MA162: Finite mathematics

Conditional Probability

Paul Koester

University of Kentucky

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Due to March 3 snow day,

SCHEDULE: ~~this~~ you are not responsible for this material.

Use these notes if you wish to attempt the bonus homework, Probability HW04.

More Black Jack

- The dealer draws two cards from a standard deck of 52 cards.
- One of his cards is placed face up, so that all players can see it.
- The other card is placed face down, so only the dealer knows which card it is.
- What is the probability the dealer is dealt $\boxed{\text{Black Jack}}$?

4 10, 4 J, 4Q, 4K

4 aces in deck

$$\frac{4 \cdot 16}{\binom{52}{2}} \approx 4.8\%$$

pick 2 of 52 cards

One A (Ace) & One card that is either a 10, J, Q, K.

- The dealer's face up card is an Ace. What is the probability the dealer has Black Jack?

Already has ace

Needs to pick one of the 16 10, J, Q, K from 51 remaining

$$\frac{16}{51} \approx 31\%$$

Conditioning

- The probability assigned to an event depends on
  - The underlying event
  - Any relevant information we happen to have
- Conditional Probabilities give us the means to take the additional information into account.
- Given two events, A and B, $P(A|B)$ is the probability of A, given that B occurs.
- If $P(A)$ and $P(B)$ are nonzero, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Smoking and lung cancer

Facts taken from `http://lungcancer.about.com/od/Lung-Cancer-And-Smoking/f/Smokers-Lung-Cancer.htm`

- 0.2% probability of lung cancer for men who never smoked
- 5.5% probability of lung cancer for male former smokers
- 15.9% probability of lung cancer for current male smokers
- 24.4% probability of lung cancer for male "heavy smokers" defined as smoking more than 5 cigarettes per day

These are examples of conditional probabilities. Determine a reasonable sample space, and determine events so that each of the above probabilities can be expressed as a conditional probability.

L = man develops lung cancer.

N = Non smoker

F = Former smoker

S = Smoker.

H = Heavy smoker

$P(L|N) = .002$

$P(L|F) = .055$

$P(L|S) = .159$

$P(L|H) = .244$

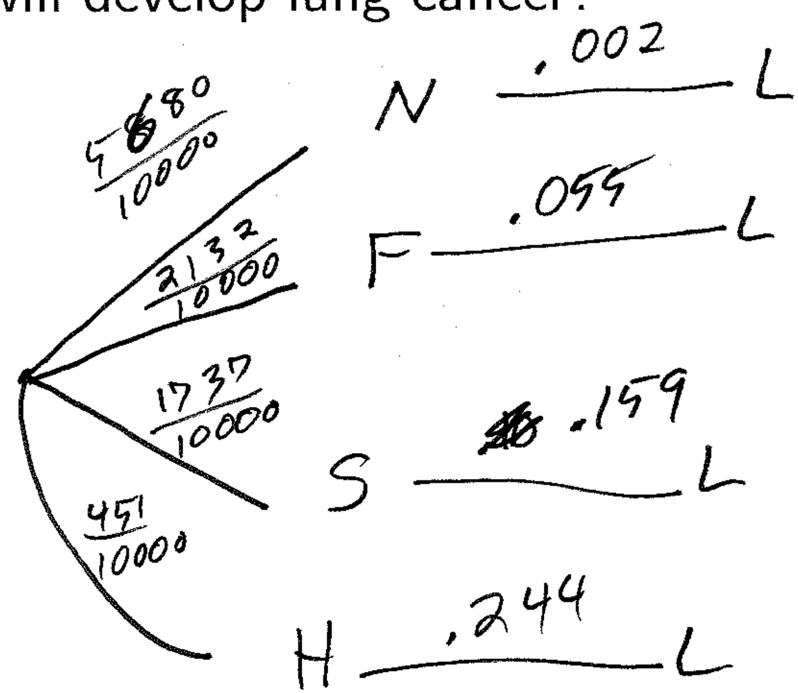
Smoking and lung cancer

A sample of 10,000 men.

Sum must be 10,000

- 5680 of these men never smoked
- 2132 are former smokers
- 1737 are current "light" smokers
- 451 The remaining men are "heavy" smokers.

What is the probability a randomly chosen male from this group will develop lung cancer?



$$P(L) = \frac{5680}{10000} \cdot .002 + \frac{2132}{10000} \cdot .055 + \frac{1737}{10000} \cdot .159 + \frac{451}{10000} \cdot .244$$

$$= 0.0514847$$

$$\approx 5.1\%$$

More on Conditional Probability

- More often than not, $P(A|B) = \frac{P(A\cap B)}{P(B)}$ is used to determine $P(A \cap B)$.
- Thus,

$$P(A \cap B) = P(A|B) \cdot P(B)$$

- Furthermore, to find $P(A)$, one often finds a disjoint partition of the sample space,

$$S = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$$

  with $B_i \cap B_j =$ for $i \neq j$
- Then

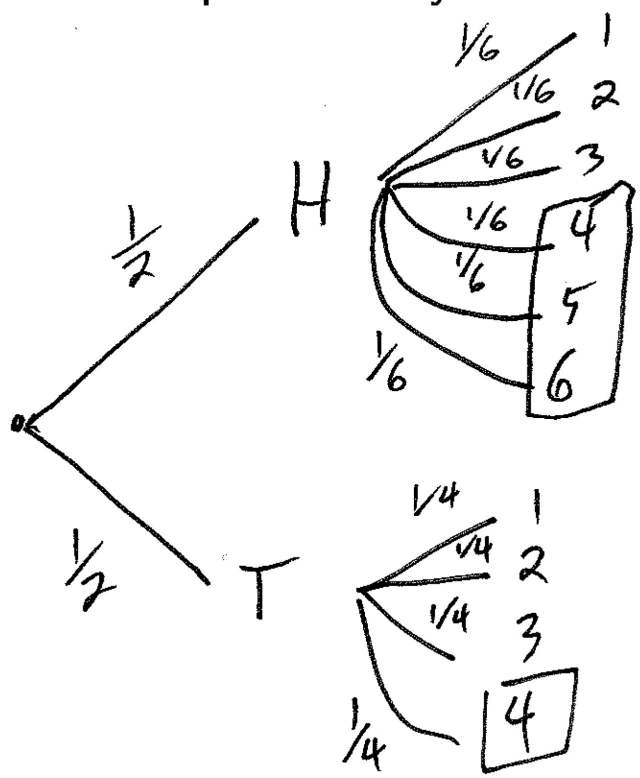
$$P(A) = \sum_{j=1}^{n} P(A|B_j) \cdot P(B_j)$$

Rolling coins

Flip a fair two sided coin.

- If heads, roll a fair 6 sided die (with faces 1, 2, 3, 4, 5, 6).
- If tails, roll a fair 4 sided die (with faces 1, 2, 3, 4).

What is the probability of rolling a number larger than 3?



$$P(>3) = P(>3|H)\,P(H) + P(>3|T)\,P(T)$$

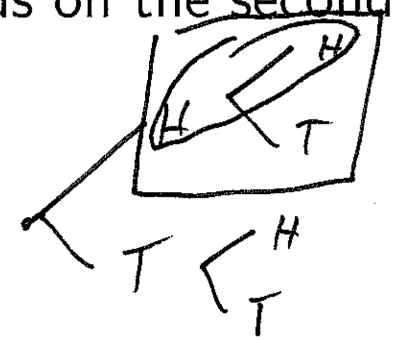
$$= \frac{3}{6}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{3}{8}$$

Flip you for it

A fair two sided coin is flipped twice.

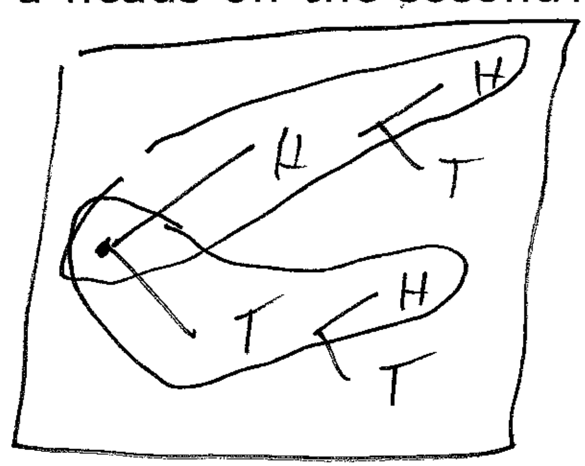
- What is the probability of getting a heads on the second, given that the first was a heads?

$P(H \text{ on } 2 \mid H \text{ on } 1) = \frac{1}{2}$



- What is the probability of getting a heads on the second?

$P(H \text{ on } 2) = \frac{1}{2}$



Independence

- The conditional probability $P(A|B)$ takes into account how the information in the event B affects the likelihood of A.
- Some information may be totally irrelevant, like the probability the coin turns up heads, given that the person flipping the coin is named Ashley.
- In this case we would say the events $A = \{\text{coin is flipped by someone named Ashley}\}$ and $B = \{\text{coin turns up heads}\}$ are independent.

Testing for independence

Given that $P(A)$ and $P(B)$ are nonzero, all three of these hold when A and B are independent:

- $P(A \cap B) = P(A)P(B)$
- $P(A|B) = P(A)$
- $P(B|A) = P(B)$

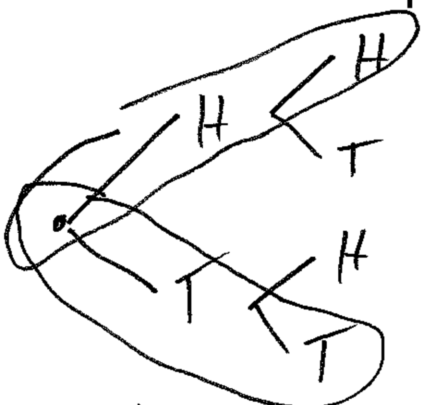
In fact, if any one of these hold, then all three hold. Thus, to verify that two events are independent, we only need to check that one of these are true.

Flip you for it

A fair two sided coin is flipped twice. A is the event "heads on the first flip." B is the event "outcomes of the two flips are the same." Are A and B independent or not?

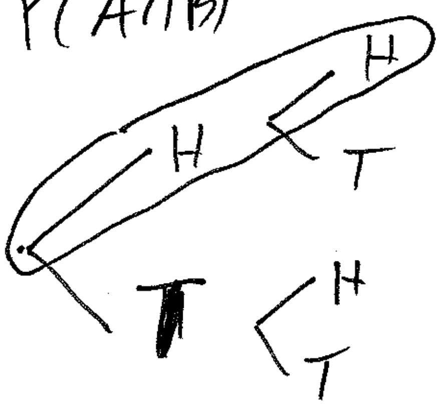
$P(A) = \frac{1}{2}$

B



$P(B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$



$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{4}$ // Independent.

Independent Cards

Standard deck of 52 cards.

- A is event "first card is Ace" $P(A) = \frac{1}{13}$
- B is event "second card is Ace" $P(B) = \frac{1}{13}$
- C is event "first card is Hearts" $P(C) = \frac{1}{4}$
- D is event "second card is Spades" $P(D) = \frac{1}{4}$.

Pairs

AB	Depend.
AC	Ind.
AD	Ind
BC	Ind
BD	Ind
CD.	Dep.

For each pair of the above events, determine if the pair of events are independent.

A & B

- $P(B|A) = \frac{3}{51}$ one ace has been drawn already. one card has been drawn already.

Not = to P(B), so A & B dependendent.

~~P(B|C)~~

A & C ⇒ suit should not affect rank, expect these independent.

More formally, $P(A \cap C) = P(\text{A of Hearts}) = \frac{1}{52} = \frac{1}{13} \cdot \frac{1}{4} = P(A)P(C)$.

So, independent.