size \$R and the end of each period for the next n periods.
- Interest rate is i per period.
- Then
$$P = R \cdot \frac{1 - (1 + i)^{-n}}{i}$$
- In Excel, P can be computed by $=$*PV(i,n,R)*.
- In WeBWorK, P can be computed by *R * PV(i,n)*.

Ex. 1: Car Loan

- Murray just purchased a car. The price of the car was $\$15,000$.
- He makes a $4000 down payment takes out a car loan to cover the rest.
- He has to make payments at the end of each month for the next 4 years.
- The interest on the loan is 6% APR compounded monthly.
- Determine the size of Murray's monthly payment.

Direct application of the loan formula. $n = 4 \cdot 12 = 48$, $i = 0.06/12 = 0.005$, *and* $P = \$11,000$ *(car is worth $15,000 but he paid $4000 up front.)*
Then

$$11000 = R \cdot \frac{1 - (1.005)^{-48}}{0.005}$$

and solving for R, *we get* $R = \$258.34$.

Ex. 1: Car Loan (Continued)

- What is the total amount of interest that Murray pays?
  Murray makes 48 payments of \$258.34, so in total he pays back $48 \cdot 258.34 = 12,400.32$. *He borrowed \$11,000, so* $\$12,400.32 - \$11,000 = \$1,400.32$ *was paid to interest.*

- How much of Murray's first payment is due to interest?
  The first payment is due at the end of the first month. He borrowed \$11,000 and interest accrues at $i = 0.005$ *per month. So his outstanding balance right before the first payment is* $\$11,000 \cdot (1.005) = \$11,055$. *Therefore, \$55 of his first payment covers interest, and the remaining* $258.34 - 55 = \$203.34$ *is applied towards reducing the principal.*

Ex. 1: Car Loan (Continued)

- It is now 2.5 years from the time Murray took out his car loan and Murray just made the 30^{th} payment on his car.
- How much would he need to pay now in order to pay off the rest of his loan[1]? *1.5 years, or 18 months, remain on the loan. The outstanding balance is therefore*

$$P = 258.34 \cdot \frac{1 - 1.005^{-18}}{0.005} = \$4,436.41$$

- What is the total amount of interest that Murray pays assuming he pays off the balance in full immediately after the 30^{th} payment? *Murray made 30 payments of \$258.34 and a payment of \$4,436.41. All in all, he paid* $30 \cdot 258.34 + 4436.41 = \$12,186.61$ *whereas he borrowed \$11,000. Thus,* $\$12,186.61 - \$11,000 = \$1,186.61$ *is paid in interest.*

[1] assuming no “early pay-off fees”

Ex. 2: Home-a-loan

- Norah has a 15 year home mortgage.
- She needs to pay $2300 at the end of each month for the next 15 years.
- The interest on the loan is 3.625% APR compounded monthly.
- She is having trouble affording the $2300 per month.
- To lower her monthly payment, she is going to refinance to a 30 year loan which has 4.5% APR compounded monthly.

Ex. 2: Home-a-loan

- Determine the size of her new monthly payment.
  First, we need to determine the amount borrowed:

  $$P = 2300 \cdot \frac{1 - (1 + 0.03625/12)^{-12 \cdot 15}}{(0.03625/12)} = \$318,985.12$$

  Now we use the loan formula a second time, but we use $P = \$318,985.12$*,* $i = 0.045/12$*, and* $n = 30 \cdot 12 = 360$*.*

  $$318,985.12 = R \cdot \frac{1 - (1 + 0.045/12)^{-360}}{(0.045/12)}$$

  Evaluating and solving for R, we find the new payment is \$1616.25.

Ex. 2: Home-a-loan

- Determine the total interest charges on the original loan. *Total paid back is* $\$2300 \cdot 12 \cdot 15 = \$414,000$ *and she had borrowed* $\$318,985.12$*, so she pays* $\$414,000 - \$318,985.12 = \$95,014.88$ *in interest*

- Determine the total interest charges on the new loan. *Total paid back is* $\$1616.25 \cdot 12 \cdot 30 = \$581,850$ *and she had borrowed* $\$318,985.12$*, so she pays* $\$581,850 - \$318,985.12 = \$262,864.88$ *in interest*

Ex. 3: Chance to buy a ranch I

- Blanch can't pass up the chance to buy a ranch.
- She will borrow \$400,000.
- She will pay back this loan by making quarterly payments at the end of each quarter for 30 years.
- Interest on the loan is 6.2% APR compounded quarterly.
- Determine the size of Blanch's quarterly payments.
  Use the loan formula with $P = 400,000$*,*
  $i = 0.062/4 = 0.0155$ *and* $n = 4 \cdot 30 = 120$

$$400,000 = R\frac{1 - (1.0155)^{-120}}{0.0155}$$

Evaluating and solving for R*, we find* $R = \$7362.63$

Ex. 3: Chance to buy a ranch II

- Determine the interest charges on the loan.
  Blanch makes 120 payments of \$7362.63, so she pays back a total of $120 \cdot 7362.63 = \$883,515.60$*. She had borrowed* $\$400,000$ *so the interest expense is* $\$883,515.60 - \$400,000 = \$483,515.60$

Ex. 3: Chance to buy a ranch I

- Blanch suspects she can drastically cut her interest expenses if she is able to make quarterly payments that are larger than required.
- Supposing that Blanch pays twice her scheduled payment each month, determine how many payments Blanch needs to make before she pays off the loan.
  The new payments are $14,725.26. Now,

$$\$400,000 = \$14,725.26\frac{1-(1.0155)^{-n}}{0.0155}$$

Multiply both sides by 0.0155 and divide both sides by 14725.26, we get

$$0.421045197 = 1-(1.0155)^{-n}$$

so $(1.0155)^{-n} = 0.578954803$ *and so* $-n \cdot \ln(1.0155) = \ln(0.578954803)$, *and so* $n = 35.53\ldots$

Ex. 3: Chance to buy a ranch II

- Determine Blanch's interest charges on the loan if she makes double payments.
  This is only an approximation. She makes 35.53 payments of $\$14,725.26$, *so she pays back* $35.53 \cdot 14725.26 = \$523,188.49$
  She borrowed $\$400,000$ *so her interest charges are $123,188.49.*

The previous solution suggested that Blanch should make 35.53 payments. In reality, she can only make a whole number of payments. Lets figure out how much she owes after the 35th payment.

The present value of the first 35 payments is

$$P = 14,725.26 \frac{1 - (1.0155)^{-35}}{0.0155} = \$395,475.59$$

whereas the present value of the entire loan is \$400,000. Therefore, the outstanding balance after the 35th payment is $\$400,000 - \$395,475.59 = \$4,524.41$. This value is measured at time zero, whereas we need to measure its value immediately after the 35th payment, so we apply an accumulation factor for 35 periods:

$$\$4,524.41 \cdot (1.0155)^{35} = \$7,751.02$$

So, in reality, Blanch would make 34 payments of size $14,725.26 and her last payment would consist of the $14,725.26 and the remaining balance of $7,751.02. Thus, the 35th payment is $22,476.28. This larger than normal payment is called a balloon payment.

What is the total interest expense, correctly dealing with this balloon payment?
Blanch made 34 payments of size $\$14,725.26$ and one payment of size $\$22,467.28$ for a total of

$$34 \cdot \$14,725.26 + \$22,467.28 = \$523,135.12$$

She had borrowed $\$400,000$ so her total interest charges are $\$123,135.12$.
Important Observations:
By doubling her quarterly payment size, the number of payments had dropped from 120 to 35. In particular, doubling the payment size reduced the number of payments by more than $1/2$.
By doubling her quarterly payment size, the interest expense had dropped from \$483,515.65 to \$123,135.12. In particular, doubling the payment size reduced the interest expense by more than $1/2$.

Annuities

- A sequence of regular cash flows of $R occurs at the end of each period for the next n periods. (R stands for "regular cash flow")
- Interest rate is i per period.
- Then the present value, P, of this annuity is

$$P = R \cdot \frac{1 - (1 + i)^{-n}}{i}$$

- In Excel, P can be computed by *=PV(i,n,R)*.
- In WeBWorK, P can be computed by *R * PV(i,n)*.
- P answers the question "What is the value of this entire stream of cash flows evaluated at the beginning"

Annuities

- Then the accumulated value, or future value, F, of this annuity is
$$F = R \cdot \frac{(1+i)^n - 1}{i}$$
- In Excel, P can be computed by $=$*FV(i,n,R)*.
- In WeBWorK, P can be computed by *R * FV(i,n)*.
- F answers the questions like "If you save \$R at the end of each year for the next n years and interest is i per year, then what is the value of your savings at the end?"

Annuities versus Loans

- Annuities and loans both involve level sized cash flows that are paid at regular time intervals
- Mathematically, they are treated the same
- Financially, the regular cash flows in a loan are being paid out, while the regular cash flows in an annuity are being received

Ex 4: FV of Annuity

- Determine the accumulated value of a 8 year annuity with level cash flows of \$1200 at the end of each quarter, provided the cash flows earn 6% annual interest compounded quarterly.
  Use the annuity formula, accumulated value form.
  $i = 0.06/4 = 0.015$, $n = 8 \cdot 4 = 32$ *and* $R = 1200$. *So*
  $$F = 1200 \cdot \frac{(1.015)^{32} - 1}{0.015} = \$48,825.95$$
- Determine the present value of the above annuity.
  Use the annuity formula, present value form.
  $i = 0.06/4 = 0.015$, $n = 8 \cdot 4 = 32$ *and* $R = 1200$. *So*
  $$P = 1200 \cdot \frac{1 - (1.015)^{-32}}{0.015} = \$30,320.57$$
  Alternatively, since we already knew the accumulated value, we could have obtained the present value by discounting the accumulated value by 32 periods:
  $$\$48,825.95 \cdot (1.015)^{-32} = \$30,320.57$$