MA162: Finite mathematics

Properties of Probabilities

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SCHEDULE: Solutions

Some Rules for Computing Probabilities

- $0 \leq P(E) \leq 1$ for any event E

- $P(S) = 1$ where S is the entire sample space

- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

- $P(E^c) = 1 - P(E)$

Balls in Urns

An urn contains some number of balls, some are red, some are white, and some are yellow.
You reach in and grab a ball at random.
You are twice as likely to grab a white ball than a red ball.
You are four times more likely to grab a yellow ball than a red ball.
What is the probability you grab a white ball?

$$P(W) + P(Y) + P(R) = 1$$

$$2P(R) + 4P(R) + P(R) = 1$$

$$7P(R) = 1$$

$$P(R) = \frac{1}{7}$$

Flipping coins

A fair coin is flipped 3 times.

HHT, HTH, THH } 3 ways to get 2 heads

- What is the probability of turning up exactly two heads?

Choose 2 of the 3 to be Heads. $\binom{3}{2} = 3$. ← This selects where heads are.

Prob of given sequence with 2 heads $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$ $\Longrightarrow$ $\frac{3}{8}$.

- What is the probability of turning up at least two heads?

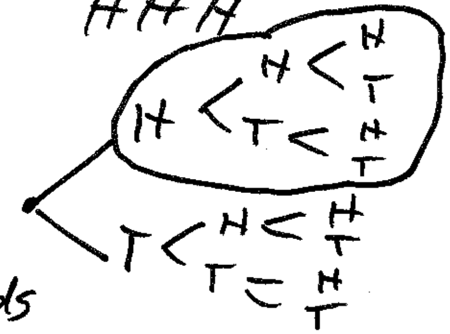
$P(\geq 2 \text{ heads}) = P(2 \text{ heads}) + P(3 \text{ heads})$

$= \frac{3}{8} + \frac{1}{8} = 1/2$

only one way to get 3 heads HHH

- What is the probability of first flip is heads?

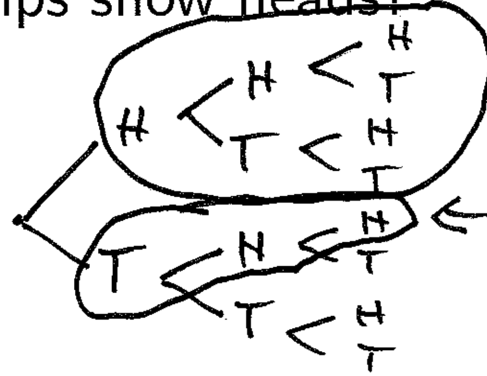
$\frac{1}{2}$ Why? Could just list all possibilities



4 of 8 start with heads

- What is the probability that the first flip shows heads OR exactly two flips show heads?

Again list all



← Heads on first $\Longrightarrow$ $\frac{5}{8}$

← Exactly 2

Too many dice?

A fair 6 sided die is rolled 50 times.

Finally, there are 50 such sequences $50 \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{49}$

- What is the probability of rolling a "2" exactly 1 time?

Prob of 2 on given roll is $\frac{1}{6}$.
Prob of not 2 on given roll is $5/6$.
A given sequence with single 2 has prob $\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{49}$

- What is the probability of rolling a "2" at least 1 time?

P(2 at least once) = 1 − P(fewer than one "2") = 1 − P(No 2)

$P(\text{No } 2) = \left(\frac{5}{6}\right)^{50}$

so $P(2 \text{ at least once}) = 1 - \left(\frac{5}{6}\right)^{50} = 0.99989$

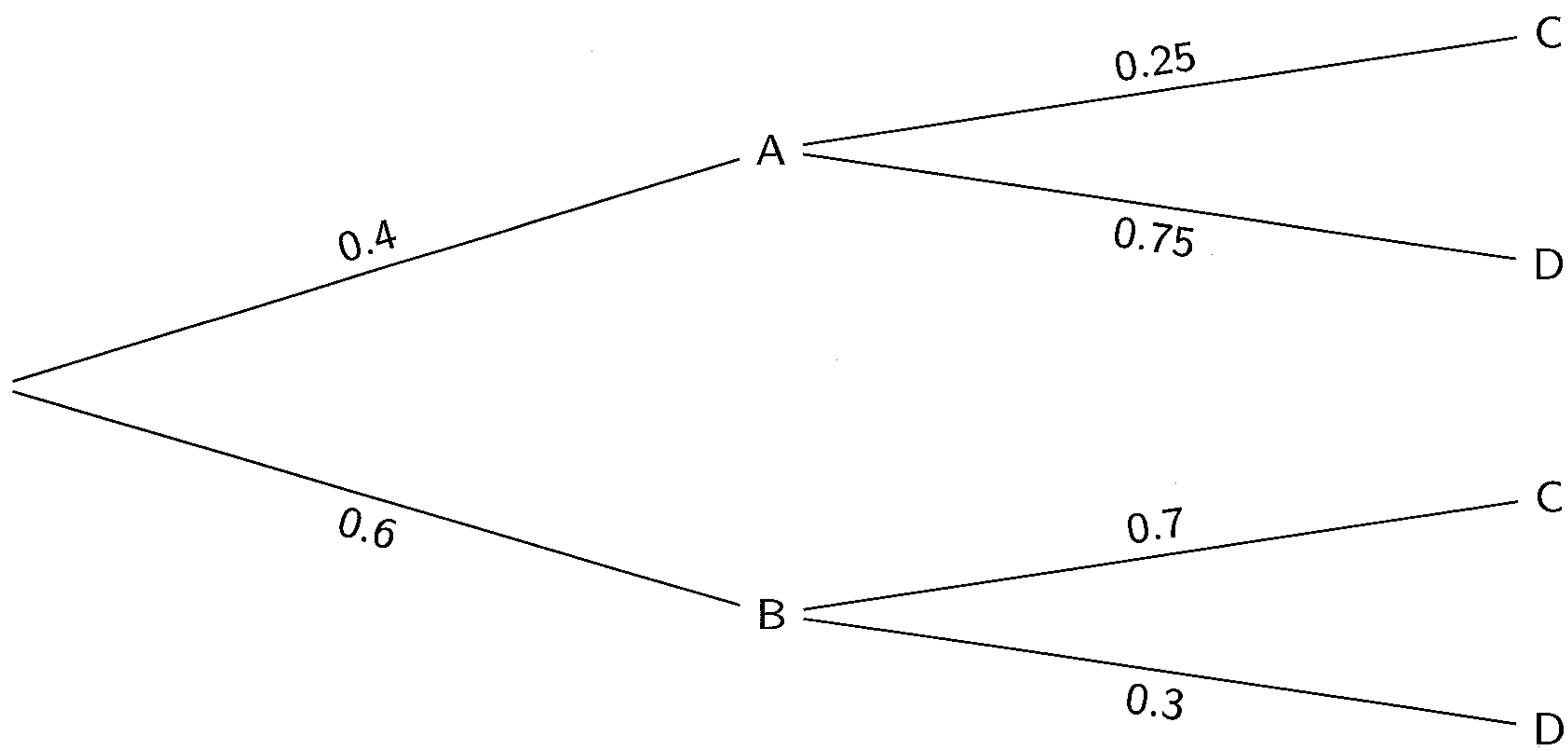
- What is the probability of rolling a "2" at most 1 time?

P(At most one two) = P(No 2) + P(Exactly one two)

$$= \left(\frac{5}{6}\right)^{50} + 50\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{49}$$

Trees

Multi-stage probability experiments are best visualized with a tree diagram.



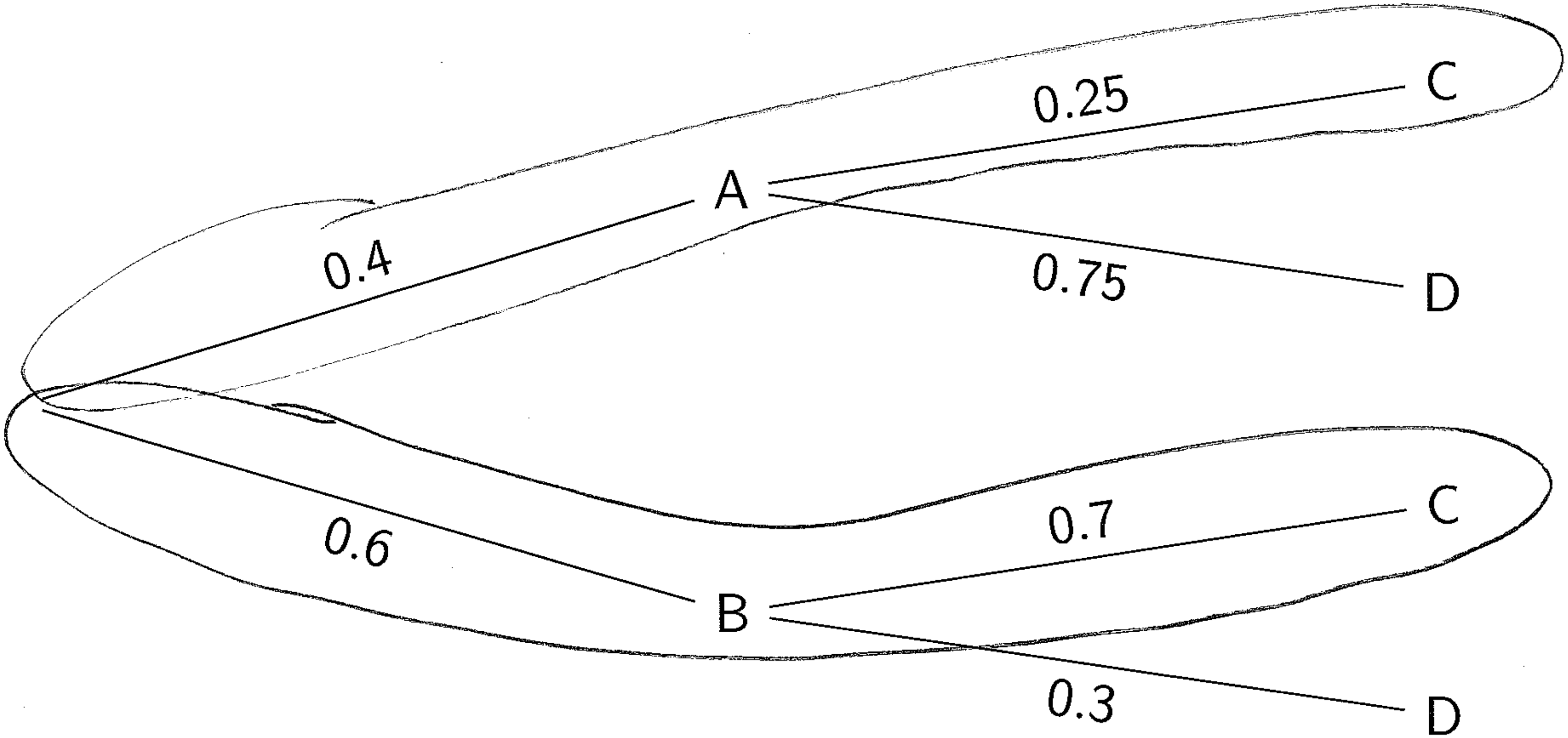
The final outcomes are represented by the nodes on the far right of the tree.
Technically, the nodes on the right are represented as intersections, but we often do not write them as intersections.
The probability of a given outcome is obtained by multiplying all of the "branching" probabilities.
For example, $P(A \cap C) = 0.4 \cdot 0.25 = 0.1$, whereas $P(A) = 0.4$

Another Probability Rule

- E is an event
- $F_1, F_2, \ldots, F_r$ are events satisfying
  - $S = F_1 \cup F_2 \cup \cdots \cup F_r$
  - Each pair F_i and F_j are mutually exclusive, for $i \neq j$
- Then $P(E) = P(E \cap F_1) + P(E \cap F_2) + \cdots + P(E \cap F_r)$

This rule is best visualized with a tree diagram.

Trees



Find the probability of C. Two ways to get to C.

$$P(C) = P(C \cap A) + P(C \cap B)$$
$$= 0.4 \cdot 0.25 + 0.6 \cdot 0.7 = 0.52.$$

More Urns

- Urn A contains 10 balls. 8 are white and 2 are red.
- Urn B contains 12 balls. 4 are white, 6 are red, and 2 are yellow.

$P(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3} = P(A)$

- Roll a single 6 sided die.
  - If value of die is 5 or 6, draw a ball at random from Urn A.
  - If value of die is 1, 2, 3, or 4, draw a ball at random from Urn B.

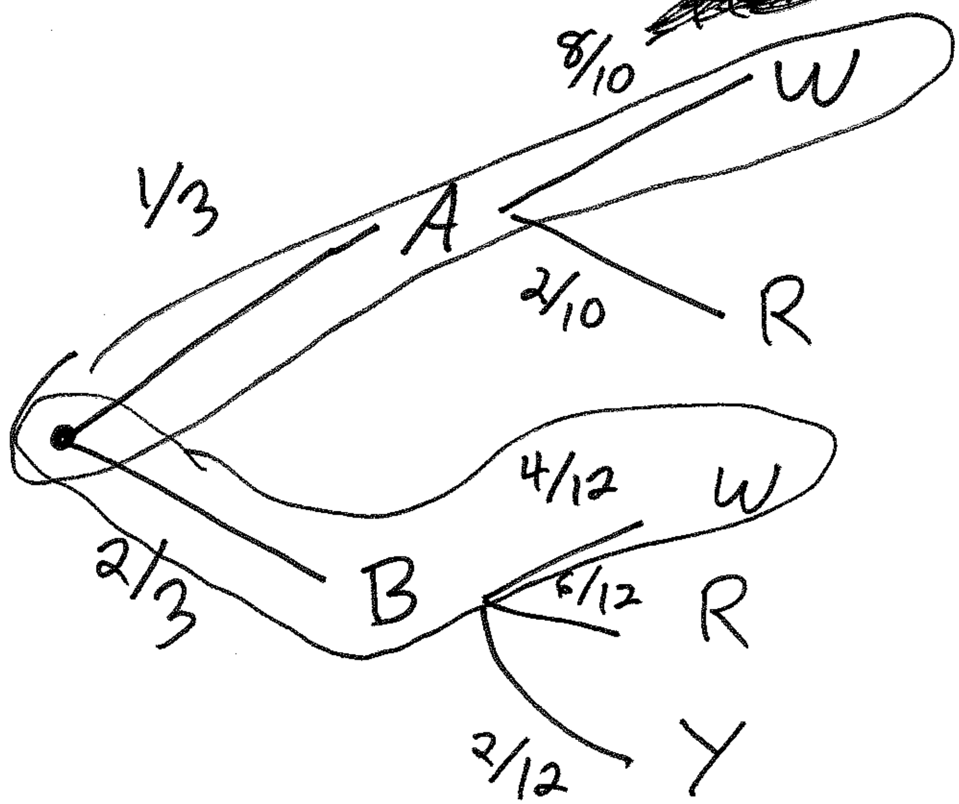
$P(B) = \frac{4}{6} = \frac{2}{3}$

- Draw a tree to represent this experiment.
- What is the probability the ball is white?

$A =$ Pick Urn A, $B =$ Pick urn B

$W =$ Pick white ball, $R =$ Pick red Ball

$Y =$ Pick yellow ball.



$$P(W) = P(A \cap W) + P(B \cap W)$$

$$= \frac{1}{3} \cdot \frac{8}{10} + \frac{2}{3} \cdot \frac{4}{12} = \frac{22}{45} = 0.488\ldots$$

Gaming

- A group of 243 video gamers were asked about video game habits.

- Question 1: Which game consoles do you own? "Own PS 3", "Own XBox 360", "Own neither", "Own both"

- Question 2: How many hours per week do you play? "no more than 2 hours", "between 2 to 6 hours", "more than 6 hours"

- Results are recorded on next page

Gaming

	PS 3 only	XBox only	Both	Neither	Total
< 2 hours	47	23	7	17	94
2 to 6 hours	34	41	11	3	89
> 6 hours	15	18	25	2	60
total	96	82	43	22	243

- Probability random gamer owns an XBox but not a PS-3?

  $82/243$

- Probability random gamer owns a PS-3? ← (~~PS~~-3 only) $\cup$ (Both systems)

  $(96+43)/243 = 139/243$

- Probability random gamer plays at least 2 hours per week?

  $(89+60)/243 = 149/243$

- Probability random gamer owns an XBox and plays two hours or less per week?

  $\dfrac{23+7}{243} = \dfrac{30}{243}$

< 2 hours + only XBox

< 2 hours + Both systems

Conditional Gaming

	PS 3 only	XBox only	Both	Neither	Total
$<$ 2 hours	47	23	7	17	94
2 to 6 hours	34	41	11	3	89
$>$ 6 hours	15	18	25	2	60
total	96	82	43	22	243

- Restrict attention only to gamers who own both consoles. What is probability they play at least two hours per week?

Only look at "Both column"

$$\frac{11+25}{43} = \frac{36}{43}$$

Notice this change the denominator.

- Restrict attention to gamers who play less than two hours per week. What is probability they own a PS 3?

$$\frac{(47+7)}{94} = \frac{54}{94}.$$