MA162: Finite mathematics

Combinatorial Probability

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SCHEDULE: Solutions.

Review of Empirical Probability

The "relative frequency" approach to probability theory suggests defining the probability of an event as

$$P(E) = \frac{n(E)}{n(S)}$$

where E is the event and S is the sample space.
Think "Probability of event is number of favorable outcomes divided by total number of outcomes"
We've already used this "relative frequency" approach. The difference between today's lesson compared to previous lessons is that we will need to rely on the more advanced counting techniques from Chapter 6.4.

Sock it to me!

- Your dresser drawer has ten white socks, four black socks, and four plaid socks.
- You reach in and pull out two socks. What is the probability that you have a matching pair?

$$P(2\text{ Whites}) + P(2\text{ Blacks}) + P(2\text{ plaids})$$

$$= \frac{\binom{10}{2}}{\binom{18}{2}} + \frac{\binom{4}{2}}{\binom{18}{2}} + \frac{\binom{4}{2}}{\binom{18}{2}} = \frac{\frac{10\cdot 9}{2} + \frac{4\cdot 3}{2} + \frac{4\cdot 3}{2}}{\frac{18\cdot 19}{2}} = \frac{1}{3}$$

- You reach in and pull out three socks. What is the probability that you can form a matching pair from these three socks?

$$P(\text{Matching pair}) = 1 - P(\text{No match}) = 1 - P(\text{All 3 different})$$

$$= 1 - \frac{10\cdot 4\cdot 4}{\binom{18}{3}} = 1 - \frac{10\cdot 4\cdot 4}{\left(\frac{18\cdot 17\cdot 16}{3\cdot 2\cdot 1}\right)} \approx 0.804$$

Drawing Cards

Two cards are drawn from a standard deck of 52 cards. (A recent HW question asked similar questions for a single card. Notice the question for 2 cards is much more involved!)

- What is the probability the two cards have the same suit?

1st card doesn't matter. 2^{nd} card drawn from 51, 12 of which match first card's suit, $12/51$

- What is the probability that a card has a number between 3 and 7? ⇒ First between 3&7, or second between 3&7 or both between 3&7. Look @ opposite!

$P(\text{At least one between } 3\&7) = 1 - P(\text{Neither between } 3\&7) = 1 - \frac{\binom{32}{2}}{\binom{52}{2}}$

32 cards not between 3&7.

- What is the probability that the two card hand contains a King or a Spade? ~~Tho~~ Too many cases to consider!

$P(King \cup Spade) = 1 - P((King \cup Spade)^c) = 1 - P(King^c \cap Spade^c)$

- What is the probability that the two card hand contains a King and a Spade?

$$1 - P(\text{King}^c \cap \text{Spade}^c) = 1 - \frac{\binom{36}{2}}{\binom{52}{2}}.$$

4 King, 13 spade,

1 King of spade,

so $4+13-1 = 16$ cards

are K or S, so

$52-16 = 36$ neither.

$$P(K \cap S) = P(K) + P(S) - P(K \cup S)$$

$$= \left(1 - P(K^c)\right) + \left(1 - P(S^c)\right) - \left(1 - \frac{\binom{36}{2}}{\binom{52}{2}}\right)$$

48 non Kings

$$= \left(1 - \frac{\binom{48}{2}}{\binom{52}{2}}\right) + \left(1 - \frac{\binom{39}{2}}{\binom{52}{2}}\right)$$

39 non spades

$$- \left(1 - \frac{\binom{36}{2}}{\binom{52}{2}}\right) = 1 - \left(\frac{\binom{48}{2} + \binom{39}{2} - \binom{36}{2}}{\binom{52}{2}}\right).$$

Poker Hands

- A 5 card hand is drawn from a standard deck of 52 cards.
- Determine the probability of drawing a "4 of a kind"

$$\frac{\binom{13}{1}\binom{4}{4}\cdot\binom{12}{1}\cdot\binom{4}{1}}{\binom{52}{5}} = \frac{13\cdot 12\cdot 4}{\binom{52}{5}}$$

- Now determine the probability of a "Full house" (3 of the cards have one rank and the other 2 cards have one rank)

$$\frac{\binom{13}{1}\binom{4}{3}\cdot\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{13\cdot 4\cdot 12\cdot 6}{\binom{52}{5}}$$

Black Jack

- Two cards are drawn from a standard deck of 52 cards.
- Determine the probability of being dealt a "BlackJack." (This means that one of the cards is an Ace and the other card has face value of 10.) 4 Aces, 16 cards have value 10. (4 "10", 4 "J", 4 "Q", 4 "K").

Need ace + 10.

$$\frac{4 \cdot 16}{\binom{52}{2}} \approx 4.8\%$$

- Now suppose 2 decks are shuffled together, so that there are 104 cards. What is the probability of being dealt a "BlackJack" from this larger deck. Now have 8 aces + 32 10s.

$$\frac{8 \cdot \cancel{16}\, 32}{\binom{104}{2}} \approx 4.77\%$$

Urns

- A bag contains 5 white balls, 12 red balls, and 9 green balls.
- You reach in and draw out 5 balls. 26 balls total.
- What is the probability that 3 of the balls are red and 2 are green? 3 of 12 red chosen

2 of 9 green chosen.

$$\frac{\binom{12}{3}\binom{9}{2}}{\binom{26}{5}}$$

$\leftarrow$ 5 of 26 chosen.

Multiple Guess

- An MA 123 exam consists of 20 questions, multiple choice
- There are 5 choices for each question

$$\text{Prob}(\text{Given question right}) = \frac{1}{5}$$
$$\text{Prob}(\text{Given question wrong}) = \frac{4}{5}$$

- A student randomly guesses on all of the questions.
- What is probability student gets exactly 5 of the questions correct?

$$\underset{\text{Choose which 5 of the 20 are right.}}{\binom{20}{5}} \underbrace{\left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^{15}}_{\text{Prob a given sequence of 5 right, 15 wrong.}}$$

- What is the probability this student gets at least an 80% on the exam? 16, 17, 18, 19, or 20 correct.

$$\binom{20}{16}\left(\frac{1}{5}\right)^{16}\left(\frac{4}{5}\right)^{4} + \binom{20}{17}\left(\frac{1}{5}\right)^{17}\left(\frac{4}{5}\right)^{3} + \binom{20}{18}\left(\frac{1}{5}\right)^{18}\left(\frac{4}{5}\right)^{2}$$
$$+ \binom{20}{19}\left(\frac{1}{5}\right)^{19}\left(\frac{4}{5}\right)^{1} + \binom{20}{20}\left(\frac{1}{5}\right)^{20}$$

Binomial Distribution

- The "Multiple Guess" problem is an example of a Binomial Probability.
- An experiment is to be repeated n times.
- At each stage, experiment may result in a "success" with probability p or "failure" with probability $q = 1 - p$.
- The outcome at any given stage does not influence the outcome at any other stage.
- The probability of exactly k successes is then

$$\binom{n}{k} p^k q^{n-k}$$

Let the good times roll...

- A fair six sided die is rolled 200 times. $p = \frac{1}{6}, \quad q = \frac{5}{6}.$
- What is the probability of rolling a "6" exactly 40 times?

$$\binom{200}{40}\left(\frac{1}{6}\right)^{40}\left(\frac{5}{6}\right)^{160}$$

- What is the probability of rolling a "6" between 38 and 40 times?

$$\binom{200}{38}\left(\frac{1}{6}\right)^{38}\left(\frac{5}{6}\right)^{162} + \binom{200}{39}\left(\frac{1}{6}\right)^{39}\left(\frac{5}{6}\right)^{161} + \binom{200}{40}\left(\frac{1}{6}\right)^{40}\left(\frac{5}{6}\right)^{160}$$

Original Slides included material on Geometric distribution, sampling with & without replacement.

~~The~~ In the interest of time, this material has been removed.