Exam 2

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. The domain of the function $f(x, y) = \sqrt{x-1} + \sqrt{y-2} + \sqrt{3-x} + \sqrt{4-y}$ is:
   - A. The region between two vertical lines
   - B. The region between two horizontal lines
   - **C. A square in the xy plane.**
   - D. The first quadrant
   - E. None of the above

2. The equation of the tangent line to the curve $\mathbf{r}(t) = \langle \cos(t), \sin(2t), e^{3t} \rangle$ at $t = 0$ is:
   - A. $\mathbf{r}(t) = \langle 2t, 1, 1+3t \rangle$
   - B. $\mathbf{r}(t) = \langle 2, t, 1+3t \rangle$
   - C. $\mathbf{r}(t) = \langle 1, 2t, 1+2t \rangle$
   - **D.** $\mathbf{r}(t) = \langle 1, 2t, 1+3t \rangle$
   - E. $\mathbf{r}(t) = \langle 1, t, 1+2t \rangle$

3. If $f(x, y) = x^2 - 2y^2$, $x = u - v$ and $y = u + v$ then $\partial f(x, y)/\partial v$ is equal to:
   - A. $-2(u^2 + 6uv + v^2)$
   - **B.** $-(6u + 2v)$
   - C. $6u + 2v$.
   - D. $-(2u + 6v)$
   - E. $2(u^2 + 6uv + v^2)$

4. Suppose that z satisfies the equation $z^2 + xy - y^3 = 3$. Assuming that this defines z as an implicit function of x, y, determine $\partial z/\partial x + \partial z/\partial y$ at the point $(x, y, z) = (1, 2, 3)$.

   A. $-3/2$

   B. $19/6$

   C. -19

   D. -9

   E. $3/2$

5. The Laplacian of a function $f = f(x, y)$ is defined to be $f_{xx} + f_{yy}$. Which of the following functions has Laplacian equal to zero?

   A. $f = x^2 - 2y^2 + 2xy$

   B. $f = x^2 - 2y^2 + 3xy$

   C. $f = -3x^2 - 3y^2 + 6xy$

   D. $f = 3x^2 + 3y^2 + 6xy$

   E. $f = 3x^2 - 3y^2 - 6xy$

6. A curve C is the intersection of the surfaces $F = x^2 + 4y^2 - 8 = 0$ and $G = z - 2x^2 - y^2$. The tangent line to C at the point $P = (2, 1, 9)$ has a direction vector equal to

   A. $\langle 8, -4, 56 \rangle$

   B. $\langle 8, 2, -1 \rangle$

   C. $\langle 4, 8, 0 \rangle$

   D. $\langle 2, 1, 9 \rangle$

   E. $\langle 12, 10, -1 \rangle$

7. Let $f(x,y,z) = xy^2 + yz^2 + zx^2$. The directional derivative of $f(x,y,z)$ at the point $P = (-1,1,2)$ in the direction of $v = \langle 1,2,-2\rangle$ is:

   A. -1

   B. -3

   C. 1

   D. 3

   E. 5

8. Let $f(x,y) = 2xe^y - 3ye^x + x - y$. The directional derivative of $f(x,y)$ at the point $(0,0)$ is equal to zero for which of the following directions?

   A. $\langle 1,1\rangle$

   B. $\langle 4,4\rangle$

   C. $\langle 3,3\rangle$

   D. $\langle -3,4\rangle$

   E. $\langle 4,3\rangle$

9. If $\int_0^2 f(x)\,dx = 5$ and $\int_2^3 g(y)\,dy = 7$ and let $I = \int\int_R f(x)g(y)\,dA$ where $R = [0,2]\times[2,3]$. The value of I is:

   A. 12

   B. 6

   C. 35

   D. 24

   E. *Not enough information to decide*

10. Let $I = \int\int_R (2x+3y)\,dA$ where R is the region defined by $0 \le x \le 5$ and $0 \le y \le 2x$. The value of I is:

    A. 80

    B. $1250/3$

    C. $80/3$

    D. 31

    E. 20

Free Response Questions

11. (10 points) Consider the ellipsoid $z^2 + x^2 + 2y^2 = 10$. The goal of this problem is to determine the maximum and minimum values of the function $f(x, y, z) = x + y + z$ on the ellipsoid.

    Use the Lagrange multiplier method to obtain your answer. Solutions using other methods will receive no credit.

    (a) (3 points) Set up the needed partial derivative equations for the Lagrange multiplier method.

    Solution: Let $g(x, y, z) = x^2 + 2y^2 + z^2 - 10$. Since

    $$\nabla f = \langle 1, 1, 1 \rangle, \quad \nabla g = \langle 2x, 4y, 2z \rangle$$

    the Lagrange equations are

    $$1 = 2\lambda x \qquad \text{(1 points)}$$
    $$1 = 4\lambda y \qquad \text{(1 points)}$$
    $$1 = 2\lambda z \qquad \text{(1 points)}$$

    (b) (5 points) Determine the critical points as deduced from the Lagrange Multiplier method.

    Solution: There are several possible ways to solve the equations in (a) and the constraint equations. Here is one. Solving for λ we see that

    $$1/(2x) = 1/(4y) = 1/(2z), \quad \text{so} \quad 2x = 4y = 2z$$

    because the first set of three expressions all equal λ. Note that $\lambda = 0$ is not an admissible value. Thus $y = x/2$, $z = x$ so substituting into the constraint equation, we get an equation for x:

    $$(5/2)x^2 = 10$$

    or

    $$x = \pm 2.$$

    Hence, the critical points are

    $$\pm(2, 1, 2)$$

    Scoring suggestion:

    Use constraint equation: (1 points)
    Solve for one of the variables (1 points)
    Use linear equations to find remaining variables (2 points)
    Correct critical points (1 points)

(c) (2 points) Identify the absolute maximum and absolute minimum of $f(x, y, z)$ on the ellipsoid.

Solution: The absolute maximum is

$$f(2, 1, 2) = 5 \qquad \text{(1 points)}$$

and the absolute minimum is

$$f(-2, -1, -2) = -5 \qquad \text{(1 points)}$$

12. (15 points) Suppose $f(x,y) = kx^2 + 2kxy + 3y^2 + x^3$ be a function of x, y.

Here k is a parameter.

Answer the following questions.

(a) (3 points) Explain why $P = (0,0)$ is a critical point for the function $f(x,y)$ for all possible real values of k.

Solution: From

$$f_x(x,y) = 2kx + 2ky + 3x^2 \qquad \text{(1 points)}$$
$$f_y(x,y) = 2kx + 6y \qquad \text{(1 points)}$$

we see that

$$f_x(0,0) = f_y(0,0) = 0$$

for all values of k. (1 points)

(b) (3 points) Calculate all the second partial derivatives of the function $f(x,y)$.

Solution:

$$f_{xx}(x,y) = 2k + 6x \qquad \text{(1 points)}$$
$$f_{xy}(x,y) = 2k \qquad \text{(1 points)}$$
$$f_{yy}(x,y) = 6 \qquad \text{(1 points)}$$

(c) (2 points) Show that the determinant of the Hessian matrix, namely

$$D = f_{xx}f_{yy} - f_{xy}^2$$

evaluated at $P = (0,0)$ is $D = 12k - 4k^2$.

Solution:

$$D = (2k + 6x)(6) - 4k^2 \qquad \text{(1 points)}$$

so

$$D(0,0) = 12k - 4k^2 = 4k(3-k) \qquad \text{(1 points)}$$

(d) (4 points) Apply the second derivative test to find all values of k for which $f(x,y)$ has a local minimum at $P=(0,0)$. Be sure to explain how the test is used.

Solution: Local maxima and minima occur when $D>0$, so we must have

$$0<k<3 \quad \text{(2 points)}$$

To have a local minimum, we need $f_{xx}(0,0)>0$. Since $f_{xx}(0,0)=2k$, we need $k>0$. (1 points)
Hence, a local minimum occurs at $P=(0,0)$ for $0<k<3$.

(e) (3 points) Apply the second derivative test to find all values of k for which $f(x,y)$ has a saddle point at $P=(0,0)$. Be sure to explain how the test is used.

Solution: f will have a saddle point at $(0,0)$ if $D<0$ (1 points). Since $D=4k(3-k)$ this will be true for $k<0$ or $k>3$. (2 points)

13. (10 points) Assume that z depends on x, y and that x, y depend on u, v which are presumed independent.

(a) (2 points) Suppose that $z = x^2y + 3xy^2 + xy$. Calculate $\partial z/\partial x$ and $\partial z/\partial y$.

Solution:

$$\frac{\partial z}{\partial x} = 2xy + 3y^2 + y \qquad \text{(1 points)}$$

$$\frac{\partial z}{\partial y} = x^2 + 6xy + x \qquad \text{(1 points)}$$

(b) (4 points) Suppose that $x = u + v^2$ and $y = uv$. Determine the partial derivatives of x and y with respect to u, v.

Solution:

$$\frac{\partial x}{\partial u} = 1 \qquad \text{(1 points)} \qquad \frac{\partial x}{\partial v} = 2v \qquad \text{(1 points)}$$

$$\frac{\partial y}{\partial u} = v \qquad \text{(1 points)} \qquad \frac{\partial y}{\partial v} = u \qquad \text{(1 points)}$$

(c) (2 points) If $u = -1$ and $v = 2$, then determine the values of x, y.

Solution:

$$x = (-1) + 2^2 = 3 \qquad \text{(1 points)}$$

$$y = (-1)(2) = -2 \qquad \text{(1 points)}$$

(d) (2 points) Using the same values of u, v determine the value of $\partial z/\partial u$. Your answer must be a real number and must have supporting work.

Solution: By the chain rule

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

With $x = 3$ and $y = -2$, we have

$$\frac{\partial z}{\partial x} = 2(3)(-2) + 3(-2)^2 + (-2) = -2$$
$$\frac{\partial z}{\partial y} = (3)^2 + 6(3)(-2) + 3 = -24$$

while with $u = -1$ and $v = 2$,

$$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial y}{\partial u} = 3$$

Hence,

$$\frac{\partial z}{\partial u} = (-2)(1) + (-24)(2) = -50$$

Scoring:

No justification: 0 points
Partial credit: 1 point
Correct and fully supported answer: 2 points

14. (15 points) Answer the following questions.

(a) (7 points) Determine an equation of the tangent plane to the surface

$$x^2 + y^3 + xyz = 15$$

at the point $P = (1, 2, 3)$. Give your answer in the form $Ax + By + Cz = D$.

Solution: The gradient of the function $f(x, y, z) = x^2 + y^3 + xyz$ will be normal to the surface (1 points).

Since

$$\nabla f = \langle 2x + yz, 3y^2 + xz, xy \rangle \qquad \text{(2 points)}$$

we have

$$\nabla f(1, 2, 3) = \langle 8, 15, 2 \rangle \qquad \text{(2 points)}$$

so the tangent plane has equation

$$8(x - 1) + 15(y - 2) + 2(z - 3) = 0 \qquad \text{(1 points)}$$

or

$$8x + 15y + 2z = 44 \qquad \text{(1 points)}$$

(b) (5 points) Determine a linear approximation for the function

$$f(x,y) = ye^x - xy + y^2$$

at the point $(0,2)$

Solution: Note that $f(0,2) = 6$. Compute the partial derivatives and evaluate at $(0,2)$:

$$f_x(x,y) = ye^x - y \quad \text{(1 points)}$$
$$f_y(x,y) = e^x - x + 2y \quad \text{(1 points)}$$

so

$$f_x(0,2) = 0 \quad \text{(1 points)}$$
$$f_y(0,2) = 5 \quad \text{(1 points)}$$

and hence

$$L(x,y) = 5y - 4 \quad \text{(1 points)}$$

(c) (3 points) Using your answer from part (b), estimate the value of $f(0.1, 1.9)$ where f is as in part (b). To receive full credit, use the linear approximation method and show all work. An unsupported numerical answer will receive no credit.

Solution:
We'll approximate $f(0.1, 1.9)$ by $L(0.1, 1.9)$. (1 points)
We compute:

$$L(0.1, 1.9) = 6 + 5(-0.1) \quad \text{(1 points)}$$
$$= 5.5 \quad \text{(1 points)}$$

The implied use of the linear approximation method is worth 1 point whether stated or not.