Exam 4

Name: ______________________________ Section and/or TA: ________________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Please give exact rather than numerical answers ($\sqrt{2}$, not 1.414). Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	15	10	100

Multiple Choice Questions

1. If $\mathbf{v} = \langle 1, -2, 3\rangle$ and $\mathbf{w} = \langle -1, 1, 2\rangle$ and $\mathbf{p} = \langle 2, 2, -3\rangle$ then $(\mathbf{v} + \mathbf{p}) \times \mathbf{w} =$
   - A. $\langle -7, 10, 2\rangle$
   - B. $\langle -8, 0, 3\rangle$
   - **C.** $\langle 0, -6, 3\rangle$
   - D. -3
   - E. $\langle 0, 6, -3\rangle$

2. For what value(s) of c does the line $\langle x, y, z\rangle = \langle 2t, ct, 3t\rangle$ lie in the plane $x + 3y + 5z = 0$?
   - A. $-13/3$
   - B. $-11/5$
   - C. $17/3$
   - **D.** $-17/3$
   - E. No value possible.

3. The tangent line to the space curve $\mathbf{r}(t) = \langle e^t, e^{2t}, 5 + t\rangle$ at $t = 0$ meets the xy plane at the point:
   - **A.** $(-4, -9, 0)$
   - B. $(6, 11, 0)$
   - C. $(4, 9, 0)$
   - D. $(-5e, -11e^2, 0)$
   - E. $(5e, 9e^2, 0)$

4. Let $\mathbf{F} = \langle x^2y,\ yz,\ z^2x \rangle$. Then curl $(\mathbf{F})$ is equal to:

   A. $\langle -y, -z^2, -x^2 \rangle$

   B. $\langle x^2 - 2\,zx, -2\,xy + y, z^2 - z \rangle$

   C. $-y - x^2 - z^2$

   D. $x^2 - 2zx - 2xy + y + z^2 - z$

   E. $\langle 2xy, z, 2zx \rangle$

5. The surface $xyz + 2yz + x^2 = 19$ has a normal line T at $P = (1, 2, 3)$. Then T meets the xy plane at point Q which is:

   A. $(8, 9, 6)$

   B. $(8, 9, 0)$

   C. $(-3, -5/2, 0)$

   D. $(6, 5, 0)$

   E. $(-5/2, -3, 0)$

6. The integral $\int_0^1 \int_0^x \int_0^{x+y} (xy + 2yz)\, dz\, dy\, dx$ is equal to:

   A. $19/30$

   B. $7/60$

   C. $1/20$

   D. $9/20$

   E. None of the above

7. Let $x(u,v) = ue^{u+v}$ and $y(u,v) = ve^{2u}$. Then the Jacobian determinant $\left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right|$ at $u = 1, v = 0$ is:

   A. e

   B. $3e^2$

   C. $2e^3$

   D. $2e$

   E. $6e^4$

8. Let C be the curve $\mathbf{r}(t) = \langle t^2, \cos(t), \sin(t)\rangle$ defined on the interval $t = 0$ to $t = \pi$. Let $\mathbf{F} = \langle x, y, z\rangle$ be a vector field.

   Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equal to:

   A. $\pi^4/2$

   B. $\pi + \pi^4/2$

   C. $8\pi^4$

   D. $2\pi + 8\pi^4$

   E. $2 + \pi^3/3$

9. Consider the line integral $\int_C (2x + y)\, dx + (x + z)\, dy + (y - 2z)\, dz$ where C is some curve joining the points $A = (0, 0, 0)$ and $B = (1, 5, 5)$. The value of the integral is:

   A. -18

   B. 56

   C. 6

   D. 32

   E. There is not enough information

10. Let $\mathbf{F}$ denote a vector field and let f define a scalar function of three variables. Which of the following expression *is* a meaningful expression?

    A. div (div $\mathbf{F}$).

    B. curl (div $\mathbf{F}$)

    C. grad (grad $\mathbf{F}$)

    D. grad (grad f)

    E. curl (curl $\mathbf{F}$)

Free Response Questions

11. (10 points) Let S denote the parametric surface defined by the parametrization

$$\mathbf{r}(u,v) = \langle u^2+1, u\cos(v), u\sin(v)\rangle.$$

Answer the following questions.

(a) (3 points) Show that the cross product $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v$ is $\langle u, -2u^2\cos(v), -2u^2\sin(v)$. Be sure to show your work.

Solution:

$$\mathbf{r}_u = \langle 2u, \cos(v), \sin(v)\rangle \qquad \text{(1 points)}$$

$$\mathbf{r}_v = \langle 0, -u\sin(v), u\cos(v)\rangle \qquad \text{(1 points)}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & \cos(v) & \sin v \\ 0 & -u\sin(v) & u\cos(v) \end{vmatrix} \qquad \text{(1 points)}$$

$$= u\mathbf{i} + -2u^2\cos(v)\mathbf{j} - 2u^2\sin(v)\mathbf{k}$$

(b) (5 points) Calculate

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left|\left\langle u, -2u^2\cos(v), -2u^2\sin(v)\right\rangle\right|$$

Solution:

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2 + 4u^4\cos^2(v) + 4u^4\sin^2(v)} \qquad \text{(3 points)}$$

$$= \sqrt{u^2+4u^4} \qquad \text{(1 points)}$$

$$= u\sqrt{1+4u^2} \qquad \text{(1 points)}$$

assuming that $u \geq 0$.

(c) (2 points) Determine the area of the surface S parameterized by $\mathbf{r}(u,v)$ if $1 \leq u \leq 2$ and $1 \leq v \leq 16$.

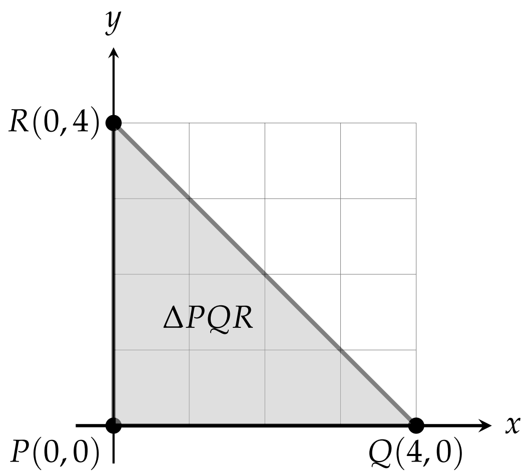
Solution:

$$\begin{aligned}
\iint_S dS &= \int_1^2 \int_1^{16} u\sqrt{1+4u^2}\, dv\, du && \text{(1 points)} \\
&= 15 \int_1^2 u\sqrt{1+4u^2}\, du \\
&= 15 \int_5^{17} \frac{1}{8}\sqrt{w}\, dw \\
&= \frac{15}{8}\frac{2}{3}\left[w^{3/2}\right]_{w=5}^{w=17} \\
&= \frac{5}{4}\left(17\sqrt{17} - 5\sqrt{5}\right) && \text{(1 points)}
\end{aligned}$$

12. (15 points) Consider the function

$$f(x, y) = 10 - x^2 - y^2 + 2\,x + 2\,y.$$

This problem asks you to determine the absolute max/min values of the function $f(x, y)$ over the triangular region $\triangle PQR$ where $P = (0, 0)$, $Q = (4, 0)$ and $R = (0, 4)$.



Answer the following questions.

(a) (2 points) Determine if $f(x, y)$ has any critical point(s) inside the $\triangle PQR$. List all such points along with value(s) of $f(x, y)$ at them.

> **Solution:** Compute
>
> $$\begin{aligned} f_x(x, y) &= -2x + 2 = 2(1 - x) \\ f_y(x, y) &= -2y + 2 = 2(1 - y) \end{aligned} \quad \text{(1 points)}$$
>
> There is a single criticial point at $(1, 1)$ with
>
> $$f(1, 1) = 12 \quad \text{(1 points)}$$

(b) (3 points) Determine all the critical points of $f(x, y)$ restricted to the line PQ. List all such points along with value(s) of $f(x, y)$ at them. Also, list the endpoints of PQ and the values of f at the endpoints.

> **Solution:** On PQ we have
>
> $$f(x, y) = g_1(x) = 10 - x^2 + 2x, \quad g_1'(x) = 2(1 - x).$$
>
> There is a critical point at $x = 1$ so we evaluate
>
> $$g_1(0) = 10, \quad g_1(1) = 11, \quad g_1(4) = 2 \quad \text{(3 points)}$$

(c) (3 points) Determine all the critical points of $f(x,y)$ restricted to the line QR. List all such points along with value(s) of $f(x,y)$ at them. Also, list the endpoints of QR and the values of f at the endpoints.

Solution: The line QR is the line $y = 4 - x$. So, along QR,

$$g_2(x) = f(x, 4-x) = -2x^2 + 8x + 2, \quad g_2'(x) = -4x + 8$$

with critical point at $x = 2$. So, we evaluate

$$g_2(0) = 2, \quad g_2(2) = 10, \quad g_2(4) = 2$$ (3 points)

(d) (3 points) Determine all the critical points of $f(x,y)$ restricted to the line RP. List all such points along with value(s) of $f(x,y)$ at them. Also, list the endpoints of RP and the values of f at the endpoints.

Solution: Along RP we have

$$f(x,y) = f(0,y) = g_3(y) = 10 - y^2 + 2y, \quad g_3'(y) = 2 - 2y.$$

There is a critical point at $y = 1$. Hence, we evaluate

$$g_3(0) = 10, \quad g_3(1) = 11, \quad g_3(4) = 2$$ (3 points)

(e) (4 points) Using all the critical points found above, determine the absolute maximum and absolute minimum of $f(x,y)$ on the region $\triangle PQR$. Be sure to list the points together with the values of $f(x,y)$ at them.

Solution: The absolute minimum is 2 and occurs at the points $(4,0)$, $(0,4)$. (2 points) The absolute maximum is 12 and occurs at the interior critical point $(1,1)$. (2 points)

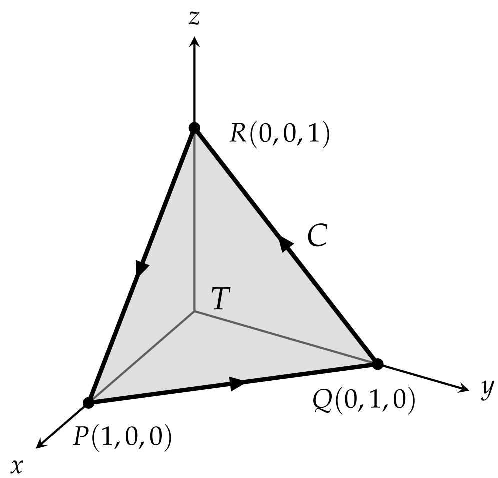
13. (15 points) The goal of this problem is to use Stokes' Theorem to evaluate the line integral

$$\oint_C \left((x+2y^2)\mathbf{i} + (y+z^2)\mathbf{j} + (z+2x^2)\mathbf{k}\right)\cdot d\mathbf{r}$$

where C is the boundary of the triangle T with vertices

$$P=(1,0,0) \quad Q=(0,1,0) \quad R=(0,0,1)$$

oriented counterclockwise when viewed from above.



Answer the following questions.

(a) (5 points) Let

$$\mathbf{F} = (x+2y^2)\mathbf{i} + (y+z^2)\mathbf{j} + (z+2x^2)\mathbf{k}$$

Calculate curl $\mathbf{F}$. Simplify your answer.

Solution: Compute

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y^2 & y+z^2 & z+2x^2 \end{vmatrix} \qquad \text{(2 points)}$$

$$= (-2z)\mathbf{i} + (-4x)\mathbf{j} + (-4y)\mathbf{k} \qquad \text{(3 points)}$$

(b) (4 points) Determine an equation of the plane passing through the three points P, Q, R.

Solution:

$$x+y+z=1 \quad \text{(4 points)}$$

(by inspection) or use any two of $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RP}$ to find a normal and use the usual recipe for the equation of a plane.

(c) (3 points) Let T denote the plane triangle enclosed by the three points P, Q, R. Find a parametrization for T.

Solution: From the equation of part (b), we have $z = 1 - x - y$ (1 points). The region T lies over the triangle $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$. Hence we can parameterize by (x, y):

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (1 - x + y)\mathbf{k} \quad \text{(2 points)}$$

(d) (3 points) Use Stokes' Theorem to set up the double integral over **F** which evaluates the given line integral. Do not compute the integral.

Solution:

From Stokes' Theorem we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_T \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS$$

so we need to compute the integral of the vector field over T. To compute the surface integral, use the parameterization

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (1 - x - y)\mathbf{k}$$

to compute

$$\begin{aligned} \mathbf{r}_x(x, y) &= \mathbf{i} - \mathbf{k} \\ \mathbf{r}_y(x, y) &= \mathbf{j} - \mathbf{k} \qquad \text{(1 points)} \\ \mathbf{r}_x \times \mathbf{r}_y &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

One can check "by eyeball" that this is the correct outward normal. Hence

$$\begin{aligned} \iint_T \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS &= \int_0^1 \int_0^{1-x} ((-2z)\mathbf{i} + (-4x)\mathbf{j} + (-4y)\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dy\, dx \\ &= \int_0^1 \int_0^{1-x} (-2z - 4x - 4y)\ dy\, dx \end{aligned}$$

(2 points)

14. (10 points) The goal of this problem is to use the Divergence Theorem, to calculate the surface integral $\iint_R \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^3 - x, y^3 - y, z^3 - z \rangle$ and R denotes the surface of the box $[0,1] \times [0,1] \times [0,1]$ with normals oriented outward.

Answer the following.

(a) (3 points) Calculate div $(\mathbf{F})$.

Solution:

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= \frac{\partial}{\partial x}\left(x^3 - x\right) + \frac{\partial}{\partial y}\left(y^3 - y\right) + \frac{\partial}{\partial z}\left(z^3 - z\right) && \text{(1 points)} \\ &= 3\left(x^2 + y^2 + z^2 - 1\right) && \text{(2 points)} \end{aligned}$$

(b) (4 points) According to the Divergence Theorem, the surface integral above is equal to a volume integral. Set up this volume integral as an iterated integral with proper limits.

Solution:

$$\begin{aligned} \iint_R \mathbf{F} \cdot d\mathbf{S} &= \int_C \operatorname{div}(\mathbf{F})\, dV && \text{(1 points)} \\ &= \int_0^1 \int_0^1 \int_0^1 3(x^2 + y^2 + z^2 - 1)\, dx\, dy\, dz && \text{(3 points)} \end{aligned}$$

(c) (3 points) Evaluate the triple integral.

Solution: From the inside out:

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 3(x^2 + y^2 + z^2 - 1)\, dx\, dy\, dz &= \int_0^1 \int_0^1 \left[x^3 + (3y^2 + 3z^2 - 3)x\right]_{x=0}^{x=1} dy\, dz \\ &= \int_0^1 \int_0^1 \left[1 + 3(y^2 + z^2 - 1)\right] dy\, dz \\ &= \int_0^1 \left[y + y^3 + 3z^2 y - 3y\right]_{y=0}^{y=1} dz \\ &= \int_0^1 3z^2 - 1\, dz \\ &= \left[z^3 - z\right]_{z=0}^{z=1} \\ &= 0 \end{aligned}$$

1 point for each of three steps in evaluating the triple integral