Math 213 Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used other than a one-page "cheat sheet" of notes, formulas, etc., written or typeset on one or both sides of an 8-1/2″ × 11″ sheet of paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 6 free-response questions. Please follow these guidelines to receive maximum credit.

- Each question is followed by a space to write your answer. Please write your answer *neatly* in the space provided.
- Show all work to receive full credit on the free response problems. You will be graded on the clarity of your presentation as well as the correctness of your answers.
- Give exact answers, rather than decimal equivalents, unless otherwise instructed (e.g., $\sqrt{2}$, not 1.414).

Question	1	2	3	4	5	6	**Total**
Possible	10	18	18	18	18	18	100
Score							

1. (Limits and Continuity - 10 points) Find:

   (a) (6 points) Find $\lim_{(x,y)\to(0,0)} \left(\dfrac{x^3+y^3}{x^2y+xy^2} \right)$ or show that it does not exist.

Solution: Use the "line test." If $f(x,y) = \dfrac{x^3+y^3}{x^2y+xy^2}$ then

$$f(x,mx) = \frac{x^3+m^3x^3}{mx^3+m^2x^3} = \frac{1+m^3}{m+m^2}.$$

Thus along the line $y = x$ $f(x,x) = 1$ but along the line $y = 2x$, $f(x,2x) = 9/6 = 3/2$. Since f has different values along different lines, the limit cannot exist.

   (b) (4 points) Find the set of all points (x,y) for which the function

$$f(x,y) = \ln(9 - x^2 - y^2)$$

is continuous.

Solution: The domain of f consists of those points (x,y) with $9 - x^2 - y^2 > 0$ or $x^2 + y^2 < 9$. f the the composition of a polynomial, which is continuous, and the log function, which is continuous on its domain. Hence, f is continuous on the open disc

$$D = \{(x,y) : x^2 + y^2 < 9\}.$$

2. (Partial Derivatives - 18 points)

   (a) (9 points) Show that the function $u(x, y) = \ln(x^2 + y^2)$ satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

by computing u_{xx}, u_{yy} and their sum. It will help to know that

$$u_x(x, y) = \frac{2x}{x^2 + y^2}, \quad u_y(x, y) = \frac{2y}{x^2 + y^2}.$$

Solution:

$$u_{xx}(x, y) = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} \text{ (3 points)}$$

$$u_{yy}(x, y) = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \text{ (3 points)}$$

Hence (3 points)

$$u_{xx} + u_{yy} = 2\, \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

(b) (9 points) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$x^2 - y^2 + z^2 - 2z = 4.$$

Solution: Differentiating the equation with respect to x we get

$$\begin{aligned} 2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} &= 0 \\ 2x &= (2 - 2z)\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial x} &= \frac{2x}{2 - 2z} \end{aligned}$$

(4 points)

Similarly, differentiating with respect to y, we get

$$\begin{aligned} -2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} &= 0 \\ (2z - 2)\frac{\partial z}{\partial y} &= 2y \\ \frac{\partial z}{\partial y} &= \frac{2y}{2z - 2} \end{aligned}$$

(4 points)

Bonus point for correct answers (1 points)

3. (18 points - Tangent Planes, Linear Approximation)

(a) (9 points) Find the equation of the tangent plane to the surface $z = x/y^2$ at the point $(-4, 2, -1)$. Express your answer in the form $z = ax + by + c$.

(1 point each)

Solution:

$$f(-4,2) = -1$$

$$f_x(x,y) = y^{-2} \qquad f_x(-4,2) = \frac{1}{4}$$

$$f_y(x,y) = -2xy^{-3} \qquad f_y(-4,2) = 1$$

Hence, the tangent plane has equation

$$z = -1 + \frac{1}{4}(x+4) + 1(y-2) \quad \text{(2 points)}$$

or

$$z = \frac{1}{4}x + y - 2. \quad \text{(2 points)}$$

(b) (9 points) Find the linear approximation to the function

$$f(x,y) = 2 - xy\cos \pi y$$

at $(1,2)$ and use it to estimate $f(1.02, 1.97)$

Solution: 1 point each

$$f(1,2) = 2 - 2\cos(2\pi) = 0$$

$$f_x(x,y) = -y\cos(\pi y) \qquad f_x(1,2) = -2$$

$$f_y(x,y) = -x\cos(\pi y) + xy\pi\sin(\pi y) \qquad f_y(1,2) = -1$$

Hence

$$L(x,y) = 0 + (-2)(x-1) + (-1)(y-2) \quad \text{(2 points)}$$

So, finally

$$\begin{aligned} f(1.02, 1.97) &\simeq L(1.02, 1.97) \\ &= (-2)(0.02) + (-1)(-0.03) = -0.01. \quad \text{(2 points)} \end{aligned}$$

4. (Chain Rule, Directional Derivatives - 18 points)

(a) (9 points) Suppose that

$$w = xy^2$$

and

$$x = r\cos\theta, y = r\sin\theta.$$

Use the chain rule to find $\partial w/\partial r$ when $r = 2$, $\theta = \pi/4$. Note that any other solution method will receive no credit.

Solution: By the chain rule

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\\ &= y^2 \cdot \cos\theta + 2xy\sin\theta\\ &= r^2\sin^2\theta\cos\theta + 2r^2\sin^2\theta\cos\theta\\ &= 3r^2\sin^2\theta\cos\theta\end{aligned}$$

Hence $\partial w/\partial r(2, \pi/4) = 12 \cdot (1/\sqrt{2})^2 \cdot (1/\sqrt{2}) = 6/\sqrt{2} =$ It is also correct to evaluate $\partial w/\partial x$ and $\partial w/\partial y$ at $(\sqrt{2}, \sqrt{2})$, evaluate $\partial x/\partial r$ and $\partial y/\partial r$ at $(2, \pi/4)$, and then use the chain rule at the last step to find $\partial w/\partial r$.

Suggested rubric:
Correct statement of chain rule (2 points)
Correct formulas for four derivatives (4 points)
Correct evaluation of derivatives (2 points)
Answer (1 points)

(b) (9 points) Find the maximum rate of change of the function $f(x,y) = 4y\sqrt{x}$ at the point $(4,1)$ and find the direction in which it occurs.

Solution:

$$(\nabla f)(x,y) = \left(\frac{2y}{\sqrt{x}}\right)\mathbf{i} + \left(4\sqrt{x}\right)\mathbf{j} \qquad \text{(2 points)}$$

$$(\nabla f)(4,1) = \mathbf{i} + 8\mathbf{j} \qquad \text{(2 points)}$$

$$|(\nabla f)(4,1)| = \sqrt{65} \qquad \text{(2 points)}$$

Hence, the maximum rate of change of f at $(4,1)$ is $\sqrt{65}$ (2 points) in the direction of the vector

$$\frac{1}{\sqrt{65}}\mathbf{i} + \frac{8}{\sqrt{65}}\mathbf{j} \quad \text{(1 points)}$$

5. (Maxima and Minima)

(a) (9 points) Find the local maximum and minimum values and saddle points for the function

$$f(x,y) = x^3 - 3x + 3xy^2$$

For each critical point, write down the Hessian matrix for the critical point, and give a reason why the critical point is a local maximum, a local minimum, or a saddle.

Solution: First, find the critical points:

$$f_x(x,y) = 3x^2 + 3y^2 - 3$$
$$f_y(x,y) = 6xy$$

Observe that $f_x(x,y) = 0$ on the circle $x^2 + y^2 = 1$, while $f_y(x,y) = 0$ along the x- and y-axes. So the four critial points are at the intersections, i.e., $(\pm 1, 0)$ and $(0, \pm 1)$. (3 points)

To analyze the critical points, compute the Hessian matrix by finding the second derivatives:

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = 6y, \quad f_{yy}(x,y) = 6x$$

so

$$\text{Hess} f(x,y) = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix} \quad D = 6(x^2 - y^2) \quad \text{(2 points)}$$

- $(1,0)$: $\text{Hess}(f)(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

  $D = 36$ and $f_{xx}(1,0) = 6$ so $(1,0)$ is a *local minimum* (1 points)

- $(-1,0)$: $\text{Hess}(f)(-1,0) = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$

  $D = 36$ and $f_{xx}(1,0) = -6$ so $(-1,0)$ is a *local maximum* (1 points)

- $(0,1)$: $\text{Hess}(f)(0,1) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$

  $D = -36$ so $(0,1)$ is a *saddle point* (1 points)

- $(0,-1)$: $\text{Hess}(f)(0,-1) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$

  $D = -36$ so $(0,-1)$ is a *saddle point* (1 points)

(b) (9 points) Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$. *Hint*: You can minimize the distance *squared* to make computations easier.

Solution: The distance squared from a point (x, y, z) on the cone to the point $(4, 2, 0)$ is given by

$$(x-4)^2 + (y-2)^2 + z^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 \quad \text{(2 points)}$$

where in the last step we used the equation of the cone to eliminate the z variable. So, we seek to minimize

$$f(x, y) = (x-4)^2 + (y-2)^2 + x^2 + y^2.$$

First, find the critical point(s):

$$f_x(x, y) = 2(x-4) + 2x = 4x - 8 \qquad \text{(1 points)}$$
$$f_y(x, y) = 2(y-2) + 2y = 4y - 4 \qquad \text{(1 points)}$$

Solving these equations for (x, y) we get a single critical point $(x, y) = (2, 1)$ (2 points). To find the z coordinate we use the equation of the cone:

$$z^2 = x^2 + y^2 = 5$$

so $z = \pm\sqrt{5}$. (1 points)

Hence, the two closest points at $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$. (2 points)

6. (Lagrange Multipliers - 18 points) Find the extreme values of the function $f(x,y) = xe^y$ subject to the constraint $x^2 + y^2 = 2$. Note that a solution by any method other than Lagrange multipliers will receive no credit.

Solution: The objective function is $f(x,y) = xe^y$ and the constraint function is $g(x,y) = x^2 + y^2$. (2 points) Computing the gradients and setting $\nabla f = \lambda \nabla g$ we get

$$e^y = 2\lambda x \quad \text{(2 points)} \tag{1}$$
$$xe^y = 2\lambda y \quad \text{(2 points)} \tag{2}$$
$$x^2 + y^2 = 2 \quad \text{(1 points)} \tag{3}$$

Note that e^y is never zero so $\lambda \neq 0$. Substituting (1) into (2) to eliminate e^y, we get

$$2\lambda x^2 = 2\lambda y$$

or $y = x^2$. Using this relation in (3) we get $y + y^2 = 2$ or $y^2 + y - 2 = 0$. The two roots are $y = 1$ and $y = -2$, but $y = -2$ is not admissible since $y = x^2$. Thus we need to test $(1,1)$ and $(-1,1)$. (8 points)

(x,y)	$f(x,y) = xe^y$
$(1,1)$	e
$(-1,1)$	$-e$

From these computations we conclude that the maximum value of $f(x,y)$ with constraint $x^2 + y^2 = 2$ is e, and the minimum value is $-e$. (3 points)