Empirical Likelihood Inference on Regression Quantiles of Residual Lifetimes

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Abstract

We propose and investigate an estimating equation for the quantile regression model for the residual lifetimes in this paper. It is closely related to the quantile regression model of Koenker (2002). However, our approach (1) deals with a conditional quantile given that subjects survived beyond $t_0$; (2) employs the so-called inverse censoring probability weighting method to accommodate censored data, (3) uses the empirical likelihood method for the inference of regression coefficients.

The empirical likelihood approach has several advantages: (1) there is no need to estimate the variance/covariance at all, which may become prohibitively complicated for other procedures that requires the estimation of such; (2) When inverting the tests to obtain confidence regions/intervals, this empirical likelihood inherits all the good properties of a likelihood ratio test; (3) Free software implementation of the test is readily available.

Key words: Censored quantile regression; Chi square distribution; Wilks theorem; Confidence interval.

1 Introduction

In medical research an ultimate question of interest is the residual lifetime of a patient given the patient’s prognostic factors and treatment choices. When the choice of first-line therapy is concerned, the residual life expectancy at the time of diagnosis is of interest. When the choice of adjuvant therapy is concerned in particular for patients in remission, the residual life expectancy at a certain time point in the followup is of interest. Increasingly more attention has been paid to the conditional residual life expectancy at a certain follow up time point in the latter case as more patients survive cancers initially and are subject to long-term
courses of secondary therapies. For example, in recent placebo-controlled randomized Phase III clinical studies on breast cancer (Goss et al., 2003; Coombes et al., 2004) an aromatase inhibitor, either letrozole or exemestane, was examined as a secondary course of drug in estrogen receptor positive patients who had been on tamoxifen for up to 5 years without recurrence of the original diseases. A straightforward way of measuring the benefits of this new secondary course of drug is in terms of prolonging the patient’s residual life expectancy, given the fact that she has survived for, say, 5 years after originally being treated. A physician would need to know it to advise a patient who would be interested first in participating in this type of study and later in taking the drug, if the efficacy is proven. In this paper we propose a nonparametric inference method on quantiles of residual lifetimes at both the time of diagnosis and a certain followup time point using empirical likelihood.

An alternative measure of drug benefits is the mean of the residual lifetime. But questionable assumptions are often needed on the censoring pattern relative to the survival times (especially in the tail) in order to make the mean residual lifetime estimable. Also in comparing survival experience of two groups the mean is only appropriate if the difference is uniform across the distribution. For example, suppose that a treatment in a cancer trial is anticipated to have a high initial treatment related mortality and to benefit only those who survive the initial period. The difference in the mean survival does not adequately summarize the treatment effect, while a set of quantiles can. We refer to Koenker (2005) for a general introduction to quantiles and quantile regression. Hence quantiles including the median are more natural quantitative measures of residual lifetimes. Furthermore, the assumptions needed for censoring pattern to work with median is much relaxed compared to the mean.

However, not much work on the quantiles of residual lifetimes is found in the literature. One can refer to published Kaplan-Meier plots and may adopt existing methods such as an adjusted version of the Kaplan-Meier method (Kaplan and Meier, 1985) or, when covariates are present, Cox proportional hazards model (Cox, 1972 and 1975) to indirectly infer the remaining life times. However, they are often cumbersome and not straightforward, especially when the residual lifetime at a certain followup time is concerned. Jeong, Jung and Costantino (2008) proposed a score type test for median residual life function in one or two sample settings based on Martingales. Jung, Jeong and Bandos (2009) defined regression
quantiles of residual lifetimes and proposed a score type test using estimating equations. While the score type tests do not require estimation of the underlying probability density function of failure time under censoring, hence overcoming the shortcomings of other existing methods, it still requires the estimation of the variance/covariance of its estimator. Such estimation under censoring is complicated, sometimes becoming prohibitively so.

In contrast the test we propose is based on empirical likelihood. Empirical likelihood was first proposed by Thomas and Grunkemeier (1975) to obtain better confidence intervals in connection with the Kaplan-Meier estimator. Owen (1988, 1990) and many other have developed it into a general nonparametric methodology. It parallels the theory of the parametric likelihood ratio test, except the parametric likelihood is replaced by a nonparametric one. In uncensored cases, it has been shown to inherit all the desirable properties of parametric likelihood ratio test: the test statistic is internally studentizing, hence not requiring the estimation of the variance/covariance, it has a limiting central chi-squared distribution under the null, and confidence interval/region obtained by inversely applying the test is range respecting. The book of Owen (2001) summarized many of the results (Chapter 6 in particular). Other relevant papers include Murphy and van der Vaart (1997), Pan and Zhou (2002), Zhou (2005). The test we propose in this paper, share these desirable properties.

When inference on the mean residual life time is concerned, Qin and Zhao (2007) and Zhao and Qin (2006) also proposed an empirical likelihood method. Their test, however, does not internally studentizing, therefore the test statistic do not have a limiting regular chi-squared distribution and needs to be estimated.

( delete?? It also can be adapted to inference on the mean residual time. We will discuss it in section???. The rest of paper consists ???)

2 Quantiles of Residual Lifetimes

We first discuss a univariate case of the quantiles of residual lifetimes at a certain followup time $t_0$, which lead to an estimating equation that inspires our definition in section 3. For a given CDF $F(t)$, the $\tau$-th quantile residual lifetime at $t_0$ is, by definition, the number $\theta_\tau = \theta_\tau(t_0)$ that solves the following

$$
\frac{1 - F(t_0 + \theta_\tau)}{1 - F(t_0)} = 1 - \tau .
$$
After a simple manipulation we see that $\theta_\tau$ is the solution to the equation

$$0 = F(t_0 + \theta_\tau) - (1 - \tau)F(t_0) - \tau.$$  

For a given $\tau$ we define a function $g_b(t)$ as

$$g_b(t) = I[t \leq (t_0 + b)] - (1 - \tau)I[t \leq t_0] - \tau.$$  

Then, the hypothesis $H_0 : \theta_\tau(t_0) = b$ is equivalent to

$$H_0 : \int_{-\infty}^{\infty} g_b(t)dF(t) = 0.$$  

Suppose we have an iid sample of lifetimes $T_1, \cdots, T_n \sim F(t)$ without censoring. The estimating equation for the $\tau$-th residual lifetime quantile is then

$$0 = \sum_{i=1}^{n} g_b(T_i) = \sum_{i=1}^{n} \{I[T_i \leq (t_0 + b)] - (1 - \tau)I[T_i \leq t_0] - \tau\},$$  

or equivalently

$$0 = \int g_b(t)d\hat{F}_n(t),$$  

where $\hat{F}_n(t)$ denote the empirical distribution based on $T_i$. In other words, the sample estimator of the $\tau$-th residual lifetime quantile based on the $T_1, \cdots, T_n$ is $b = \hat{\theta}_n$ that solves this equation. We can use the estimating equation and test the hypothesis by an empirical likelihood ratio test (see Owen, 2001) in connection with (2) or (3). Suppose now we do not observe all the $T_i$ but rather have right censored data $Z_i = \min(T_i, C_i)$, $\delta_i = I[T_i \leq C_i]$ instead, where $C_i$ are the censoring times. The only modification to the above estimating equation (3) is to replace the empirical distribution $\hat{F}_n(t)$ with the Kaplan-Meier estimator $\hat{F}_{KM}(t)$ based on $(Z_i, \delta_i)$ (Kaplan and Meier 1958). Equivalently the estimating equation (2) becomes

$$0 = \sum_{i=1}^{n} w_ig_b(Z_i)$$  

where $w_i$ is the probability mass the Kaplan-Meier estimator puts on observation $Z_i$.

After a simple manipulation the estimating equation (2) can be simplified to

$$0 = \sum_{i=1}^{n} I[T_i > t_0]\{\tau - I[T_i \leq (t_0 + b)]\}$$  

4
or as a minimization problem

$$\min_b \sum_{i=1}^n I[T_i > t_0] \{T_i - (t_0 + b)\} \{\tau - I[T_i \leq (t_0 + b)]\}.$$  \hfill (6)

Recall a so-called “check function” of Koenker and Basset (1978) and its derivative:

$$\rho_\tau(t) = t(\tau - I[t \leq 0]) \quad \psi_\tau(t) = (\tau - I[t \leq 0]).$$

We see that (5) and (6) are respectively

$$0 = \sum_{i=1}^n I[T_i > t_0] \psi_\tau \{T_i - (t_0 + b)\}, \quad \min_b \sum_{i=1}^n I[T_i > t_0] \rho_\tau \{T_i - (t_0 + b)\}. \quad \hfill (7)$$

In a similar fashion, with right censored data, (4) is equivalent to

$$0 = \sum_{i=1}^n w_i I[Z_i > t_0] \psi_\tau \{Z_i - (t_0 + b)\}.$$

Compare (7) to the estimating equation for the $\tau^{th}$ quantile (Koenker and Basset 1978) we see that there are two modifications for the residual quantile: namely add indicator functions $I[T_i > t_0]$ and replace $T_i$ by $T_i - t_0$ in $\psi$ or $\rho$. Furthermore, the censoring is taken care of by the weights $w_i$. This motivates our definition of residual life quantile regression below.

### 3 Regression Quantiles of Residual Lifetimes

The $\tau^{th}$ regression quantile, $\beta$, is (see Koenker 2002) defined by

$$\min_\beta \sum_{i=1}^n \rho_\tau(T_i - x_i^T \beta).$$

As motivated by analysis in section 2, the regression residual quantile is defined by

$$\min_\beta \sum_{i=1}^n I[T_i > t_0] \rho_\tau(T_i - t_0 - x_i^T \beta)$$

assume $T_i$ are all observed.

The formal definition of the residual quantile regression model may be given as follows:

$$(T_i - t_0) = x_i^T \beta + \epsilon_i, \quad \text{given } T_i > t_0$$

where $\epsilon_i$ are independent and have zero $\tau^{th}$ quantile, and $T_i$ are responses with covariates $x_i$. 

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However, the linear models for the lifetimes traditionally will apply a log (or other) transformation to the responses (lifetimes). This is the so called Accelerated Failure Time (AFT) model. We thus assume the model

$$\log(T_i - t_0) = x_i^\top \beta + \epsilon_i, \quad \text{given } T_i > t_0$$

(8)

and we are lead to the definition of the estimator as

$$\min_{\beta} \sum_{i=1}^n I[T_i > t_0] \rho_r \{\log(T_i - t_0) - x_i^\top \beta\} .$$

Under random right censoring we do not observe all the $T_i$ but right censored data $Z_i = \min(T_i, C_i)$ and $\delta_i = I[T_i \leq C_i]$. The discussion in section 2 lead us to define the censored regression residual quantile estimator based on $(Z_i, \delta_i, x_i)$ as

$$\min_{\beta} \sum_{i=1}^n w_i I[Z_i > t_0] \rho_r \{\log(Z_i - t_0) - x_i^\top \beta\}$$

(9)

where $w_i$ are the probability that a Kaplan-Meier estimator based on $(Z_i, \delta_i)$ puts on the location $Z_i$. This is the so called inverse probability of censoring weighting method (ICPW). See Bathke, Kim and Zhou (2008) for a discussion of this the related empirical likelihood analysis. When $t_0 = 0$ and all $Z_i$‘s positive, this coincide with Bathke, Kim and Zhou (2008).

We note that $\beta(\tau, t_0)$ represents covariate effects that are quantile specific and potentially different by $t_0$. The dependency of the covariate effects on the followup time $t_0$ distinguishes the proposed quantile regression of residual lifetime at $t_0$ from the regular quantile regression.

Several works exist in literature, when $t_0$ is the time origin of the survival times ($t_0 = 0$) or the lower limit of $T$ and the conditional quantile given $x$, $Q_\tau(x)$, is to be estimated. Ying, Jung, and Wei (1995) proposed a method for the median, assuming $T_i$ and $C_i$ be independent (unconditionally). Honoré, Khan and Powell (2002) extended Powell (1984, 1986)’s method developed under fixed censoring $C_i$ to random censoring, also assuming $T_i$ and $C_i$ be independent (unconditionally). The more stringent assumption of unconditional independency was relaxed by Portnoy (2003). Portnoy (2003) uses a novel “recursive reweighting” scheme which is in the same spirit of the redistribution-of-mass idea of Efron (1967), and estimates $Q_\tau(x)$ for all $0 \leq \tau \leq 1$, assuming $T_i$ and $C_i$ be independent conditioning on $x_i$, if $Q_\tau(x)$ is linear for all $0 \leq \tau \leq 1$. The recursive approach, however, makes it challenging to establish the asymptotic normality theory. Recently Peng and Huang (2009) proposed a martingale-based
estimating procedure, which performs similarly to Portnoy’s estimator. Both Portnoy’s and Peng and Huang’s approaches rely on a strong global assumption that all the lower quantiles are linear in \( x \) in order to estimate the conditional \( \tau \)-th quantile. Using a kernel Wang and Wang (2009) developed a local Kaplan-Meier method, which relaxes the global assumption and requires the linearity only in \( Q_\tau(x) \), the conditional quantile of interest. The use of the kernel in estimating the local Kaplan-Meier, however, limits the method to be only applicable when the dimension of \( x_i \) is low and larger sample size.

If \( t_0 > 0 \) and \( t_0 \) is not the lower limit of \( T \), one may adapt the above described existing methods to estimate \( Q_{\tau,t_0}(x) \). However, inference is tricky with the existing methods. The asymptotic covariance matrices of the estimators take complex forms and are estimated via resampling in practice. However, due to censoring the quantile of interest may not be estimable in all resampled data, particularly with Portnoy (2003)’s method, and it is not clear what to do in those cases. The empirical likelihood methods that we propose in the next section do not require the estimation of the asymptotic covariance matrix nor do resampling of the data. As the followup time \( t_0 \) is fixed, we simply write \( \beta(\tau,t_0) = \beta \).

4 Empirical Likelihood Inference

Zhou, Kim and Bathke (2008) proposed two empirical likelihood methods, case-wise and residual-wise, for accelerated failure time model and regular censored quantile regression. They extended the distinction between two different data generating models, correlation and regression model, that was first identified in linear regression by Freedman (1981), to the censored regression. They identified that, unlike in linear regression where the different data generating models only require different assumptions, different estimators are required in the censored regression. The case- and residual-wise methods were proposed by constructing empirical likelihood case- and residual-wise respectively, correspondingly to the different data generating mechanisms of accelerated failure time correlation and regression models. We refer to Zhou, Kim and Bathke (2008) for details and focus on the extension of the methods to the problem under consideration.
4.1 Inverse Censoring Probability Weighted Estimator and Casewise Empirical Likelihood

We assume a correlation data generating model for $(T_i, x_i)$: $n$ independent identically distributed vector $(T_i, x_i)$ are generated from the model (8). We observed $(Z_i, \delta_i, x_i)$ where $Z_i = \min(T_i, C_i)$, $\delta_i = I[T_i \leq C_i]$ and $T_i$ and $C_i$ are independent. We define the estimator $\hat{\beta}$, the quantile regression on residual times after $t_0$ for the model (8) by a minimization of the sample function (9). We refer to Zhou, Kim and Bathke (2008) for the discussion of how the IPCW weights take care of the censoring. We note that whenever $Z_i \leq t_0$ then the $i^{th}$ term is zero due to the indicator function. In other words, $I[Z_i > t_0] \log(Z_i - t_0)$ should be understood as zero whenever $Z_i \leq t_0$.

We formulate the equivalent estimating equation for the $\hat{\beta}$. Formally taking the derivative with respect to $\beta$ in (9) we get the estimating equation for the residual lifetime quantile regression

$$0 = V(\beta) = \sum_{i=1}^{n} w_i I[Z_i > t_0] x_i \psi_r(\log(Z_i - t_0) - x_i^\top \beta) .$$

(10)

When solving the above equation to obtain the estimator $\hat{\beta}$, we can make use the general weighted quantile regression program in the R package quantreg of Koenker, with the weights $w_i I[Z_i > t_0]$ and responses $\log(Z_i - t_0)$.

Reflecting the independent identically distributed pairs $(T_i, x_i)$, we let $p_i$ denote the probability mass placed on the observation $(Z_i, \delta_i, x_i)$ and construct the likelihood case-wise as follows:

$$EL(p) = \prod_{i=1}^{n} \{p_i \delta_i \{ \sum_{Z_j > Z_i} p_j \}^{1-\delta_i} ,$$

(11)

where $p_i \geq 0$ and $\sum p_i = 1$. The maximization of (11) with respect to $p_i$ is well known to be achieved by (the jumps of) the Kaplan-Meier estimator computed from $(Z_i, \delta_i)$. We denote this maximum value achieved as $EL(KM)$.

We now consider maximizing (11) with respect to $p_i$ under an extra constraint:

$$0 = \sum_{i=1}^{n} p_i I[Z_i > t_0] x_i \psi_r(\log(Z_i - t_0) - x_i^\top \beta) .$$

(12)

We denote the maximum value of $EL$ attained under this extra constraint (12) by $EL(p|\beta)$. Parallel to the parametric case, we define the likelihood ratio by

$$R(\beta) = EL(p|\beta) / EL(KM) .$$

(13)
The following theorem shows that this empirical likelihood ratio for the median residual lifetimes inherits the desirable properties of parametric counterpart.

**Theorem** Consider the residual quantile regression model (8). Suppose we observe right censored data as defined above.

Suppose $\beta = \beta_0$, $EX_i^2 < \infty$ and $\int \frac{dF(t)}{1-G(t-)} < \infty$ where $G(t)$ denotes the distribution of censoring times $C_1, \cdots, C_n$. Then as $n \to \infty$, for the empirical likelihood ratio defined in $(E: $casewiseEL$)$ and $(??ratio))$, we have

$$-2 \log R(\beta) \longrightarrow \chi^2_q$$

in distribution where $q$ denotes the dimension of $\beta$.

The iid-ness of $C_i$ is not essential, as long as the censoring is independent of $(T_i, x_i)$. In this case, we will need to require $\int \frac{dF(t)}{1-G_i(t-)} \leq K < \infty$ in the above theorem. The CDF $F$ denotes the marginal distribution of $T_i$.

A 100(1 - $\alpha$)% confidence interval or region is given by inversely applying the likelihood ratio as follows: $\{\beta : -2 \log R(p|\beta) \leq c_{1-\alpha}\}$ where $c_{1-\alpha}$ is the $(1 - \alpha)$-th quantile of a $\chi^2$ distribution with $q$ degrees of freedom. A publicly downloadable software implementation of the empirical likelihood ratio tests with censored data is `emplik`. It is an extension package to be used with the R software (R Development Core Team 2008, http://cran.wustl.edu). In particular, the function `el.cen.EM2` inside the package `emplik` carries out the above test.

**Remark 1:** We note that the estimating equation (10) for $\hat{\beta}$ corresponds to the constraint equation (12) with $p_i$ replaced by $w_i$, the jumps of the Kaplan-Meier estimator. Hence the confidence interval or region defined above is centered at $\hat{\beta}$ in the sense that $-2 \log R(\hat{\beta}) = 0$.

**Remark 2:** We note some special cases. If the followup time $t_0 = 0$ or it is the lower bound of $T_i$, (??) is equivalent to the median regression estimation of Bathke, Zhou and Kim (2008). If $X_i \equiv 1$ (or constant) and $\hat{F}_n(t)$ is replaced by $\hat{F}_{KM}(t)$, then the estimating equation (10) reverts back to the median residual life estimator we discussed in section 2. When there is no censoring, and $t_0 = 0$ and forget about the log transform, then we get the regular quantile regression of Koraker (2006).

**Remark 3:** Similar analysis for the mean residual time can be derived accordingly. In particular, when $\psi_m(t) = t$ in the equation (10) we obtain the mean residual regression estimator. Inference on the conditional mean residual function can be carried out similarly.
5 Examples and Simulations

5.1 Simulations and Enhancement

A simulation study was performed to compare the two sample testing procedure from Jeong et al. (2008) and one based on the empirical likelihood approach. For both of the two groups simulated, failure times were generated identically from the Weibull distribution with censoring proportions of 0%, 10%, 20% and 30% similarly as in Jeong et al. (2008). The proportion of retaining the null hypothesis of the equality of the two medians were compared for different sample sizes at various time points.

One of the distinct feature of empirical likelihood is that it may be further improved by a Bartlett correction (Romano). However, the correction in the censored data case are not well understood so far. We consider mean adjusted empirical likelihood as an enhancement. A related but separate issue is the discretness of quantile. In uncensored case Owen (1988) noted that when a population quantile is concerned, empirical likelihood reproduces precisely the so-called sign-test or binomial-method interval, which has the coverage accuracy of \( n^{-1/2} \). Chen and Hall (1993) smoothed the constraint function and obtained smoothed empirical likelihood confidence intervals with the coverage error of order \( n^{-1} \). Due to the discrete nature of the regression quantile function, straight empirical likelihood has similar performance and Whang (2006) considered smoothing the constraint function similarly as to Chen and Hall (1993), (theoretically) attaining the coverage error of order \( n^{-1} \). In the case we considered in this paper with random right censoring the proposed method presumably has the same problem and is conservative as shown in table 1, albeit less compared to Jeong et al. (2008)’s method. On the other hand smoothing \( \psi_{1}^{*}(\cdot) \) as proposed in Chen and Hall (1993) may be conceptually troublesome, although improving the performance. With the smoothing it is not clear what is the parameter under question and sample quantile is not anymore the estimator, although the smoothed estimator is asymptotically equivalent to the sample quantile. Also the resulting confidence interval/region is not associated with sample quantile as the way straight empirical likelihood confidence interval/region is (see Remark 1). Therefore we instead consider a mean...
adjustment for the empirical likelihood ratio:

\[-2 \log R^*(b) = \frac{d}{a} \{-2 \log R(b)\}\]

where \(a\) is the bootstrap mean of \(-2 \log \text{ empirical likelihood ratio at } b\) and \(d\) is the dimensionality of \(b\). The adjustment was proposed by Chen and Cui (2006) as empirical Bartlett correction of empirical likelihood for smooth functionals. Due to the discreteness in the quantile, the adjustment may not attain the accuracy better than \(n^{-1}\) but nevertheless improved the performance as shown in Table 1. In general, the results from Jeong et al.’s method tend to be slightly more conservative than ones from the EL approach, especially when the sample size becomes large.

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Table 1: Empirical coverage probabilities for the Jeong et al. (2008) and Empirical Likelihood method. Null hypothesis.

5.2 Examples

In the first example we take the data set cancer from the R package survival. It contains 228 survival times of lung cancer patients from Mayo Clinic with 63 right censored observations.

We fit a median regression on residual lifetime after one year (365.25 days), using the age and sex of the patient as covariates.

Intercept = 1.77341897, slope for age= 0.00816709, slope for sex=0.06587046

These are similar to the regular median regression, i.e. with \(t_0 = 0\).
Intercept=2.157169901, slope for age =0.003291117, slope for sex= 0.085008842.

For the second example we consider a lung cancer data set (Maksymiuk et al., 1994) that has been analyzed by Ying et al. (1995) using median regression, and by Huang et al. (2005) using a least absolute deviations method in the accelerated failure time (AFT) model. In this study, 121 patients with limited-stage small-cell lung cancer were randomly assigned to one of two different treatment sequences A and B, with 62 patients assigned to A and 59 patients to B. Each death time was either observed or administratively censored, and the censoring variable did not depend on the covariates treatment and age. Denote \( X_{1i} \) the treatment indicator variable, and \( X_{2i} \) the entry age for the ith patient, where \( X_{1i} = 1 \) if the patient is in group B. Let \( Y_i \) be the base 10 logarithm of the ith patient’s failure time. The median observed survival time is 511 days. The data set is available in the R package \texttt{rankreg}.

\begin{table}
\centering
\begin{tabular}{lrrrr}
\hline
\textit{t}_0 & \text{month} & \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 \\
\hline
0 & 2.603342985 & -0.263000044 & 0.003836832 & \\
2 & 2.525266352 & -0.302518690 & 0.004506769 & \\
4 & 2.427864714 & -0.356523567 & 0.005461105 & \\
6 & 2.11445574 & -0.44987254 & 0.01060704 & \\
8 & 1.77747523 & -0.51178286 & 0.01605759 & \\
10 & 1.57846108 & -0.56747545 & 0.01852359 & \\
12 & 1.1760552 & -0.4966368 & 0.0247365 & \\
\hline
\end{tabular}
\caption{Estimated slope for regression model with median residual lifetimes after \textit{t}_0. The treatment effects, \( \hat{\beta}_1 \), increases over \textit{t}_0. (lung cancer data)}
\end{table}

The third example comes from a breast cancer study (NSABP Protocol B-04) as described in Jeong et al. (2008). The data includes ?? node positive patients and ?? node-negative patients.

From Table 2 we see that the two approaches provided almost identical results for the 95% confidence intervals for the ratio of the two medians.

\begin{table}
\centering
\begin{tabular}{llllll}
\hline
\textit{t}_0 & Median Residual Lifetime & Ratio & 95\% CI & \\
 & Node-Negative & Node-Positive & J & J & EL & EL \\
\hline
0 & 12.46 (11.2,13.5) & 6.87 (6.4,7.4) & 0.55 & (0.49, 0.63) & (0.49, 0.63) & \\
2 & 12.44 (11.2,13.6) & 6.93 (5.9,8.1) & 0.56 & (0.47, 0.70) & (0.47, 0.70) & \\
4 & 13.05 (11.8,14.8) & 8.24 (6.8,10.2) & 0.63 & (0.49, 0.81) & (0.49, 0.81) & \\
6 & 13.40 (12.5,14.3) & 8.75 (7.7,10.6) & 0.65 & (0.54, 0.81) & (0.56, 0.82) & \\
8 & 12.91 (11.9,13.8) & 10.19 (8.8,11.6) & 0.79 & (0.66, 0.93) & (0.67, 0.93) & \\
10 & 12.48 (11.2,13.7) & 9.66 (8.2,11.8) & 0.77 & (0.62, 1.00) & (0.62, 1.00) & \\
12 & 11.85 (10.6,13.0) & 9.66 (7.5,12.6) & 0.82 & (0.63, 1.08) & (0.63, 1.08) & \\
\hline
\end{tabular}
\caption{Median residual lifetime and ratio for node-negative and node-positive patients. (breast cancer data)}
\end{table}
Table 3. Estimated median residual lifetimes in node-negative and node-positive groups, the ratios, and 95% confidence intervals for the ratios (NSABP B-04 data)

6 Discussion

Other type of AFT estimators, BJ and or rank based.

7 Appendix

We list here R code for some of the computation in the examples. \texttt{WRegRes( ), WRegResTest( )}

\textit{Proof of Theorem 1}

\[ \sum_{i=1}^{n} g(U_i)p_i = \theta \quad \text{or} \quad 0 = \sum_{i=1}^{n} g(U_i, \theta)p_i \]  

where \( g(t) \) is a given function such that \( 0 < \text{Var} g(Y) < \infty \) and the \( \theta \) is the value we wish to test.

Assume

\[ \int \frac{g^2(t)}{1 - G(t -)} dF(t) < \infty, \]

where \( G(\cdot) \) is the distribution of the censoring variable \( C_i \). This is to guarantee the variance of \( \int g(t) d\hat{F}_{KM}(t) \) is finite.

The Empirical Likelihood Theorem asserts that under the null hypothesis, \( H_0 : \theta = \mathbb{E}g(T) \), then \( -2 \log \text{empirical likelihood ratio} \) has an asymptotic chi squared distribution.

\textbf{Theorem} Consider the right censored data and its empirical likelihood defined above. Suppose \( \mathbb{E}g(Y) = \theta \). Assume also that condition (6) holds. Then we have

\[ W(\theta) = -2 \log \frac{\max EL}{EL(KM)} \xrightarrow{p} \chi^2_1 \quad \text{as} \quad n \to \infty, \]

where the maximum in the numerator is carried out over all probabilities \( p_i \) that satisfy (3).

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\textbf{References}


