MA162: Finite mathematics

Gauss Jordan

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SCHEDULE: Solutions

Last Time

- We introduced the Gauss-Jordan elimination method for solving systems of linear equations.
- In all of those problems, the system of equations had a unique solution.
- Today, we'll see how to deal with the two other cases: No solution and Infinitely many solutions

Applied Problem

Mr. and Mrs. Garcia have a total of $\$100,000$ to be invested in stocks, bonds, and a money market account. The stocks have a rate of return (RoR) of 12%/year, the bonds pay 8%/year and the money market pays 6%/year. The Garcias have stipulated that the amount invested in stocks should be equal to the sum of the amount invested in bonds plus twice the amount invested in money markets. How should the Garcias allocate their resources if they require an annual income of $\$12,000$ from their investments?

This looks very similar to example from last time. But very small changes in numbers can result in VERY DIFFERENT behavior of solutions.

Garcia's System of Equations

Let S denote the amount in stocks, B the amount in bonds, and M the amount in money markets.

Three equations:
The Garcia's capital: $S + B + M = \$100,000$
Required return: $0.12S + 0.08B + 0.06M = \$12,000$
Desired Diversification: $S = B + 2M$

Rewrite the last one as $S - B - 2M = 0$, and build the matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0.12 & 0.08 & 0.06 & 12000 \\ 1 & -1 & -2 & 0 \end{array}\right]$$

Now apply Gauss-Jordan

Doing Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0.12 & 0.08 & 0.06 & 12000 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_2 - 0.12R_1]{R_3 \mapsto R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & \overset{-0.04}{\cancel{-0.03}} & -0.06 & 0 \\ 0 & -2 & -3 & -100000 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto \frac{1}{-0.04} R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & 1 & 1.5 & 0 \\ 0 & -2 & -3 & -100000 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & -100000 \end{array}\right]$$

The bottom row reads $0S + 0B + 0M = -100000$. In other words, $0 = -100,000$!!

The practical interpretation is that there is no way to satisfy all of the Garcia's requirements. They'll either have to invest more, lower their required return, change their desired diversification, etc.

For this particular problem, you may notice something is amiss right from the start. They invested \$100,000 & wanted return of \$12,000, or 12%. Since the stock, bond, & mm RoR were 12%, 8%, 6%, the only way to get 12% is to invest everything in stock. (If they invest anything in bonds or mm, their RoR will be $< 12\%$). But if all goes to stock, then $S = B + 2M$ can't possibly be satisfied.

Identifying Systems with no solution

- Begin using the Gauss-Jordan method on the augmented matrix

- If, at any point in the calculation, you come across a row with all zeros on the left of the vertical bar but a nonzero entry on the right side of the same row, then STOP. Your system is inconsistent. In other words, it has no solution.

Revising the Garcia's Expectations

Mr. and Mrs. Garcia have a total of $\$100,000$ to be invested in stocks, bonds, and a money market account. The stocks have a rate of return (RoR) of 12%/year, the bonds pay 8%/year and the money market pays 6%/year. The Garcias have stipulated that the amount invested in stocks should be equal to the sum of the amount invested in bonds plus twice the amount invested in money markets. How should the Garcias allocate their resources if they require an annual income of $\$10,000$ from their investments?

Thus, the Garcias have decided to lower their expected return.

Doing Gauss-Jordan on revised system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0.12 & 0.08 & 0.06 & 10000 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_2 - 0.12R_1]{R_3 \mapsto R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & -0.04 & -0.06 & -2000 \\ 0 & -2 & -3 & -100000 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto \frac{1}{-0.04} R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & 1 & 1.5 & 50000 \\ 0 & -2 & -3 & -100000 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & 1 & 1.5 & 50000 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Like before, there is no pivot column in the bottom row. However, things are very different this time, as the bottom row simply reads $0 = 0$, a true, but not useful, assertion.

The only thing that changed between this & previous problem was desired return $\$12{,}000 \longrightarrow \$10{,}000$.
Since the ~~w~~ left side of matrix is still the same, we end up using the exact same row ops.

Finishing the calculation

Lets continue with Gauss-Jordan:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100000 \\ 0 & 1 & 1.5 & 50000 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -0.5 & 50000 \\ 0 & 1 & 1.5 & 50000 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put the variables back in

$$S - 0.5M = 50000$$
$$B + 1.5M = 50000$$

t is a "parameter" or "free variable"

Rewrite as follows:

$$S = 50000 + 0.5t$$
$$B = 50000 - 1.5t$$
$$M = t$$

Interpreting the Result

$$S = 50000 + 0.5t$$
$$B = 50000 - 1.5t$$
$$M = t$$

The Garcias have flexibility in choosing their allocation.
For example, if they put \$10,000 in the money market, then they should put $S = \$50,000 + 0.5 * \$10,000 = \$55,000$ in stocks and $B = \$50,000 - 1.5 * \$10,000 = \$35,000$ in bonds.
Algebraically, the value of t could be anything. It is a free parameter.

Interpreting the Result

Realistically, each of S, B, and M should be nonnegative (in this class we don't allow short-sells), in which case t can take values between 0 and \$33,333.33. Don't worry if you don't know what a short sale is.
(If $t < 0$ then Garcias put negative amount in money market. If $t > \$33,333.33$ then they put negative amount in bonds.)

Exercise: Find the allocation with the maximal amount invested in stocks

Why 0 to 33,333.33?

Need $M \geq 0 \Rightarrow t \geq 0.$

Need $S \geq 0 \Rightarrow 50000 + \frac{1}{2}t \geq 0 \Rightarrow \frac{1}{2}t \geq -50{,}000$
$\Rightarrow t \geq -100{,}000.$

Subsumed by $t \geq 0$.

Need $B \geq 0 \Rightarrow 50000 - \frac{3}{2}t \geq 0 \Rightarrow \frac{3}{2}t \leq 50000$
$\Rightarrow t \leq \frac{100000}{3} = 33{,}333.33$

So $t \geq 0$ & $t \leq 33{,}333.33$

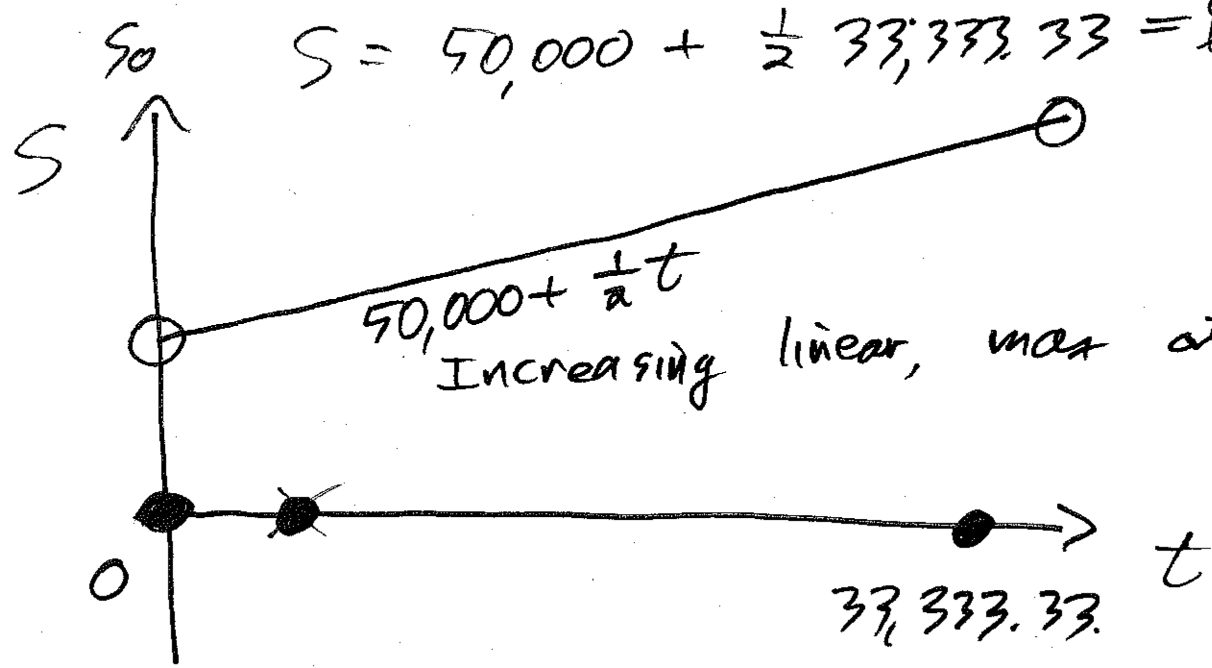
Max amount in stocks:

$S = 50{,}000 + 0.5t$

↑ Positive slope ⇒ Largest when t large.

largest allowed t is $t = 33{,}333.33$

$S = 50{,}000 + \frac{1}{2}\,33{,}333.33 = \$66{,}666.67$



Increasing linear, max at right endpoint

Row-Reduced Form of a Matrix

A matrix is in row-reduced form if

- Each row consisting entirely of zeros lies below any other row with nonzero entries
- The first nonzero entry in a row (reading from left to right) is a 1 (called a leading 1)
- In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row
- If a column contains a leading 1, then the other entries in that column are zeros

Gauss-Jordan puts matrices in Row-Reduced form. You know that you are <u>done with Gauss-Jordan</u> when your matrix is in Row-Reduced form. But still have to interpret result!!

Are these reduced?

For each of the matrices, decide if they are in row-reduced form. If not, state why and put it into row-reduced form.
In each case, also identify which matrices correspond to inconsistent systems (no solution) and which correspond to underdetermined systems (infinitely many solutions).

Circled entries are pivots

(A) $\left[\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$

- Not Row-reduced, since nonzero in $\square$ above pivot.
- Inconsistent, since bottom row reads $0 = 1$.

Inconsistent: Bottom row reads $0 = 1$.

(B) $\left[\begin{array}{ccc|c} \boxed{2} & 0 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Not row-reduced, since $\square$ is a pivot, but entry is not $= 1$.

Consistent

(C) $\left[\begin{array}{ccc|c} 1 & 0 & \boxed{1} & 6 \\ 0 & 1 & \boxed{2} & 8 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Not row reduced, $\square$ entries need to be zero, since above or below pivot.

(D) $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 6 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 0 & 0 & 1 & 5 & 10 \end{array}\right]$

Not row reduced. "All zero" rows must appear at the bottom.

(A) In practice, we wouldn't bother to further reduce (A) since it is inconsistent. But to practice w/ Row-Reduced form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is now in Row-reduced form.

Note: to check if reduced, look only at left side of |

(B) $$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Now, its row-reduced.

(C)
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 \mapsto R_2 - 2R_3]{R_1 \mapsto R_1 - R_3} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Reduced.

To solve, put variables back in:

$x = 5, \quad y = 6, \quad z = 1.$

(D)

Solving systems with infinitely many solutions

Determine the solution to this system.

System D, after swapping R_2 & R_3.

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 6 \\ 0 & 0 & 1 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in Row-reduced form!

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 6 \\ 0 & 0 & 1 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

~~x+y~~ x & z are pivot variables.
y & w are free variables.

How to interpret?? ① Put variables back in.

$$\begin{aligned} x + 3y + w &= 6 \\ z + 5w &= 10 \end{aligned} \Longrightarrow \begin{aligned} x &= 6 - 3y - w \\ z &= 10 - 5w. \end{aligned}$$

② Put free on right side

③ Introduce parameters for the free variables

say $y = s$, $w = t$

$$\begin{cases} x = 6 - 3s - t \\ y = s \\ z = 10 - 5t \\ w = t \end{cases}$$

This is the solution.