MA162: Finite mathematics

Permutations and Combinations

Paul Koester

University of Kentucky

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SCHEDULE: Solutions

6.4: Permutations

- How many different ways can the letters O, P, and E be re-arranged?

Method 1 Exhaustive List

OPE, POE, EOP   6 possibilities.
OEP, PEO, EPO

Method 2

First letter has 3 choices, 2nd has 2 choices, last has 1 choice
$3 \cdot 2 \cdot 1 = 6$

- How many different ways can the letters in the word "MAINE" be re-arranged?

Exhaustively listing possibilities is not feasible.

But: 5 choices for first letter
4 for second
3 for third
2 for 4th
1 for last

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! =$ "5 factorial"

6.4: Another Permutation

- Bob, Jane, Mike, and Sally are elected to serve on a committee.
- One will have to serve as president, one as vice-president, one as treasurer, and one as secretary.

  Assuming each person is assigned only one office.
- In how many ways can the positions be assigned to Bob, Jane, Mike, and Sally?

4 choices for Prez.
3 for VP.
2 for Treas.
1 for Sec.

$4 \cdot 3 \cdot 2 \cdot 1 = 4!$

6.4: Yet Another Permutation

- You have 100 songs in your i-Tunes library.
- How many ways can you create a playlist with 12 songs?

- Assume the same song cannot be chosen twice.

You have 100 choices for first song.
99 for second
98 for third.
⋮
89 choices for 12^{th}.

$$100 \cdot 99 \cdot 98 \cdots 89 = \frac{100!}{88!} = \frac{100!}{(100-12)!}$$

In Webwork, could type as 100 P 12

- If you allow repeating songs, then # of choices is 100^{12}

6.4: Permutations: More formally

- Given a set of distinct objects, a permutation of this set is an arrangement which specifies an ordering on the elements of the set.
- The number of permutations of a set of n distinct items is

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

- The above expression is called n factorial and is denoted $n!$.

6.4: Permutations: More formally

- Given a set of distinct objects, a permutation of this set is taken r at a time is an arrangement which chooses r of the elements of the set and specifies an ordering of these r items..

- The number of permutations of a set of n distinct items taken r at a time is

$$P(n, r) = \frac{n!}{(n-r)!}$$

In Webwork, type as

n P r

For example 500 P 13

6.4: Permutations: Not Necessarily Distinct Items

- You have a group of 3 boys and 4 girls. Suppose only gender matters.

  BBBGGGG
  BBGBGGG, etc.

- They are to stand in line.

- How many arrangements are there? If 7 distinct "objects" then 7!

$$\frac{7!}{3!\ 4!}$$

Can't tell the 3 boys apart, so divide off 3!

Can't tell the 4 girls apart, so divide off 4!

6.4: Permutations: Not Necessarily Distinct Items

- You have a set with n items.
- n_1 of the items are of alike of one kind. n_2 items are alike and of another kind. ... n_r items are alike of yet another kind.
- All of the items are one of the r types, so that

$$n_1 + n_2 + \ldots + n_r = n$$

- The number of these n objects taken n at a time is then

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

6.4: Permutations: Rearranging Letters

- In how many ways can the letters in KENTUCKY be rearranged?

If all 8 letters were different, would be $8!$

But K used twice, each of the other 6 are used once.

$$\frac{8!}{2!\,1!\,1!\,1!\,1!\,1!\,1!} = \frac{8!}{2!}$$

- A poor musician has forgotten how to play her favorite melody. She thought it was AACEGEA, but that doesn't sound right. She knows that she has the right notes, and the right number of repeats of each note, She's just putting them in the wrong order. In how many ways can she rearrange those notes?

$$\frac{7!}{3!\,2!\,1!\,1!} = \frac{7!}{3!\,2!}$$

7 letters.

A: 3 times

E 2 times

C once

G once.

6.4: Combinations - when order doesn't matter

- There are 10 new songs you'd like to download.

- You just received a gift card which will allow you to download 7 songs.

- In how many ways can you choose 7 out of these 10 songs?

If order mattered, this would be $10\ P\ 7$, or $\frac{10!}{3!}$

- NOTE: The order in which you download the songs does not matter!

But the order in which we download the songs does not matter. There were $7!$ ways to order a given set of 7, so divide off $7!$.

$$\frac{10!}{7!\ 3!} = \binom{10}{7} =$$ "10 choose 7" In Webwork, $10\ C\ 7$.

6.4: Combinations - in general

- A combination of n distinct items taken r at a time is an un-ordered selection of r of these n items.

- The number of such combinations is

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

In Webwork n C r

- The expression $\binom{n}{r}$ is pronounced "n choose r." This is also called a "binomial coefficient."

6.4: Counting Cards

- A standard deck of cards has 52 cards.
- A standard Poker hand has 5 of these cards.
- How many 5 card hands are possible?

Choose 5 from 52

$$\binom{52}{5} = 2,598,960.$$

6.4: Counting Cards

- A standard deck of cards has the following cards:
  A♡ 2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡
  A♢ 2♢ 3♢ 4♢ 5♢ 6♢ 7♢ 8♢ 9♢ 10♢ J♢ Q♢ K♢
  A♣ 2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣
  A♠ 2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠
- See
  `http://en.wikipedia.org/wiki/List_of_poker_hands`
  for definitions of poker hands.

- Try counting the number of each type of poker hand yourself. This is a great way to practice the ideas from this chapter.

- Check your work against wikipedia's, but be warned that not everything you read on *wikipedia* is reliable!

6.4: Full house or 4 of a kind?

- A full house is a hand with 3 of one kind and 2 of another kind.

$$\binom{13}{1}\binom{4}{3}\cdot\binom{12}{1}\binom{4}{2} = 3744$$

choose "rank" for 3 of a kind

choose 3 suits for 3 of a kind

choose ~~rank~~ rank for pair.

Choose 2 suits for pair.

- A four of a kind is a hand with 4 of one kind, and some other card.

$$\binom{13}{1}\binom{4}{4}\cdot\binom{12}{1}\binom{4}{1} = 624.$$

Choose rank for 4 of a kind

choose 4 suits for 4 of a kind

choose suit for last card.

Choose rank for remaining card.

- Which is more common? Full house or 4 of a kind?

6.4: Counting Cards

- A hand in 7 card stud consists of 7 cards, in which 3 cards are face-up (visisble to all players) and 4 cards are face-down.

- How many distinct hands can be dealt to a player in 7 card stud?

Approach 1: Choose 4 of 52 to be face down. $\binom{52}{4}$
Then, from remaining 48, choose 3 for face up. $\binom{48}{3}$. $\Longrightarrow$ $\binom{52}{4}\binom{48}{3} =$
4,682,459,600

- NOTE: It is NOT $\binom{52}{7}$.

Approach 2: Choose the 7 cards, $\binom{52}{7}$.
Now choose 3 of these 7 to be face up, $\binom{7}{3}$

$\binom{52}{7}\binom{7}{3} = 4,682,459,600$

Same!