MA162: Finite mathematics

Gauss Jordan

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SCHEDULE: Solutions

Last Time

- We solved several systems of two linear equations in two unknowns

- We also solved one system of three linear equations in three unknowns

- With the three by three systems and larger systems, it is very easy to get lost without a systematic approach.

Gauss-Jordan Method

- Given a system of linear equations, the Gauss-Jordan Method will replace our system with another system.
- The new system will be
  - equivalent to the original system, i.e., the two systems will have the same solutions
  - easier to solve than the original system
- This is accomplished by performing row operations

Row Operations

Performing these operations on a system of linear equations will not change the solution set:

- Row Op 1: Swap the order of any two equations
- Row Op 2: Replace any equation by a nonzero constant multiple of itself
- Row Op 3: Replace an equation by the sum of that equations with a constant multiple of any other equation

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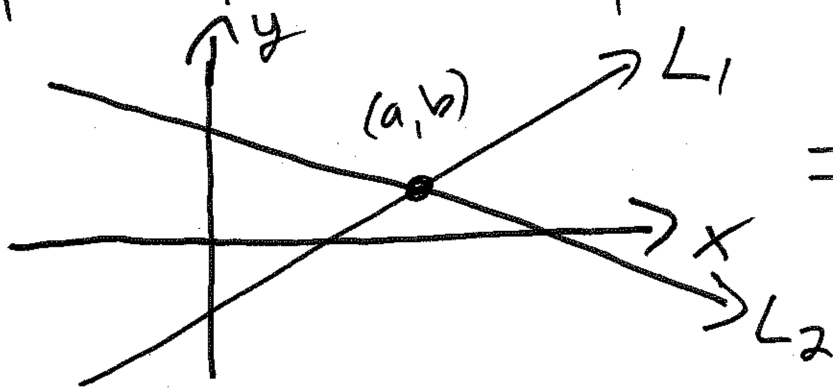
- Row Ops 1 & 2 do not change the individual lines.
- Row Op 3 manages to rotate lines about the intersection point.



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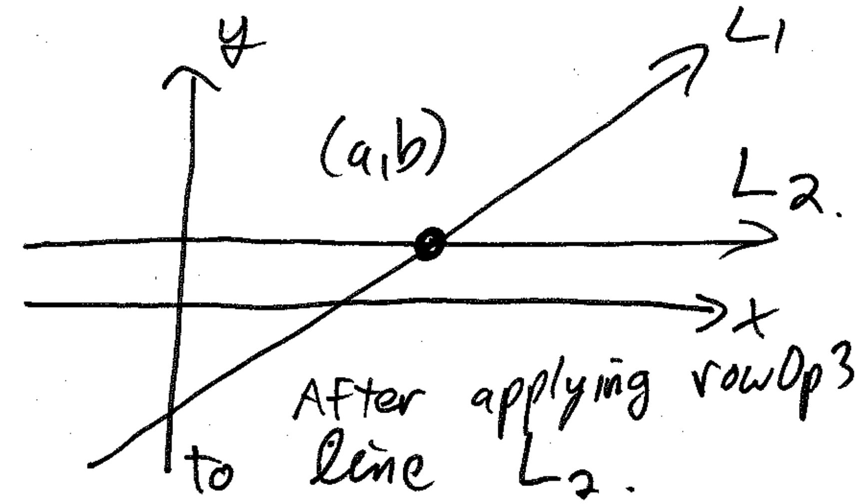

The Challenge

- We could apply the row operations in an ad hoc manner to convert our system into another, equivalent system.
- But the goal is to choose the "right" row operations so that the resulting system is very easy to solve.

Gauss-Jordan: Warming up with 2 equations, 2 unknowns

Solve the system of equations: Pivot Entry

$$\begin{array}{rcl} \boxed{x} + y & = & 2 \\ x + 2y & = & 4 \end{array}$$

↓ Use pivot to annihilate all entries directly below.

Solution: First replace row 2 with row 2 - row 1: $R_2 \mapsto R_2 - R_1$

$$\begin{array}{rcl} \boxed{x} + y & = & 2 \\ \boxed{y} & = & 2 \end{array}$$

Move down & right to find next pivot. Now all pivots have been found

Now replace row 1 with row 1 - row 2: $R_1 \mapsto R_1 - R_2$

Use last pivot to annihilate entries above.

$$\begin{array}{rcl} x + & = & 0 \\ y & = & 2 \end{array}$$

Verify $(x, y) = (0, 2)$ is a solution by plugging into the original.

Gauss-Jordan and Augmented Matrices

The previous system can be written more compactly as

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 4 \end{array}\right]$$

We can perform the row operations directly on the augmented matrix: First pivot

$$\left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R2 \mapsto R2 - R1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & 2 \end{array}\right] \xrightarrow{R1 \mapsto R1 - R2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$$

Second pivot

Now put variables back in:

$$\begin{array}{rcrcl} x & + & & = & 0 \\ & & y & = & 2 \end{array}$$

Using Gauss-Jordan: 3 equations, 3 unknowns

Solve the system of equations:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 6 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

First, put into augmented matrix

Pivot Entry

New pivot

$R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 4 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Next pivot

$$\xrightarrow{(R_3)/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_2 - 2R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now back sub

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow$$

Put variables back in:

$$x = -1, \quad y = 4, \quad z = 1$$

Outline of Gauss-Jordan

Part 1: Start at the top left of the matrix.

- In the current row, find first column with nonzero entry. Divide entire row by this entry, so that first nonzero entry in row is a 1.
- This entry found above is called a *pivot*.
- Use the pivot and row operation three for each row below, making the entries below the pivot equal to 0.
- Now move down to the next row and repeat the process

When done, matrix will have an upper triangular form.

Outline of Gauss-Jordan

Part 2: Now that matrix is in upper triangular form, go to the bottom row.

- In the current row, the pivot entry, i.e, the first column entry with a nonzero entry.
- Use this pivot and row operation three for each row above, making the entries above the pivot equal to 0.
- Now move up to the next row and repeat the process

For the problems we deal with today, when done, your matrix will look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

so the solution is $x = a$, $y = b$, and $z = c$.

Applied Problem

Mr. and Mrs. Garcia have a total of $\$100,000$ to be invested in stocks, bonds, and a money market account. The stocks have a rate of return (RoR) of 12%/year, the bonds pay 8%/year and the money market pays 6%/year. The Garcias have stipulated that the amount invested in stocks should be equal to the sum of the amount invested in money markets and 3 times the amount invested in the bonds. How should the Garcias allocate their resources if they require an annual income of $\$10,000$ from their investments?

Garcia's System of Equations

Let S denote the amount in stocks, B the amount in bonds, and M the amount in money markets.

Three equations:
The Garcia's capital: $S + B + M = \$100,000$
Required return: $0.12S + 0.08B + 0.06M = \$10,000$
Desired Diversification: $S = 3B + M = 0$

Rewrite the last one as $S - 3B - M = 0$, and build the matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0.12 & 0.08 & 0.06 & 10000 \\ 1 & -3 & -1 & 0 \end{array}\right]$$

Now apply Gauss-Jordan

Pivot

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 100000 \\ .12 & .08 & .06 & 10000 \\ 1 & -3 & -1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - .12R_1} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 100000 \\ 0 & \textcircled{-.04} & -.06 & -2000 \\ 0 & -4 & -2 & -100000 \end{array}\right]$$

$$\xrightarrow{(R_2)/-.04} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 100000 \\ 0 & \textcircled{1} & 1.5 & 50000 \\ 0 & -4 & -2 & -100000 \end{array}\right] \xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 100000 \\ 0 & \textcircled{1} & 1.5 & 50000 \\ 0 & 0 & \textcircled{4} & 100000 \end{array}\right]$$

$$\xrightarrow{(R_3)/4} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 100000 \\ 0 & \textcircled{1} & 1.5 & 50000 \\ 0 & 0 & \textcircled{1} & 25000 \end{array}\right] \xrightarrow[R_2 - 1.5R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 0 & 75000 \\ 0 & \textcircled{1} & 0 & 12500 \\ 0 & 0 & \textcircled{1} & 25000 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 62500 \\ 0 & \textcircled{1} & 0 & 12500 \\ 0 & 0 & \textcircled{1} & 25000 \end{array}\right] \Longrightarrow$$

Put variables back in:

$S = \$62{,}500$

$B = \$12{,}500$

$M = \$25{,}000$

Garcias should put \$62,500 into stocks, \$12,500 into Bonds & \$25,000 into Money Market

Applied Problem

What if the Garcias want to consider other possibilities, like

- if they invest $\$120,000$ and want return of $\$10,000$
- if they invest $\$100,000$ and want return of $\$12,000$
- if they invest $\$100,000$ and want return of $\$9,000$

You could repeat the previous calculation for each of these right hand sides. BUT, if we know all of the cases they want to consider in advance, we can do it all in a single calculation by using a larger augmented matrix:

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0.12 & 0.08 & 0.06 & 10000 & 10000 & 12000 & 9000 \\ 1 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the coefficients of the variables did not change, we will use the exact same row operations as before.

Applied Problem

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0.12 & 0.08 & 0.06 & 10000 & 10000 & 12000 & 9000 \\ 1 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \mapsto R_2 - 0.12R_1]{R3 \mapsto R_3 - R1} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0 & -0.04 & -0.06 & -2000 & -4400 & 0 & -3000 \\ 0 & -4 & -2 & -100000 & -120000 & -100000 & -100000 \end{array}\right]$$

$$\xrightarrow{R2 \mapsto -(R_2)/4} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0 & 1 & 1.5 & 50000 & 110000 & 0 & 75000 \\ 0 & -4 & -2 & -100000 & -120000 & -100000 & -100000 \end{array}\right]$$

$$\xrightarrow{R3 \mapsto R_3 + 4R2} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0 & 1 & 1.5 & 50000 & 110000 & 0 & 75000 \\ 0 & 0 & 4 & 100000 & 320000 & -100000 & 200000 \end{array}\right]$$

$$\xrightarrow{R3 \mapsto (R_3)/4} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 100000 & 120000 & 100000 & 100000 \\ 0 & 1 & 1.5 & 50000 & 110000 & 0 & 75000 \\ 0 & 0 & 1 & 25000 & 80000 & -25000 & 50000 \end{array}\right]$$

$$\xrightarrow[R_1 \mapsto R_1 - R_3]{R_2 \mapsto R_2 - 1.5R_3} \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 75000 & 40000 & 125000 & 50000 \\ 0 & 1 & 0 & 12500 & -10000 & 37500 & 0 \\ 0 & 0 & 1 & 25000 & 80000 & -25000 & 50000 \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 62500 & 50000 & 87500 & 50000 \\ 0 & 1 & 0 & 12500 & -10000 & 37500 & 0 \\ 0 & 0 & 1 & 25000 & 80000 & -25000 & 50000 \end{array}\right]$$

Now we can read off the answers for each of the Garcia's scenarios. Notice the second and third cases are going to cause trouble, as they require the Garcias to invest negative amounts.