MA 213 — Calculus III Fall 2017
Final Exam December 11, 2017

Name: KEY

Section: ____________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

Exam Scores

Do not write in the table below

Question	Score	Total
1		9
2		9
3		9
4		9
5		9
6		9
7		9
8		10
9		9
10		9
11		9
Total		100

Free Response. Show your work!

1. (9 points) Find an equation for the plane through the points $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.

$(0,1,1) = A$, $(1,0,1) = B$, $(1,1,0) = C$

$\vec{AB} = \langle 1,-1,0 \rangle \qquad \vec{AC} = \langle 1,0,-1 \rangle$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$\therefore a = b = c = 1$ To find d, use say $(0,1,1)$: $1x + 1y + 1z = 2$

$$\therefore \boxed{x + y + z = 2}$$

2. (9 points) Find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$ in the xy-plane.

$\vec{a} = \langle 2,-1 \rangle$ perpendicular to $2x - y = 3$

$\vec{b} = \langle 3,1 \rangle$ perpendicular $3x + y = 7$

$\vec{a} \cdot \vec{b} = 6 - 1 = 5 \qquad |\vec{a}| = \sqrt{5} \qquad |\vec{b}| = \sqrt{10}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{5} \cdot \sqrt{10}} = \sqrt{\frac{5}{10}} = \frac{1}{\sqrt{2}}$$

$$\therefore \boxed{\theta = \frac{\pi}{4}}$$

OR:

$y = 2x - 3$

Parametric form:

$\langle t, 2t-3 \rangle$

$y = 7 - 3x$

Parametric form:

$\langle t, 7-3t \rangle$

So vectors $\parallel$ to lines are $\langle 1,2 \rangle$ and $\langle 1,-3 \rangle$

$\vec{a} \cdot \vec{b} = -5$

$|\vec{a}| = \sqrt{5} \qquad |\vec{b}| = \sqrt{10}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|a||b|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

This gives $\theta = 3\pi/4$ but acute angle is $\pi/4$

$xe^x = x + x^2 + \cdots$

Free Response. Show your work!

3. (9 points) Find the curvature of the plane curve $y = xe^x$ at the point $(0, 0)$.

$$\kappa(x) = \frac{|y''|}{[1+(y')^2]^{3/2}}$$

$y'(x) = e^x + xe^x$ $\quad y'(0) = 1$

$y''(x) = 2e^x + xe^x$ $\quad y''(0) = 2$

$$\kappa(0) = \frac{|2|}{[1+1^2]^{3/2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

4. (9 points) Find an equation for the tangent plane to the surface $z = x\sin(x+y)$ at the point $(-1, 1, 0)$. Write the equation in the form $x + by + cz = d$.

$$\frac{\partial z}{\partial x} = \sin(x+y) + x\cos(x+y) \qquad \frac{\partial z}{\partial x}(-1,1) = \sin(0) + (-1)\cos(0) = -1$$

$$\frac{\partial z}{\partial y} = x\cos(x+y) \qquad \frac{\partial z}{\partial y}(-1,1) = (-1)\cos(0) = -1$$

Tgt plane at $(a, b, f(a,b))$: $\quad z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

In our case:

$$z - 0 = (-1)(x+1) + (-1)(y-1)$$

$$\therefore z = -x-1-y+1 \Rightarrow z = -(x+y)$$

$$\therefore \boxed{x+y+z=0}$$

Free Response. Show your work!

5. (9 points) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$x^2 + 2y^2 + 3z^2 = 1.$$

$2x + 6z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-2x}{6z} = \boxed{\frac{-x}{3z}}$

$4y + 6z\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{4y}{6z} = \boxed{\frac{-2y}{3z}}$

6. (9 points) Find all the points on the line $x = 3$ at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

$\nabla f(x,y) = \langle 2x-2, 2y-4 \rangle = (2x-2)\hat{i} + (2y-4)\hat{j}$

① Along $x=3$ $\quad \nabla f(3,y) = 4\hat{i} + (2y-4)\hat{j}$

② $\nabla f(3,y)$ is a multiple of $\hat{i}+\hat{j}$ if

$4 = 2y-4$

$8 = 2y$

$y = 4$

$\therefore$ There is only one such point, $\boxed{(x,y) = (3,4)}$

Free Response. Show your work!

7. (9 points) Find the critical points of $f(x, y) = x^2 + y^4 + 2xy$ and classify them as local maximum, local minimum, or saddle point.

$$\frac{\partial f}{\partial x} = 2x + 2y \qquad \frac{\partial f}{\partial y} = 4y^3 + 2x$$

(1) C.P. : $\frac{\partial f}{\partial x} = 0 \Rightarrow y = -x \qquad \frac{\partial f}{\partial y} = 0 \Rightarrow 4(-x)^3 + 2x = 0$

$$2x = 4x^3$$

$$2x(2x^2 - 1) = 0$$

$$\therefore x = 0, \; x = \pm \frac{1}{\sqrt{2}}$$

C.P. $(0,0), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(2) Hessian matrix : $\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2$

$H(f)(x,y) = \begin{pmatrix} 2 & 2 \\ 2 & 12y^2 \end{pmatrix}$ Determinant $= 24y^2 - 4$

(3) Classification of pts :

$(0,0) \quad D = -4 \quad \boxed{\text{Saddle}}$

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad D = 24\frac{1}{2} - 4 = +8 \quad \frac{\partial^2 f}{\partial x^2}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 \quad \boxed{\text{Minimum}}$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad D = +8 \quad \frac{\partial^2 f}{\partial x^2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 \quad \boxed{\text{Minimum}}$

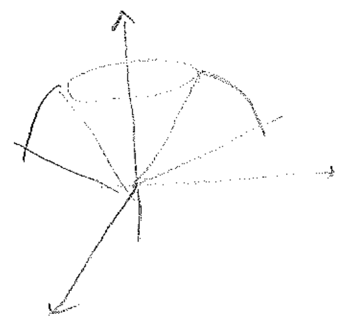
Free Response. Show your work!

8. (10 points) Use spherical coordinates (ρ, θ, ϕ) to find the volume of the part of the ball $\rho \leq 3$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

$0 \leq \rho \leq 3$

$0 \leq \theta \leq 2\pi$

$\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$



$$V = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \sin\phi \left(\int_0^3 \rho^2 d\rho \right) d\theta \, d\phi$$

$$= \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \sin\phi \, (9) \, d\theta \, d\phi$$

$$= \int_{\pi/6}^{\pi/3} 18\pi \sin\phi \, d\phi$$

$$= 18\pi \left(-\cos\phi \right) \Big|_{\pi/6}^{\pi/3}$$

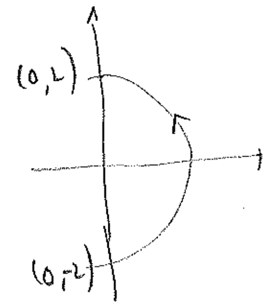
$$= 18\pi \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right)$$

$$= \boxed{9\pi(\sqrt{3} - 1)}$$

9. (9 points) Evaluate

$$\int_C xy^4\, ds,$$

where C is the right half of the circle $x^2 + y^2 = 4$.



1. $x(t) = 2\cos(t)$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

   $y(t) = 2\sin(t)$

2. $x'(t) = -2\sin(t)$ $\quad y'(t) = 2\cos(t)$

   $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = 2\, dt$

3. $\int_C xy^4\, ds = \int_{-\pi/2}^{\pi/2} 32\cos(t)\sin^4(t) \cdot 2\, dt$

   $u = \sin(t)$

   $du = \cos(t)\, dt$

t	u
$-\pi/2$	-1
$\pi/2$	$+1$

$= \int_{-1}^{1} 64u^4\, du$

$= \frac{64u^5}{5}\Big|_{-1}^{1}$

$= \boxed{\frac{128}{5}}$

Free Response. Show your work!

10. (9 points) Let $\mathbf{F}(x,y) = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$. Find a potential function for $\mathbf{F}(x,y)$ and evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any path from $(1,0)$ to $(2,1)$.

(1) $\frac{\partial f}{\partial x} = 2xe^{-y}$ (2) $\frac{\partial f}{\partial y} = 2y - x^2e^{-y}$

From (1): (integrate in x)

(3) $f(x,y) = x^2e^{-y} + C(y)$

using (3) in (2):

$-x^2e^{-y} + C'(y) = 2y - x^2e^{-y}$

$\therefore C'(y) = 2y$

$C(y) = y^2 + C$

$\therefore f(x,y) = x^2e^{-y} + y^2 + C$

Hence $\int_C \vec{F} \cdot d\vec{r} = f(2,1) - f(1,0)$

$= (4e^{-1} + 1) - (1)$

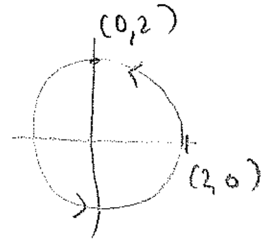
$= \boxed{4e^{-1}}$

Free Response. Show your work!

11. (9 points) Use Green's Theorem to evaluate

$$\int_C y^3\,dx - x^3\,dy,$$

where C is the positively oriented circle $x^2 + y^2 = 4$.



$$P(x,y) = y^3 \qquad Q(x,y) = -x^3$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3x^2 - 3y^2 = -3(x^2+y^2)$$

By Green's theorem,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D -3(x^2+y^2)\,dA$$

polar coordinates

$$= \int_0^{2\pi}\int_0^2 -3r^2\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{3r^4}{4}\right]\Big|_0^2 \,d\theta$$

$$= 2\pi \cdot (-3\cdot 4)$$

$$= \boxed{-24\pi}$$