Math 213 Exam 3

Name: ______________________ Section: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used other than a one-page "cheat sheet" of notes, formulas, etc., written or typeset on one or both sides of an 8-1/2″ × 11″ sheet of paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 6 free-response questions. Please follow these guidelines to receive maximum credit.

- Each question is followed by a space to write your answer. Please write your answer *neatly* in the space provided.
- Show all work to receive full credit on the free response problems. You will be graded on the clarity of your presentation as well as the correctness of your answers.
- Give exact answers, rather than decimal equivalents, unless otherwise instructed (e.g., $\sqrt{2}$, not 1.414).

Question	1	2	3	4	5	6	**Total**
Possible	10	18	18	18	18	18	100
Score							

1. (Coordinate Systems - 10 points) Find:

   (a) (4 points) Find the rectangular coordinates of the point whose spherical coordinates are $(\rho, \theta, \phi) = (4, \pi/6, \pi/4)$

Solution:

$$x = 4\sin(\pi/4)\cos(\pi/6) = 2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)$$
$$y = 4\sin(\pi/4)\sin(\pi/6) = 2\sqrt{2}(1/2)$$
$$z = 4\cos(\pi/4) = 2\sqrt{2}$$

1 point for work
1 point for each correct answer

$x = \underline{\sqrt{6}}$

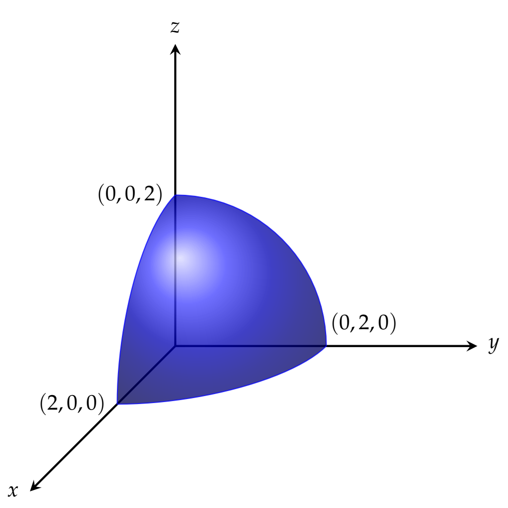
$y = \underline{\sqrt{2}}$

$z = \underline{2\sqrt{2}}$

   (b) (6 points) Sketch the solid described in spherical coordinates by the inequalities

$$0 \le \rho \le 2, \quad 0 \le \theta \le \pi/2, \quad 0 \le \phi \le \pi/2$$

and describe the solid. Be sure to label intercepts with the x, y, and z axes.



Solution: The solid is the intersection of a ball of radius 2 centered at $(0, 0, 0)$ and restricted to the first octant $x \ge 0, y \ge 0, z \ge 0$.

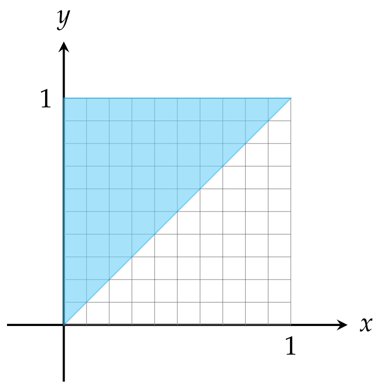
Description:
1 point for identifying the solid as a ball
1 point for specifying its center
1 point for locating it in the first octant

Figure:
1 point for correct figure (quarter sphere)
1 point for correct quadrant
1 point for *all* intercepts

2. (Iterated Integrals -18 points) The purpose of this problem is to compute the iterated integral

$$\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$$

(a) (6 points) Sketch the region of integration on the axes provided.



3 points for correct region (e.g., upper rather than lower triangle)

3 quality points (!)

(b) (6 points) Write down an iterated integral equivalent to the given one but with the orders of integration in x and y reversed.

Solution:

$$\int_0^1 \int_0^y \cos(y^2)\, dx\, dy$$

2 points each for correct x and y limits
1 point for correct order
1 point for correct integrand

(c) (6 points) Compute the iterated integral using your result from part (b).

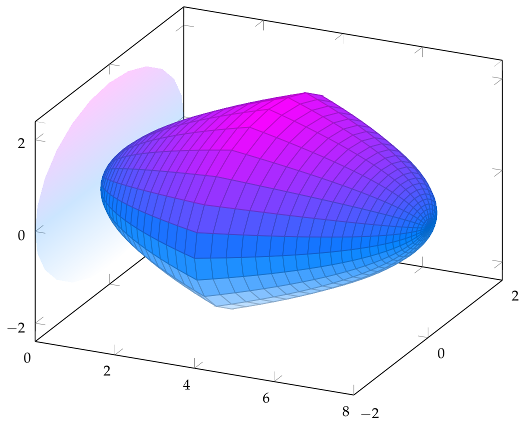
Solution:

$$\begin{aligned}\int_0^1 \int_0^y \cos(y^2)\, dx\, dy &= \int_0^1 \left(\left[x\cos(y^2) \right] \Big|_0^y \right) dy \\ &= \int_0^1 y\cos(y^2)\, dy \\ &= \frac{1}{2}\int_0^1 \cos(u)\, du \\ &= \frac{1}{2}\sin(1)\end{aligned}$$

1 point for each step above
2 points for correct answer

3. (Triple Integrals - 18 points) The purpose of this problem is to find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

(a) (6 points) Describe the region using cylindrical coordinates $x = r\cos\theta$, $z = r\sin\theta$.



$\underline{0} \leq r \leq \underline{2}$

$\underline{0} \leq \theta \leq \underline{2\pi}$

$\underline{r^2} \leq y \leq \underline{8 - r^2}$

1 point for each correct answer

(b) (6 points) Write down a triple integral in cylindrical coordinates for the volume of the solid.

Solution:

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\,dy\,dr\,d\theta$$

1 point each for limits
1 point for integrand 1
1 point for correct Jacobian factor
1 bonus point for correct answer

(c) (6 points) Evaluate the triple integral.

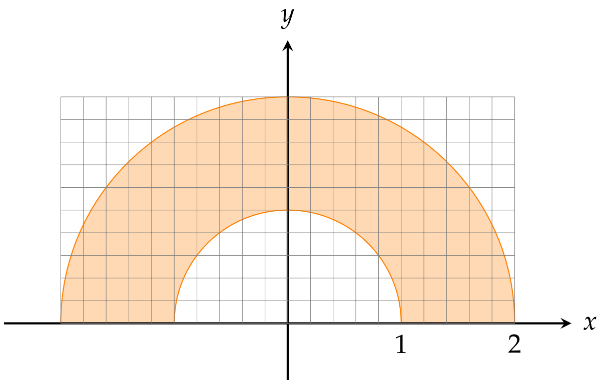
Solution:

$$\begin{aligned} \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\,dy\,dr\,d\theta &= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[4r^2 - \frac{1}{2}r^4\right]\Bigg|_0^2 d\theta \\ &= \int_0^{2\pi} 8\,d\theta \\ &= 16\pi \end{aligned}$$

2 points for each of three iterated integrals

4. (Applications of Double Integrals - 18 points) Find the mass of the lamina bounded by the semicircles $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portions of the x axis that join them, assuming that the mass density is $\rho(x,y) = \sqrt{x^2+y^2}$.

   Hint: Use polar coordinates.



Solution: In polar coordinates, the region is

$$\{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}.$$

while the density function ρ in polar coordinates is $\rho = r$.

The mass is given by

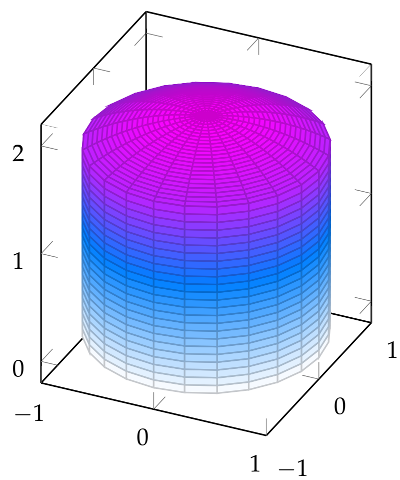
$$\begin{aligned} M &= \iint_R \rho \, dA \\ &= \int_0^\pi \int_1^2 r^2 \, dr \, d\theta \\ &= \int_0^\pi \left[\frac{r^3}{3} \right] \Big|_1^2 \, d\theta \\ &= \int_0^\pi \frac{7}{3} \, d\theta \\ &= \frac{7\pi}{3} \end{aligned}$$

3 points for correct polar region
3 point for correct integrand r
3 points for correct Jacobian factor r
4 points for correct evaluation of inner (r) integral
5 points for answer

5. (Triple Integrals - 18 points)

The purpose of this problem is to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

(a) (6 points) Describe the solid in cylindrical coordinates by filling in the table below.



$\underline{0} \leq r \leq \underline{1}$

$\underline{0} \leq \theta \leq \underline{2\pi}$

$\underline{0} \leq z \leq \underline{\sqrt{4-r^2}}$

The picture suggests

$$0 \leq z \leq \sqrt{4-r^2}$$

but the description suggests

$$-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$

Either answer should be graded correct if all parts are consistent.

1 point for each correct answer

(b) (6 points) Set up the volume integral in cylindrical coordinates.

Solution:

$$V = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

1 point each for correct limits
1 point for correct integrand
1 point for Jacobian factor
1 bonus point for correct answer

(c) (6 points) Evaluate the integral.

Solution:

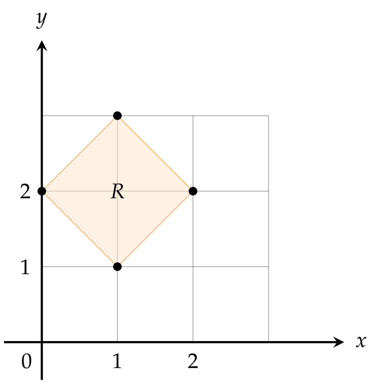
$$\begin{aligned}
\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^1 r\sqrt{4-r^2} \, dr \, d\theta \\
&= \int_0^{2\pi} \int_3^4 \frac{\sqrt{u}}{2} \, du \, d\theta \qquad u = 4 - r^2, \; du = -2r \, dr \\
&= \int_0^{2\pi} \left[\frac{1}{3} u^{3/2} \right] \Bigg|_3^4 \, d\theta \\
&= \frac{2\pi}{3} \left(8 - 3\sqrt{3} \right)
\end{aligned}$$

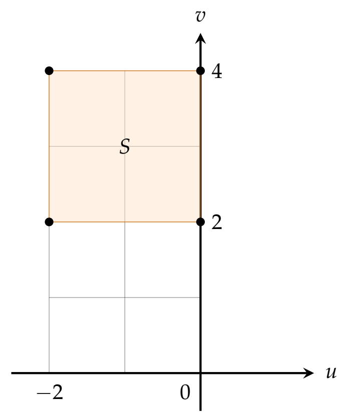
2 points for each correctly evaluated iterated integral

6. (Change of Variables - 18 points) The purpose of this problem is to evaluate the integral

$$\iint_R \frac{x-y}{x+y}\, dA$$

where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$ and $(1,3)$ in the xy plane, shown at left.





(a) (4 points) By completing the table below, show that the transformation

$$u = x - y, \quad v = x + y$$

maps R onto the region S shown at right.

x	y	u	v
0	2	-2	2
1	1	0	2
2	2	0	4
1	3	-2	4

1 point per correct table row

(b) (4 points) Using the equations $u = x - y$, $v = x + y$, solve for x and y in terms of u and v.

Solution: From these equations $u + v = 2x, \quad v - u = 2y$ so

$$x = \frac{1}{2}(u+v)$$
$$y = \frac{1}{2}(v-u)$$

2 points per correctly derived equation

(c) (4 points) Find the Jacobian determinant of the transformation from (u, v) to (x, y) found in part (b). Write your answer in the space provided, and be sure to show your work!

Solution: From the equations above

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$\frac{\partial(x, y)}{\partial(u, v)} = \underline{\frac{1}{2}}$

2 points for correct partials
2 points for answer

(d) (6 points) Use the transformation from part (b), the Jacobian determinant from part (c) and the change of variables theorem to evaluate $\iint_R \frac{x-y}{x+y}\, dA$.

Solution:

$$\begin{aligned} \iint_R \frac{x-y}{x+y}\, dA &= \int_2^4 \int_{-2}^0 \frac{u}{v} \frac{1}{2}\, du\, dv \\ &= \int_2^4 \frac{1}{v} \left[\frac{u^2}{4}\right]\Bigg|_{-2}^0 dv \\ &= -\int_2^4 \frac{1}{v}\, dv \\ &= -\ln(4) + \ln(2) \\ &= -\ln(2) \end{aligned}$$

1 point for correct expression of integrand as u/v
1 point for Jacobian factor
2 points for evaluation of inner integral
2 points for evaluation of outer integral