MA162: Finite mathematics

Financial Mathematics

University of Kentucky

January 29, 2014

SCHEDULE:

Last Time

- **Compound Interest, Compounded m times per year**
  If \$P is invested for t years at annual interest rate r compounded m times per year, then A (the accumulated value) is
  $$A = P\left(1 + \frac{r}{m}\right)^{mt}$$
- Equivalently, letting $i = r/m$ (interest per period) and $n = mt$ (total number of periods) then
  $$A = P\left(1 + i\right)^{n}$$
- **Continuously Compounded interest**
  If \$P is invested for t years at continuously compounded rate of δ per year, then
  $$A = P\,e^{\delta t}$$

Last Time

- Consider annual compound interest r compounded m times per year. The *annual effective rate of interest*, $r_e ff$ is obtained by solving this equation for r_{eff} :

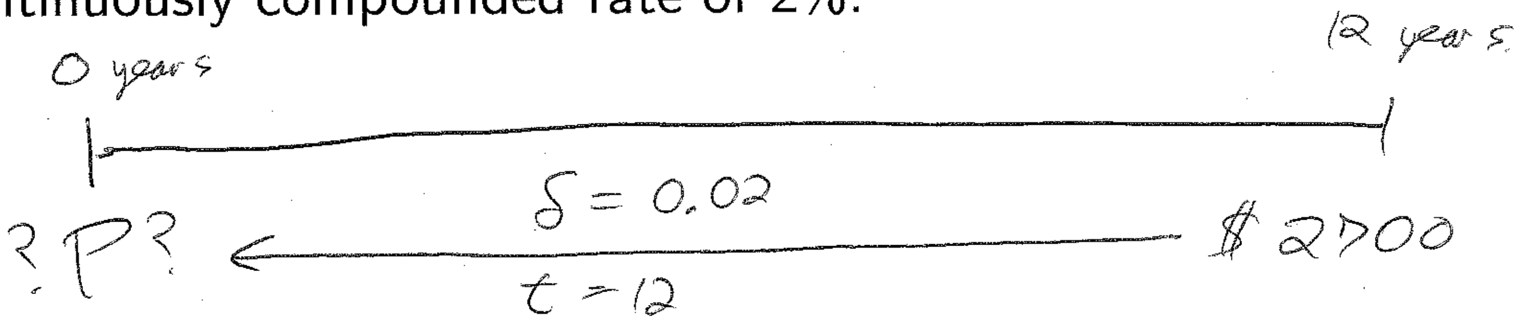
$$\left(1+\frac{r}{m}\right)^m = (1+r_{eff})$$

- Consider continuously compounded interest rate of δ. The *annual effective rate of interest*, $r_e ff$ is obtained by solving this equation for r_{eff} :

$$e^\delta = (1+r_{eff})$$

Ex. 1: Continuous Compound Interest

Find the present value of \$2700 invested for 12 years at continuously compounded rate of 2%.



$$2700 = Pe^{0.02 \cdot 12}$$

$$\frac{2700}{e^{0.02 \cdot 12}} = P$$

$$\frac{2700}{1.27124915} = P$$

$$2123.90 = P$$

Ex. 2: Effective Rate

Determine the continuously compounded rate of interest if the effective rate of interest is 6%

$r_{eff} = 0.06$

\$1 invested 1 year @ 6% annual $\Rightarrow$ 1.06

\$1 invested 1 year @ δ continuous $\Rightarrow$ e^{δ}

so $e^{\delta} = 1.06$

$$\ln(e^{\delta}) = \ln(1.06)$$

$$\delta = \ln(1.06) = 0.0582689$$

so $\delta = 5.82689\ \%$

Ex. 3: Dealing with Varying Rate

- John borrowed \$3500 on January 29, 2010. He is to repay the loan on January 29, 2014.
- His loan is an adjustable rate loan, meaning that the interest rate *can* vary at certain specific times.
- The original interest rate was 3% APR compounded quarterly.
- On July 29, 2013, the interest rate was changed to 5%.
- Assuming no more changes in the interest rate, how much must he pay back?

Jan 2010 — Jan 2013 — July 2013 — Jan 2014

$\$3500 \longrightarrow 3500\left(1+\frac{.03}{4}\right)^{14}$

$= 3885.96$

Jan 2010 to July 2013 = 3.5 years

So $3.5 \cdot 4 = 14$ quarters

$3885.96 \longrightarrow 3885.96\left(1+\frac{.05}{4}\right)^{2}$

$= 3983.72$

Note: $3983.72 = 3500\left(1+\frac{0.03}{4}\right)^{14} \cdot \left(1+\frac{.05}{4}\right)^{2}$

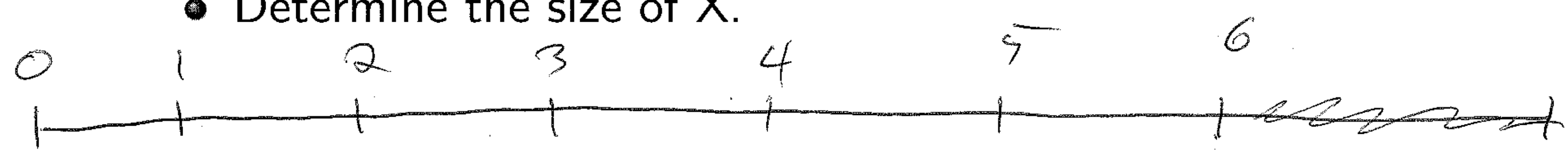
Ex. 4: Dealing with multiple cash flows

- Kendall opened a savings account 6 years ago, by invested \$5000 in an account which earns 4% APR compounded semi-annually.
- 5 years ago, she deposited an extra \$2000.
- 2 years ago, she withdrew some amount, call it X.
- Her current balance is \$4361.
- Determine the size of X.

$m = 2$

$r = 0.04$

so $i = 0.02$



$+5000 \longrightarrow 5000(1.02)^8$

$+2000 \longrightarrow 2000(1.02)^6$

$-X$

$4361/(1.02)^4 \longleftarrow 4361$

So $5000(1.02)^8 + 2000(1.02)^6 - X = 4361/(1.02)^4$

$$5858.30 + 2252.32 - X = 4028.89$$

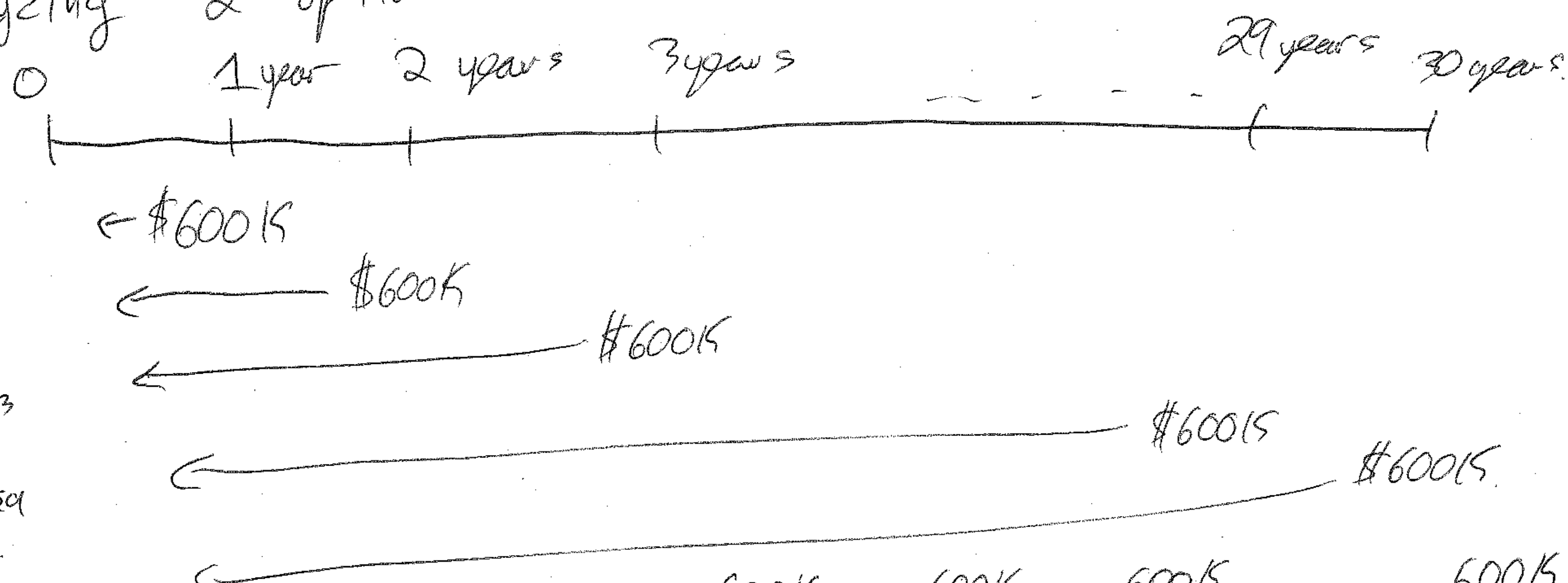
So $X = 5858.30 + 2252.32 - 4028.89$

$$X = 4081.73$$

Ex. 5: Dealing with lots of cash flows

- Loraine won the Ohio Lottery. She can choose between accepting \$9,000,000 today or she can accept \$600,000 at the end of each year for the next 30 years.
- Assume discount rate of 3% per year, compounded annually.
- Which option is "better"?

Analyzing 2nd option:



$\frac{600K}{1.03}$

$\frac{600K}{(1.03)^2}$

$\frac{600K}{(1.03)^3}$

$\frac{600K}{(1.03)^{29}}$

$\frac{600K}{(1.03)^{30}}$

So Value today $= \frac{600K}{1.03} + \frac{600K}{(1.03)^2} + \frac{600K}{(1.03)^3} + \cdots + \frac{600K}{(1.03)^{30}}$

Geometric Sum

Given a real number $a \neq 1$ and a positive integer n,

$$a + a^2 + a^3 + \ldots + a^n = \frac{a - a^{n+1}}{1 - a}$$

Consequently,

$$\frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \left(\frac{1}{1+i}\right)^3 + \ldots + \left(\frac{1}{1+i}\right)^n = \frac{\frac{1}{1+i} - \left(\frac{1}{1+i}\right)^{n+1}}{1 - \frac{1}{1+i}}$$

and with a bit of algebra, this simplifies down to

$$\frac{1 - (1+i)^{-n}}{i}$$

Loans

- An amount \$P is borrowed. (P stands for principal, or present value)
- The loan is to be repaid by making *regular* payments of size \$R and the end of each period for the next n periods.
- Interest rate is i per period.
- Then

$$P = R \cdot \frac{1-(1+i)^{-n}}{i}$$

- In Excel, P can be computed by $=PV(i,n,R)$.
- In WeBWorK, P can be computed by $R * PV(i,n)$.

For lottery, $R = 600{,}000$, $i = 0.03$, $n = 30$

$$P = 600{,}000 \frac{1-(1.03)^{-30}}{0.03} = 11{,}760{,}264.81$$

In HW, could type in $\boxed{600000 * PV(0.03, 30)}$

Ex. 6: Car Loan

- Murray just purchased a car. The price of the car was $15,000.
- He makes a $4000 down payment takes out a car loan to cover the rest.
- He has to make payments at the end of each month for the next 4 years.
- The interest on the loan is 6% compounded monthly.
- Determine the size of Murray's monthly payment.

$t=4$

$m=12 \Rightarrow \quad n = 12 \cdot 4 = 48, \quad r = 0.06, \quad i = \frac{r}{m} = 0.005$

P? Car costs ~~$15,000~~, but only borrows $11,000,

so $P = \$11,000$

Then $11{,}000 = R \dfrac{1-(1.005)^{-48}}{0.005}$

So $R = \dfrac{11000}{\left(\dfrac{1-(1.005)^{-48}}{0.005}\right)} = \$258.34$

Ex. 6: Car Loan

- What is the total amount of interest that Murray pays?

Next class.....

- How much of Murray's first payment is due to interest?