MA162: Finite mathematics

Set Theory and Counting

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February 12, 2014

SCHEDULE: From Chapter 6.1 & 6.2 in Tan's Finite Math.

Sets and Counting

- Jane has her MA 162 lecture on Monday and Wednesday and MA 162 recitation on Tuesday.
- She has her ACC 202 course on Monday, Wednesday, and Friday.
- She has her MGT 301 course on Monday evenings.
- In a given week, how many days does she have class?

Days she has MA 162 = {Mon, Tue, Wed} → 3 days

Days she has ACC 202 = {Mon, Wed, Fri} → 3 days

Days she has MGT 301 = {Mon} → 1 day

Days she has class? It's not 3+3+1 = 7!

(MA162) ∪ (~~MA~~ ACC 202) ∪ (MGT 301) =

{Mon, Tue, Wed, Fri}

4 days.

- Counting with numbers is only effective when things don't overlap.
- We need to use sets when there is the possibility of overlap.
- $\{\text{Mon}, \text{Tues}, \text{Wed}\} \cup \{\text{Mon}, \text{Wed}, \text{Fri}\} \cup \{\text{Mon}\} = \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Fri}\}$

Setting up some terminology

- A set is a collection of objects.
- The objects in a set are called elements of the set.
- "x is an element of A" is written as $x \in A$.
- These elements could be numbers, letters, names of people, places, functions, other sets, etc.
- The order of the elements does not matter. $\{1, 2, 3, 4\}$ and $\{4, 2, 1, 3\}$ are identical as sets.
- Repetition does not matter. $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 3, 3, 4, 1, 2, 2, 1\}$ are identical as sets.
- The set with no elements is called the empty set and is denoted $\emptyset$

Setting up some terminology

- A set A is a subset of the set B if every element of A is also an element of B.
- "A is a subset of B" is denoted $A \subset B$.
- Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then $A \subset B$.

New sets from old

- The union of A and B, denoted $A \cup B$ is the set of all elements belonging to at least one of A or B.
- The intersection of A and B, denoted $A \cap B$ is the set of all elements belonging to both A and B.
- The difference of A and B, denoted $A - B$ is the set of all elements belonging to A but not B.
- Given a universal set U, the complement of A, denoted A^c is the set of all elements of the universal set which do not belong to A.

De Morgan's Laws

If A and B are two sets, then

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

- Coin is flipped three times;
- $A = \{HHH, HHT, HTH, HTT\}$ = Heads on first flip
- $B = \{HHT, HTH, THH\}$ = exactly 2 heads.

A A&B A&B A B

- $A \cup B =$ {HHH, HHT, HTH, HTT, THH}
- $A \cap B =$ {HHT, HTH}

In A but not B

- $A - B =$ {HHH, HTT}
- $B - A =$ {THH}
- $A^c =$ U − A = {THH, TTH, THT, TTT}
- $B^c =$ U − B = {HHH, HTT, ~~H~~THT, TTH, TTT}

U = All outcomes for 3 flips
= {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

For the coin flipping problem find each of

- $(A \cup B)^c = \mathcal{U} - (A \cup B) = \{THT, TTH, TTT\}$
- $A^c \cap B^c = \{TTH, THT, TTT\}$

Same (should expect this, from De Morgan)

- $A^c \cup B^c = \{THH, TTH, THT, TTT, HHH, HTT\}$
- $(A \cap B)^c = \{HHH, HTT, THT, TTH, TTT, THH\}$

Same. (should expect this from De Morgan)

To include or not to include...

Consider a standard deck of 52 playing cards.

How many cards are either red or have face value equal to 10?

$R =$ "Red card" $\quad n(R) = 26 \quad$ (13 Hearts, 13 diamonds)

$T =$ "Face Value $= 10$" $\quad n(T) = 16. \quad$ (4 Jacks, 4 Queens, 4 Kings, 4 10s).

Want $n(R \cup T)$

Its not $n(R \cup T) = n(R) + n(T) = 26 + 16.$

why?

8 cards are red + have face value 10.

These are counted twice

$$n(R \cup T) = n(R) + n(T) - n(R \cap T)$$

$$= 26 + 16 - 8 = 34$$

Inclusion-Exclusion Principle

- **Two set inclusion-exclusion:**

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

- **Three set inclusion-exclusion:**

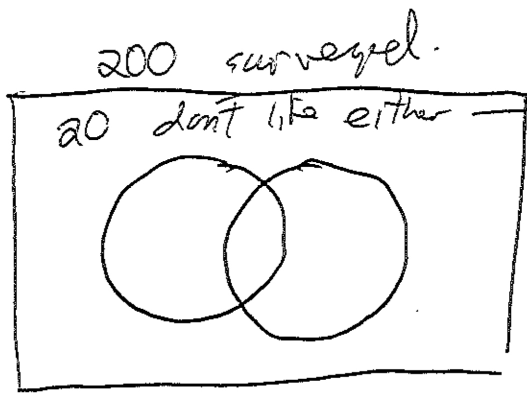
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$$
$$-n(B \cap C) + n(A \cap B \cap C)$$

- **Many set inclusion-exclusion:** The above two versions suggest a pattern for four sets, five sets, etc. The idea isn't too hard, but developing enough notation to write down a general formula is a bit of a challenge...

200 people were surveyed about dining preferences. 155 people surveyed like Mexican food. 75 people surveyed like Thai food. 20 of the people surveyed like neither Mexican food nor Thai food. How many of the surveyed people like Mexican food but not Thai food?

M = Like Mexican, $n(M) = 155$

T = Like Thai $n(T) = 75$



180 like at least one.

$n(M \cup T) = 180$.

But

$$n(M \cup T) = n(M) + n(T) - n(M \cap T)$$

$$180 = 155 + 75 - n(M \cap T)$$

So $n(M \cap T) = 230 - 180 = 50$.

Want: $n(M - T)$

$$n(M - T) = n(M) - n(T \cap M)$$

$$= 155 - 50 = 105$$

To include or not to include...

100 students were interviewed.

- 68 of them have part-time jobs.
- 61 of them are involved in extra-curricular activities at school.
- 52 of them are involved in community outreach programs.
- 41 of them have part-time jobs and are involved in extra-curricular activities
- 35 of them have part-time jobs are are involved in community outreach programs
- 31 of them are involved in extra-curricular activities and community outreach programs.
- Each student interviewed was involved in at least one of the above three activities.
- How many students from this group were involved in all three of the above activities?

$P =$ "Part-time"

$E =$ "Extra Curr"

$C =$ "community outreach"

$n(P) = 68$, $n(E) = 61$, $n(C) = 52$

$n(P \cap E) = 41$, $n(P \cap C) = 35$, $n(E \cap C) = 31$

Each of the 100 are involved in at least one activity.

$\Rightarrow n(P \cup E \cup C) = 100$.

So

$$100 = 68 + 61 + 52 - 41 - 35 - 31 + n(P \cap E \cap C).$$

$\Rightarrow n(P \cap E \cap C) = 26$.

#(involved in all 3) $= 26$.

To include or not to include...

Roll a pair of fair six sided dice with the digits 1,2,3,4,5,6. How many outcomes have either at least one die showing a 4 or the sum of the dice is equal to 9

$F =$ "At least one 4", $n(F) = 11$

$(4,1)$
$(4,2)$
$(4,3)$
$(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)$
$(4,5)$
$(4,6)$.

$n(F \cap N) = 2$.

Sum is 9 and at least one is 4 — The pairs $(4,5)$ & $(5,4)$

$N =$ "Sum is 9": $n(N) = 4$.

$$\left.\begin{array}{l} 3+6 \\ 4+5 \\ 5+4 \\ 6+3 \end{array}\right\} \text{4 of them}$$

Want $n(F \cup N)$.

$$n(F \cup N) = n(F) + n(N) - n(F \cap N)$$
$$= 11 + 4 - 2 = 13$$