MA162: Finite mathematics

Systems of Linear Equations

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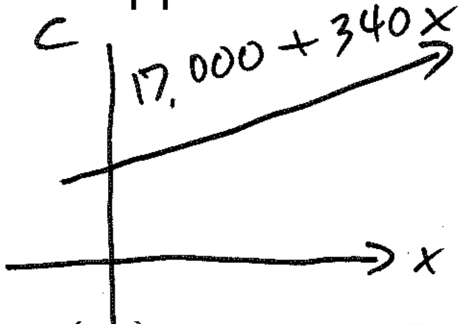
Schedule:

Cost, Revenue, Profit, Break-even I

You have 100 acre plot of land. You plan to grow and sell corn. Suppose the cost of growing x acres of corn is



$$C(x) = \$17,000 + \$340x$$

(a.) In plain English, what is the meaning of the vertical intercept?

$C(0) = 17,000.$

\$17,000 is the fixed cost, ie, the cost of doing business, even if you don't produce anything

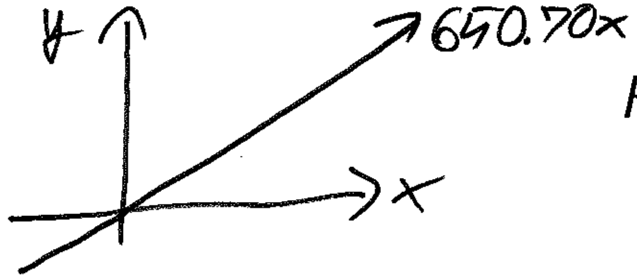
(b.) In plain English, what is the meaning of the slope?

Slope \$340/Acre.

Cost of growing each additional acre of corn.

Cost, Revenue, Profit, Break-even I

Suppose the revenue generated by selling x acres worth of corn is



$$R(x) = \$650.70x$$

(a.) In plain English, what does the slope mean?

Slope = \$650.70/acre.
Each acre of corn brings in \$650.70 of revenue.

(b.) In plain English, what does the vertical intercept mean?

$R(0) = 0$

You earn no revenue if you sell no corn

Cost, Revenue, Profit I

(a.) If you grow and sell 20 acres of corn, which is greater, the cost or the revenue?

$C(20) = 17000 + 340 \cdot 20 = \$23800$

$R(20) = 650.70 \cdot 20 = \$13014.$

Cost is higher.

(b.) If you grow and sell 70 acres of corn, which is greater, the cost or the revenue?

$C(70) = 17000 + 340 \cdot 70 = \$40{,}800$

$R(70) = 650.70 \cdot 70 = \$45{,}549.$

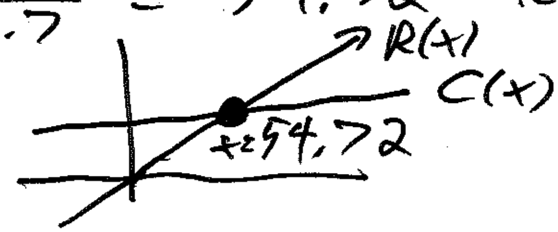
Revenue is higher.

(c.) Determine the value of x at which the revenue and cost functions are equal.

$C(x) = R(x) \Rightarrow 17000 + 340x = 650.7x \Rightarrow 17000 = 310.7x$

$\Rightarrow x = \frac{17000}{310.7} = 54.72$ acres.

Have to grow at least 54.72 acres before you earn a profit.



Supply, Demand, and Equilibrium

- Your tech company is about to unveil its new *Me-Pod*.
- Your task is to determine how much to charge for this device.
- Charge too little, then your company will sell a lot of these devices, but will generate very little revenue on each individual sale.
- Charge too much, then your company will generate a lot of revenue on each individual sale, but very few of these devices will be sold.
- What to do?

The Demand Function I

First, you ask your marketing team to determine how many of these devices your company might expect to sell at various prices. Their results:

- Expect to sell 1000 units per month if you charge \$200 per Me-Pod.
- Expect to sell 800 units per month if you charge \$250 per Me-Pod.

Let N denote the number of Me-Pods sold per month and let p denote the price per Me-Pod.

$$\text{Slope} = \frac{1000-800}{200-250} = \frac{100}{-50} = -4.$$

You find

$$N = -4p + 1800$$

So $N - 1000 = -4(p-200)$

$\Rightarrow N = -4p + 800 + 1000$

(Double check! Make sure you can determine the line if given two points)

This is the *demand function*.

How much will be sold at a given price.

The Supply Function I

Next, you ask your manufacturing division to determine how many of these devices they are willing to produce at various prices. Their answer:

- Willing to produce 600 units per month if they sell for \$200 per Me-Pod.
- Willing to produce 1000 units per month if they sell for \$250 per Me-Pod.

$$\text{Slope} = \frac{600-1000}{200-250} = \frac{-400}{-50} = 8.$$

You find

$$N = 8p - 1000$$

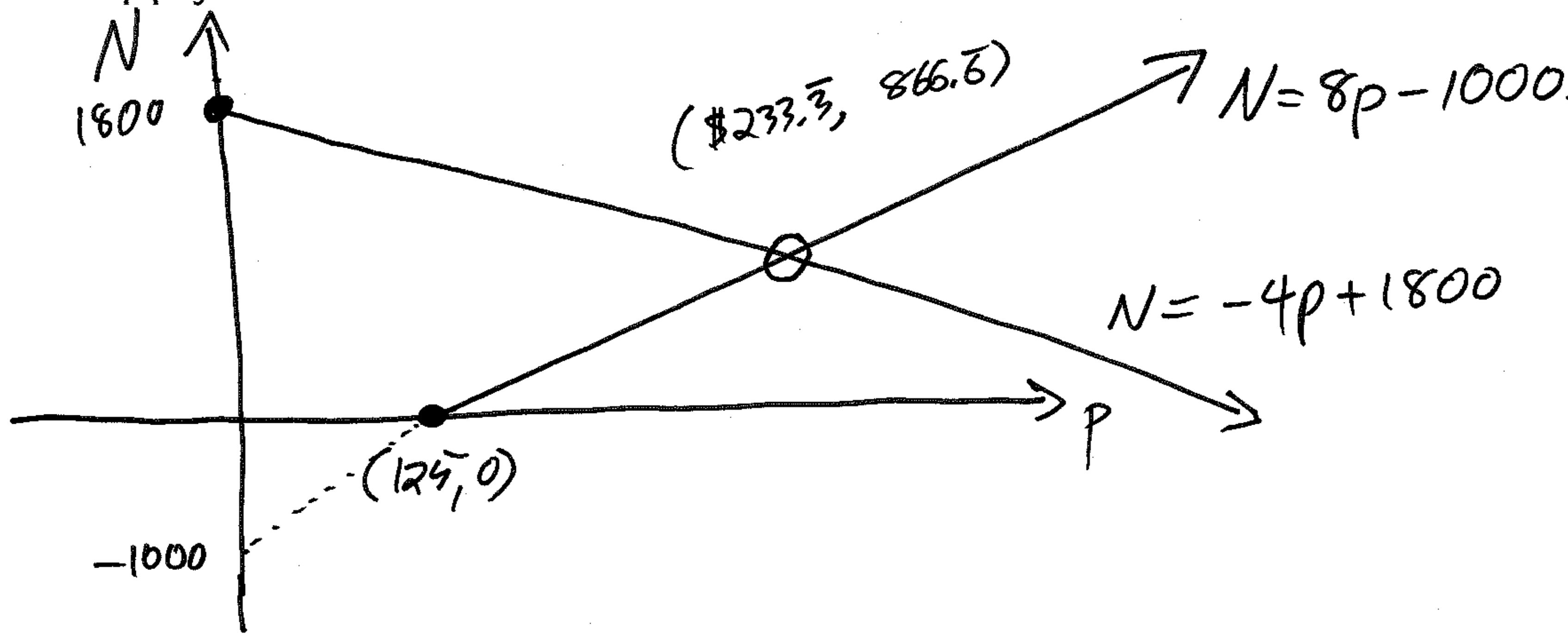
$$N - 600 = 8(p-200)$$
$$N = 8p - 1600 + 600$$
$$N = 8p - 1000.$$

This is the *supply function*.

This is how many items producer is willing to manufacture @ given price per unit.

Supply and Demand

Plot the supply and demand functions on the same set of axes



Now find the point of intersection.

$$\begin{cases} N = 8p - 1000 \\ N = -4p + 1800 \end{cases} \Longrightarrow \begin{cases} N - 8p = -1000 \\ N + 4p = 1800 \end{cases}$$

Bottom ↦ Bottom − Top

$$\begin{cases} N - 8p = -1000 \\ 12p = 2800 \end{cases} \Longrightarrow \begin{cases} N - 8p = -1000 \\ p = 233.\overline{3} \end{cases}$$

$$N = 8(233.\overline{3}) - 1000 = 866.\overline{6}$$

The point $(233.\overline{3}, 866.\overline{6})$ is the ~~Break~~ "Equilibrium point"

If charge > $\$233.\overline{3}$, then producers will produce more than consumers want to purchase.

If charge < $\$233.\overline{3}$, then consumers will want to purchase more than producers are willing to produce.

Two Linear Equations in Two Unknowns

- Many applied problems can be solved by determining the point where two lines cross.
- This is equivalent to solving a system of two linear equations in two unknown.
- $$\begin{array}{rcrcl} ax & + & by & = & c \\ dx & + & ey & = & f \end{array}$$
- Systems of equations can be solved using either
  - substitution
  - elimination
- We will focus on elimination: elimination provides an efficient method for solving very large systems (lots of equations and lots of unknowns)

Special Cases: Inconsistent Systems

Find the solution to the system of equations:

$$\begin{array}{rcrcl} x & - & y & = & 4 \\ -3x & + & 3y & = & 5 \end{array}$$

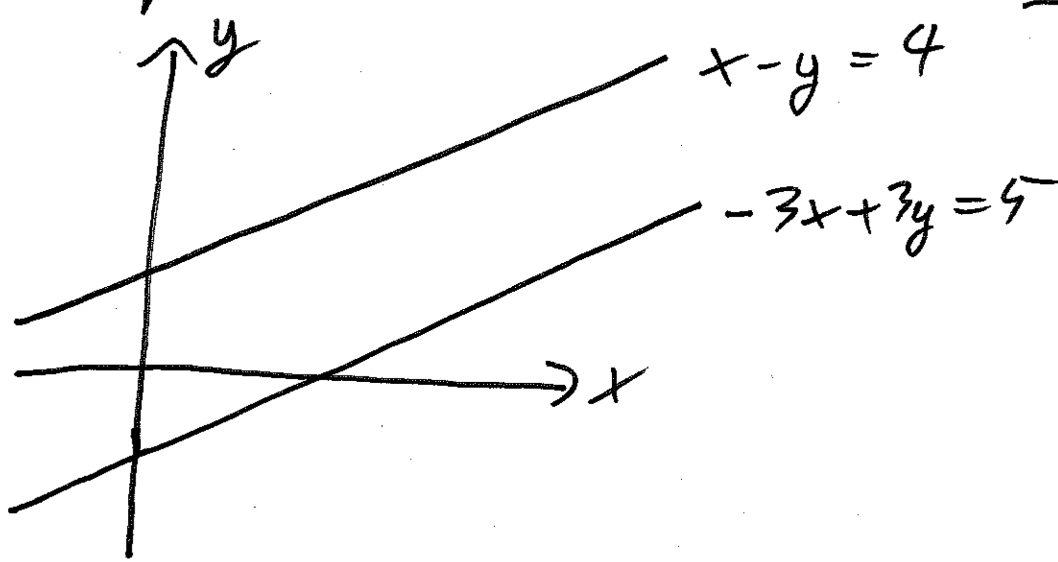
Bottom Row $\mapsto -\frac{1}{3}$ Bottom Row.

$$\Rightarrow \begin{cases} x - y = 4 \\ x - y = -5/3 \end{cases} \xrightarrow{\substack{\text{Bottom Row} \\ -\text{ Top Row}}} \begin{cases} x - y = 4 \\ 0x \quad 0y = -\frac{17}{3} \end{cases}$$

Bottom row now reads $0 = -\frac{17}{3}$, which is nonsense

This means system has $\boxed{\text{no solution.}}$

Graphically



Lines are parallel.
Never intersect
So no solution.

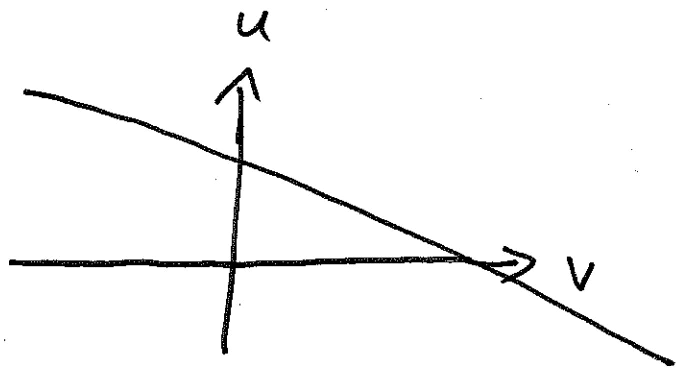
Special Cases: Underdetermined Systems

Find the solution to the system of equations:

$$\begin{array}{rcrcl} 3u & + & 6v & = & 9 \\ 2u & + & 4v & = & 6 \end{array} \xRightarrow[\text{Bottom}/2.]{\text{Top}/3} \begin{cases} u + 2v = 3 \\ u + 2v = 3 \end{cases}$$

Same Line!

Infinitely many intersections.



Solutions can be parametrized.

$u + 2v = 3 \Rightarrow u = 3 - 2v.$

So solution is $\begin{cases} \cancel{u \quad u} \quad v = v \\ u = 3 - 2v. \end{cases}$

Alternatively, could express v in terms of u.

$u + 2v = 3 \Rightarrow 2v = 3 - u \Rightarrow v = \frac{3-u}{2}.$

So $\begin{cases} u = u \\ v = \frac{3-u}{2}. \end{cases}$

In WebWork, you will be told which variable should be solved for in terms of the other.

Bigger systems

The management of Hartman Rent-A-Car has allocated \$2.25 million to buy a fleet of new automobiles consisting of compact, intermediate-size, and full-size cars. Compacts cost \$18,000 each, intermediate-size cars cost \$27,000 each, and full-size cars cost \$36,000 each. If Hartman purchases twice as many compacts as intermediate-size cars and the total number of cars to be purchased is 100, determine how many cars of each type will be purchased. (Assume that the entire budget will be used.)

C = # Compact Cars., I = # intermediate, F = # Full.

Conditions:

$$C + I + F = 100 \quad (\text{total \# cars})$$

$$18000\,C + 27000\,I + 36000\,F = 2{,}250{,}000 \quad (\text{Total Budget})$$

$$C - 2I = 0 \quad (\#C = 2\#I)$$

$$\begin{cases} C + I + F = 100 \\ C - 2I = 0 \\ 18000C + 27000I + 36000F = 2{,}250{,}000 \end{cases}$$

In next section, we'll build systematic approach.
For now, we'll try substitution.
2^{nd} Eq $\Rightarrow C = 2I$, so replace all C with $2I$.

$$\begin{cases} 3I + F = 100 \\ C = 2I \\ 18000 \cdot 2I + 27000I + 36000F = 2{,}250{,}000 \end{cases}$$

$$\Rightarrow \begin{cases} 3I + F = 100 \\ 63000I + 36000F = 2{,}250{,}000 \\ C = 2I \end{cases}$$ ← Divide by 9000

$$\begin{cases} 3I + F = 100 \\ 7I + 4F = 250 \\ C = 2I \end{cases}$$ ← $F = 100 - 3I$ use this to replace F in line 2.

$$\begin{cases} F = 100 - 3I \\ 7I + 4(100 - 3I) = 250 \\ C = 2I \end{cases} \Rightarrow \begin{cases} F = 100 - 3I \\ 7I + 400 - 12I = 250 \\ C = 2I \end{cases} \Rightarrow$$

$$\begin{cases} F = 100 - 3I \\ 5I = 150 \\ C = 2I \end{cases} \Rightarrow \begin{cases} F = 100 - 3I \\ I = 30 \\ C = 2I \end{cases}$$

$$\begin{cases} F = 100 - 3 \cdot 30 = 10 \\ I = 30 \\ C = 2 \cdot 30 = 60. \end{cases} \Rightarrow \begin{cases} F = 10 \\ I = 30 \\ C = 60. \end{cases}$$

Recommendation:

Purchase 10 full-sized cars,
30 intermediate
& 60 compact cars.