Homework 1

525 Spring 2006

Due Feb 2

1. Let $X_1, \ldots, X_n$ be iid $\sim Bernoulli(p)$. Define $\hat{p} = (1/n) \sum X_i$. (Hint: What is the distribution of $\sum X_i$?)

   a) Compute the MSE for estimating p with $\hat{p}$. Show $\hat{p}$ is a consistent estimator of p (i.e. MSE $\to 0$ as $n \to \infty$.

   b) Let $\tilde{p} = (1 + \sum X_i)/(n+2)$. Compute the MSE of $\tilde{p}$ as an estimator of p and show $\tilde{p}$ is also consistent.

   c) When $n = 100$ and $p = 0.5$, which estimator ($\hat{p}$ or $\tilde{p}$) is the better estimator in terms of MSE? When $p = 0.1$ and $n = 100$, which is the better estimator?

   d) When $n = 100$, and $p = 0.01$, which estimator is the better estimator in terms of MSE?

2. Let $X_1, \cdots, X_n$ be independent and having a common distribution $\sim N(\mu, \sigma^2)$ where both μ and σ^2 are unknown, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. (Please note, here the variance σ^2 is the parameter not the standard deviation σ)

   a) Is $1/n \sum (X_i - \mu)^2$ an estimator of σ^2? Why or why not?

   b) compute the MSE of estimator s^2 as an estimator of σ^2.

   c) Let us try a new estimator of σ^2: $\hat{\sigma}^2 = Cs^2$, for some positive constant C.

   Find the best constant C in the sense that minimizes the MSE.

3. Let $X_1, \cdots, X_n$ be iid $\sim Exp(\lambda)$, where the density of each X_i is $\lambda \exp\{-\lambda x\}$ over the range 0 to ∞. Note that the distribution of $Y = \sum X_i$ is $Gamma(n, \lambda)$ with density

$$f(y|\lambda, n) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} \exp\{-\lambda y\}$$

a) if we use $\sum X_i/n$ as the estimator of a new parmeter $\beta = 1/\lambda$. Find the MSE.

a) Show the density of $W = 1/Y$ is

$$\frac{\lambda^n}{\Gamma(n)}(1/w)^{n+1}\exp\{-\lambda/w\}$$

b) Note that, for any λ and n,

$$\int_0^\infty f(w|\lambda)dw = 1$$

use this formula to derive $E[W]$ and $V[W]$

c) Compute the MSE of using $1/\bar{x}$ as an estimator of λ. Is $1/\bar{x}$ a consistent estimator of λ?