MA 213 — Calculus III Spring 2018
Exam 3 April 11, 2018

Name: KEY

Section: ____________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

# Exam Scores

*Do not write in the table below*

| Question | Score | Total |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total | | 100 |

## Free Response. Show your work!

1. (10 points) Evaluate

$$\iint_R \frac{xy^2}{x^2+1}\,dA,$$

where $R = \{(x,y) \mid 0 \le x \le 1, -3 \le y \le 3\}$.

$u = x^2+1$
$du = 2x\,dx$

$$\int_{-3}^{3}\int_{0}^{4} \frac{xy^2}{x^2+1}\,dx\,dy = \int_{-3}^{3} y^2\left[\int_0^1 \frac{x}{x^2+1}\,dx\right]dy$$

$$= \int_{-3}^{3} y^2\left[\tfrac{1}{2}\ln(x^2+1)\right]\Big|_0^1 dy = \int_{-3}^{3} y^2\left(\tfrac{1}{2}\ln 2\right)dy$$

$$= \left(\tfrac{1}{2}\ln 2\right)\left[\frac{y^3}{3}\right]\Big|_{-3}^{3} = \left(\tfrac{1}{2}\ln 2\right)(9-(-9))$$

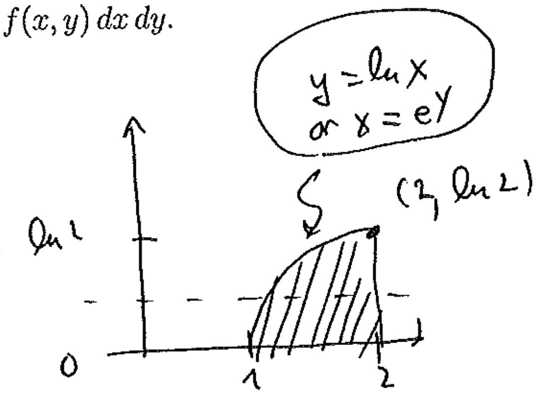
$$= \boxed{9\ln 2}$$

2. (10 points) Change the order of integration in

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx,$$

i.e. determine $A, B, C, D$ for which

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx = \int_A^B\int_C^D f(x,y)\,dx\,dy.$$




$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx =$$

$$\boxed{\int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\,dy\,dx}$$

3. (10 points) Set up an iterated integral in polar coordinates to compute the area inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$. Do not evaluate the integral.

1) Intersection pts.

$(x-1)^2+y^2 = x^2+y^2$

$1-2x=0$

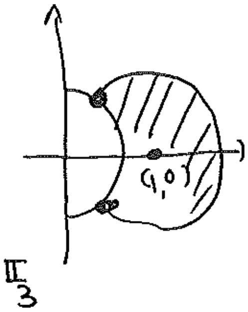
$x=\frac{1}{2}$

$x^2+y^2=1$

$\frac{1}{4}+y^2=1$

$y=\pm\frac{\sqrt{3}}{2}$

$\therefore \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ OR $r=1$, $\theta=\pm\frac{\pi}{3}$




2) Inner and outer radii

Inner: $r=1$ Outer: $(r\cos\theta-1)^2+(r\sin\theta)^2=1 \Rightarrow r^2-2r\cos\theta+1=1$

$\Rightarrow r^2-2r\cos\theta=0$

$\Rightarrow r=2\cos\theta$

Hence $\iint_R dA = \int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$

4. (10 points) Find the average distance from a point $(x,y)$ of the disk $x^2+y^2\le R^2$ to its center.

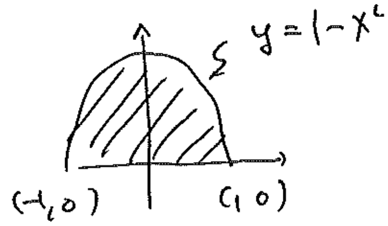
The disc has area $\pi R^2$; the distance from $(0,0)$ is $\sqrt{x^2+y^2}=r$

so average is $\frac{1}{\pi R^2}\iint r\,dA = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R r\cdot r\,dr\,d\theta$

$= \frac{1}{\pi R^2}\int_0^{2\pi}\frac{R^3}{3}\,d\theta = \frac{2\pi}{\pi R^2}\cdot\frac{R^3}{3} = \boxed{\frac{2}{3}R}$

## Free Response. Show your work!

5. (10 points) Find the mass of a lamina occupying the region $D$ in the $xy$-plane bounded by $y = 1 - x^2$ and $y = 0$ if the density is $\rho(x, y) = 3y$.



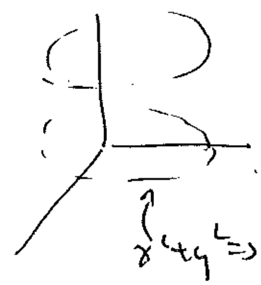

$$M = \iint_R 3y \, dA$$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} 3y \, dy \, dx = \int_{-1}^{1} \left[\frac{3y^2}{2}\right]\Big|_0^{1-x^2} dx$$

$$= \int_{-1}^{1} \frac{3}{2}(1-x^2)^2 \, dx = \int_{-1}^{1} \frac{3}{2}(1 - 2x^2 + x^4) \, dx = \frac{3}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]\Big|_{-1}^{1}$$

$$= \frac{3}{2}\left[2 - \frac{4}{3} + \frac{2}{5}\right] = \boxed{\frac{8}{5}}$$

6. (10 points) Find the surface area of the part of the plane $2x + 2y + z = 5$ inside the cylinder $x^2 + y^2 = 3$.




$$z = 5 - 2x - 2y \qquad \frac{\partial z}{\partial x} = -2 \qquad \frac{\partial z}{\partial y} = -2$$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \cancel{dx\,dy} \, dA = 3 \, \cancel{dx\,dy} \, dA$$

$$S = \iint_R dS = \iint_R 3 \, \cancel{dx\,dy} \, dA$$

$$= 3 \cdot (3\pi)$$

$$= \boxed{9\pi}$$

(the projection of the cylinder of radius $\sqrt{3}$ on the $xy$ plane has area $\pi(\sqrt{3})^2 = 3\pi$)

## Free Response. Show your work!

7. (10 points) Consider

$$\iiint_E z\,dV,$$

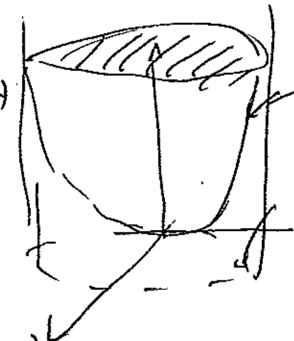
where $E$ is enclosed by

$$z = 0, \quad z = x^2 + y^2, \quad \text{and} \quad x^2 + y^2 = 4.$$

Use cylindrical coordinates to express this triple integral as an iterated integral. Do not evaluate the integral.

The top of the solid is at $z = 4$ (where paraboloid and cylinder intersect).

$x^2+y^2=4$ (cylinder



$z = x^2+y^2$ (paraboloid)

$z = 0$

$E = \{ (r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ 0 \le z \le x^2+y^2 \}$

Note $x^2+y^2 = r^2$

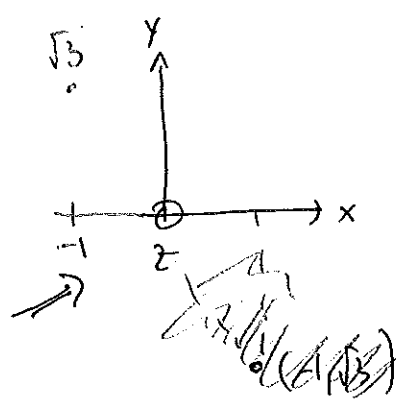
$\therefore \iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \left( \int_0^{x^2+y^2} z\,dz \right) r\,dr\,d\theta$

8. (10 points) Find the spherical coordinates $(\rho, \theta, \phi)$ of a point whose rectangular coordinates are $(-1, \sqrt{3}, 2\sqrt{3})$.

$\rho = \sqrt{1+3+4\cdot 3} = 4$

$\cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ so $\phi = \pi/6$

$\tan\theta = \frac{y}{x} = -\sqrt{3}$ so $\theta = \cancel{-\pi/3}$ $\frac{2\pi}{3}$



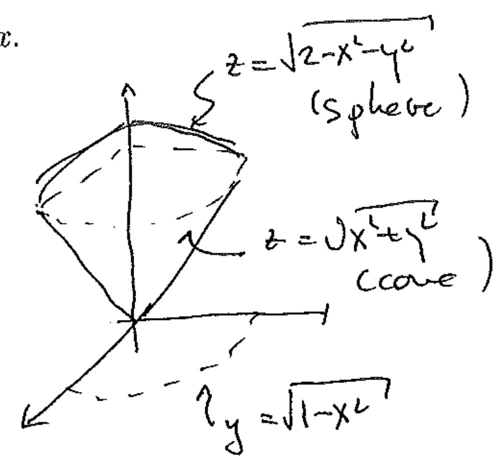

$\therefore$ $(\rho,\theta,\phi) = \cancel{(4, -\pi/3, \pi/6)}$

$(4, \frac{2\pi}{3}, \frac{\pi}{6})$

**9.** (10 points) Change to spherical coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x^2y^3\,dz\,dy\,dx.$$

Do not evaluate the integral.




- The cone (solid) occupies $0 \le \phi \le \frac{\pi}{4}$
- The sphere occupies $0 \le \rho \le \sqrt{2}$
- The solid lies over the $1^{st}$ quadrant

  $0 \le \theta \le \frac{\pi}{2}$.

So the integral is $\displaystyle\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} (\rho \sin\phi \cos\theta)^2 (\rho \sin\phi \sin\theta)^3 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

**10.** (10 points) Evaluate

$$\iiint_E (1 + ye^{x^2z})\,dV$$

where $E$ is the cube $E = [-1,1] \times [-1,1] \times [-1,1]$. [Hint: Integrate with respect to $y$ first.]

$$\iiint_E (1+ye^{x^2z})\,dV = \int_{-1}^{1}\left(\int_{-1}^{1}\left(\int_{-1}^{1} (1+ye^{x^2z})\,dy\right)dz\right)dx$$

$$= \int_{-1}^{1}\left(\int_{-1}^{1} \left[y + \frac{y^2}{2}e^{x^2z}\right]\Big|_{y=-1}^{y=1} dz\right)dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2 \, dz\, dx$$

$$= \boxed{8}$$