ASYMPTOTICS OF THE LENGTH SPECTRUM FOR HYPERBOLIC MANIFOLDS OF INFINITE VOLUME

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Abstract. We compute the leading asymptotics of the counting function for closed geodesics on a convex co-compact hyperbolic manifold in terms of spectral data and scattering resonances for the Laplacian. Our result extends classical results of Selberg for compact and finite-volume surfaces to this class of infinite-volume hyperbolic manifolds.

1. Introduction

Selberg's zeta function for a compact or finite-volume hyperbolic surface X [18] is an analytic function constructed from the lengths of closed geodesics. As such, it encodes information about the length spectrum, much as the celebrated Riemann zeta function encodes information about the distribution of prime numbers. Just as the first pole of the Riemann zeta function determines the leading asymptotics of the counting function for prime numbers, so the first zero of Selberg's zeta function determines the leading asymptotics of the counting function for 'prime' geodesics. In this note, we show how results of Patterson and Perry [17] on the divisor of Selberg's zeta function for a convex co-compact hyperbolic manifold can be used to obtain similar results for the counting function for closed geodesics on this class of infinite-volume hyperbolic manifolds.

To state our results, we first recall some basic facts about convex co-compact hyperbolic manifolds. A discrete group G of orientation-preserving isometries of real hyperbolic space $\mathbf{H}^{n+1}$ is called convex co-compact if it admits a finite-sided fundamental domain of infinite hyperbolic volume and has no parabolic elements. If, in addition, G is torsion-free, then the orbit space $X = G\backslash\mathbf{H}^{n+1}$ is a Riemannian manifold with the induced metric; such an X is a convex co-compact hyperbolic manifold. We will also assume that G is non-elementary so that there is an infinite number of distinct hyperbolic conjugacy classes. Under these hypotheses, G consists solely of hyperbolic elements and may be written as a disjoint union of hyperbolic conjugacy classes $\{\gamma\}$, each of which contains a unique primitive element γ. We denote by $\ell(\gamma)$ the length of the associated closed geodesic. We wish to study the counting function for primitive closed geodesics defined by

$$\pi_X(t) = \#\left\{\{\gamma\} : \ell(\gamma) \leq t\right\}.$$

To do so, we will use Selberg's zeta function for X as discussed in [15]. Let $\{\gamma\}$ be a listing of conjugacy classes of hyperbolic elements of G. Each primitive element γ is conjugate to the composition of an $SO(n)$ rotation with eigenvalues $\{\alpha_i(\gamma)\}_{i=1}^{n}$

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and a dilation by $\exp(\ell(\gamma))$. Selberg's zeta function is the Euler product

$$Z_X(s) = \prod_{\{\gamma\}} \prod_{k_1,\cdots,k_n \geq 0} \left[1 - \alpha_1(\gamma)^{k_1} \cdots \alpha_n(\gamma)^{k_n} \exp(-(s+|k|)\ell(\gamma))\right]. \tag{1}$$

The product is easily seen to converge for $\Re(s) > \delta(G)$, where $\delta(G)$ is the exponent of convergence for the Poincaré series

$$\sum_{\gamma \in G} \exp(-sd(w, \gamma(w))). \tag{2}$$

Here $d(\cdot,\cdot)$ denotes hyperbolic distance and w is a chosen point in $\mathbf{H}^{n+1}$; the exponent of convergence is independent of the choice of w. For the class of groups under consideration, $\delta(G) \in (0, n)$. Patterson and Perry [17] showed that the singularities of $Z_X(s)$ are determined by eigenvalues and resonances for the Laplacian on X and topological data of X. In order to find the leading asymptotics of $\pi_X(t)$, we need to know about the first pole of $Z'_X(s)/Z_X(s)$.

We will show that, as in the compact case, the first pole of Selberg's zeta function is determined by the first pole of the resolvent for the Laplacian Δ_X on X. To describe the relationship, we recall that the operator Δ_X has at most finitely many L^2-eigenvalues in the interval $[0, n^2/4)$ [8] and purely absolutely continuous spectrum of infinite multiplicity in $[n^2/4, \infty)$ [9]. It follows that the $L^2(X)$-resolvent operator $\tilde{R}_X(\lambda) = (\Delta_X - \lambda)^{-1}$ is a meromorphic function in the cut plane $\mathbf{C}\backslash[n^2/4, \infty)$ with simple poles at the eigenvalues of Δ_X whose residues are finite-rank operators whose rank equals the dimension of the corresponding eigenspace. It is convenient to introduce the quadratic transformation $\lambda = s(n-s)$ which maps the half-plane $\Re(s) > n/2$ onto the cut plane $\mathbf{C}\backslash[n^2/4, \infty)$; writing $R_X(s) = \tilde{R}_X(s(n-s))$ we have $R_X(s)$ meromorphic in $\Re(s) > n/2$. It was shown by Mazzeo and Melrose [10] that, viewed as a mapping from $C_0^\infty(X)$ to $C^\infty(X)$, the operator $R_X(s)$ admits a meromorphic continuation to $\mathbf{C}$. We will view the resolvent in this extended sense in what follows. The resolvent kernel can be written as a sum over the group G of translates of the resolvent on $\mathbf{H}^{n+1}$ which is dominated by the Poincaré series (2), so that $R_X(s)$ is entire for $\Re(s) > \delta(G)$.

The first pole of $R_X(s)$ (or, more precisely, the resolvent times an Euler Gamma function) occurs in all cases at $s = \delta(G)$, and is a simple pole with rank-one residue. Moreover, there are no other poles of the resolvent on the line $\Re(s) = \delta(G)$. If $\delta(G) > n/2$, this follows from the Elstrodt-Patterson-Sullivan theorem (see [4] and references therein and [13] for $n = 1$, and see Sullivan [19] for $n \geq 1$): there is a strictly positive L^2-eigenfunction of Δ_X with eigenvalue $\lambda_0 = \delta(G)(n - \delta(G))$, and that λ_0 is the lowest eigenvalue (ground state) of Δ_X. The absence of other poles on the line $\Re(s) = \delta(G)$ follows from the self-adjointness of Δ_X. If $\delta(G) \leq n/2$, it was shown by Patterson [14] that the resolvent multiplied by $\Gamma(s - n/2 + 1)$ again has a simple pole with rank-one residue and no other poles on the line $\Re(s) = \delta(G)$. Thus:

Proposition 1.1. *Let G be a convex co-compact, torsion-free discrete group of hyperbolic isometries. The operator-valued function $\Gamma(s - n/2 + 1) R_X(s)$ is holomorphic in $\Re(s) > \delta(G)$ and its first pole occurs at $s = \delta(G)$, with no other poles on the line $\Re(s) = \delta(G)$. The pole at $s = \delta(G)$ is simple and its residue is a rank-one operator.*

We will use the characterization of the divisor of $Z_X(s)$ obtained in [17] to show that in all cases $Z'_X(s)/Z_X(s)$ has its first pole at $s = \delta(G)$, that this pole is simple, and that its residue is one. We will then obtain the leading asymptotics of $\pi_X(t)$.

Theorem 1.1. *Let $X = G\backslash\mathbf{H}^{n+1}$ be a convex co-compact hyperbolic manifold. Then*

$$\lim_{t\to\infty} \left(\pi_X(t) / \left[\exp(\delta t) / (\delta t)\right]\right) = 1$$

where $\delta = \delta(G)$.

For infinite volume surfaces with finite geometry, including convex co-compact hyperbolic manifolds with $n = 1$, this result is due to Guillopé [6].

The plan of this paper is as follows. In section 2 we use the remarks above and results of [17] to compute the first pole of $Z'_X(s)/Z_X(s)$. In section 3 we reduce the proof of Theorem 1.1 to a standard Tauberian argument.

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2. Singularities of the Zeta Function

Our goal in this section is to prove:

Proposition 2.1. *The first pole of $Z'_X(s)/Z_X(s)$ occurs at $s = \delta(G)$, and the residue of $Z'_X(s)/Z_X(s)$ at $s = \delta(G)$ is 1. Moreover, there are no other poles of $Z'_X(s)/Z_X(s)$ on the line $\Re(s) = \delta(G)$.*

First, we consider the case $\delta(G) \geq n/2$. For a resolvent pole ζ, we define the multiplicity of ζ, M_ζ, to be the rank of the operator

$$\int_{\gamma_{\zeta,\varepsilon}} R_X(s)\, ds$$

where $\gamma_{\zeta,\varepsilon}$ is a simple closed contour enclosing ζ and no other singularity of $R_X(s)$.

Lemma 2.1. *Let X be a convex co-compact hyperbolic manifold, and suppose that $\delta(G) \geq n/2$. Then the first pole of $Z'_X(s)/Z_X(s)$ occurs at $s = \delta(G)$ and has residue one. Moreover, there are no other poles of $Z'_X(s)/Z_X(s)$ on the line $\Re(s) = \delta(G)$.*

Proof. It is shown in Theorem 6.2 of [17] that all poles ζ of $Z'_X(s)/Z_X(s)$ for $s > n/2$ arise from L^2-eigenvalues of Δ_X with eigenvalue $\zeta(n-\zeta)$, and the residue at each such ζ is the multiplicity of the corresponding eigenvalue of Δ_X. Thus, for $\delta(G) > n/2$ we observe that, by the Elstrodt-Patterson-Sullivan theorem, Δ_X has its first eigenvalue at $\delta(G)(n-\delta(G))$. Moreover, the simplicity of the first eigenvalue guarantees that $M_{\delta(G)} = 1$. If $\delta(G) = n/2$, we first observe that there are no L^2-eigenvalues since $R_X(s)$ is holomorphic for $s > \delta(G)$. Next, we recall from [17] that $Z'_X(s)/Z_X(s)$ may only have a singularity at $s = n/2$ if $R_X(s)$ has a pole at $s = n/2$, and residue of $Z'_X(s)/Z_X(s)$ at $s = n/2$ equal to $M_{n/2}$. By Proposition 1.1, such a singularity occurs, and $M_{n/2} = 1$. □

This proves Proposition 2.1 if $\delta(G) \geq n/2$. It remains to consider the case $\delta(G) < n/2$.

If $\delta(G) < n/2$ there are no L^2-eigenvalues and no resolvent pole at $s = n/2$, so it follows from Theorems 1.5 and 1.6 of [17] that the only singularities of $Z_X'(s)/Z_X(s)$ in $0 < \Re(s) < n$ must come from poles of the scattering operator $\mathcal{S}_X(s)$. Thus we must study the relationship between poles of $R_X(s)$ and those of $\mathcal{S}_X(s)$. To do this, we first review some basic facts about $\mathcal{S}_X(s)$ and its relationship to the resolvent $R_X(s)$ but refer the reader to [17] for a full discussion.

To define the scattering operator, recall that the manifold X admits a compactification to a C^∞ manifold $\bar{X}$ with boundary $\partial\bar{X}$; $\partial\bar{X}$ is a smooth compact manifold without boundary. Let ρ be a determining function for $\partial\bar{X}$, i.e., a smooth function which is strictly positive on X, vanishes to first order on $\partial\bar{X}$, and has everywhere nonvanishing differential on $\partial\bar{X}$. For $\Re(s) = n/2$, $\Im(s) \neq 0$, and a given function $f_- \in C^\infty(\partial\bar{X})$, there exist a unique C^∞ solution u of the eigenvalue equation

$$(\Delta_X - s(n-s))\, u = 0$$

having the asymptotic form

$$u \sim \rho^s f_+ + \rho^{n-s} f_- + \mathcal{O}\left(\rho^{n/2+1}\right);$$

see [2, 7, 11]. It follows that the mapping $f_- \mapsto f_+$ is a well-defined linear mapping from $C^\infty(\partial\bar{X})$ to itself. In our normalization, the scattering operator is given by

$$\mathcal{S}_X(s)\, f_- = 2^{n-2s} \frac{\Gamma(n/2 - s)}{\Gamma(s - n/2)} f_+$$

where $\Gamma(z)$ is the Euler Gamma function; the Γ-factors divide out trivial poles and zeros which are artifacts of the covering space $\mathbf{H}^{n+1}$. This linear operator is actually a pseudodifferential operator on $C^\infty(\partial\bar{X})$ which depends trivially on the choice of ρ; moreover it extends to a meromorphic family of pseudodifferential operators for $s \in \mathbf{C}$, and up to holomorphically invertible factors is a meromorphic family of Fredholm operators. Thus one can use the theory of [5] to conclude that the multiplicity of zeros and poles of $\mathcal{S}_X(s)$ at $s = \zeta$ is given by the integer

$$\nu_\zeta = \operatorname{Tr}\left(\frac{1}{2\pi i}\int_{\gamma_{\zeta,\varepsilon}} \mathcal{S}_X(s)^{-1}\, \mathcal{S}_X'(s)\, ds\right)$$

where $\gamma_{\zeta,\varepsilon}$ is a simple closed contour surrounding ζ and no other singularity of $\mathcal{S}_X(s)$. Indeed, $\nu_\zeta = \nu_\zeta^+ - \nu_\zeta^-$ where ν_ζ^+ (resp. ν_ζ^-) is a nonnegative integer which counts the multiplicity of the zero (resp. pole) at $s = \zeta$ (see [17] for a complete discussion). From the theory of [5] and the functional equation $\mathcal{S}_X(s)\,\mathcal{S}_X(n-s) = I$, (here I is the identity operator), it follows that $\nu_{n-\zeta} = -\nu_\zeta$.

Theorems 1.5 and 1.6 of [17] imply that poles of $Z_X'(s)/Z_X(s)$ in the range $0 < \Re(s) < n/2$ can only occur if the scattering operator has ν_ζ nonzero for some such ζ, and such poles have residue $-\nu_\zeta$. Moreover, in the case $\delta(G) < n/2$, the scattering operator is holomorphic in a half-plane $\Re(s) > \delta(G)$ which contains the critical line $\Re(s) = n/2$, so that $\nu_\zeta \geq 0$ for ζ in this half-plane. It follows that for $\Re(\zeta) < n/2$, $\nu_\zeta = -\nu_\zeta^-$.

Thus, to complete the proof of Proposition 2.1, we must show that the scattering operator has poles in this region only if $R_X(s)$ has such poles, that the resolvent has a unique pole at $s = \delta(G)$ of multiplicity one on the line $\Re(s) = \delta(G)$, and that this pole induces a pole of the scattering operator of multiplicity one. It will then follow from Proposition 1.1 that $Z'_X(s)/Z_X(s)$ has its first pole at $s=\delta(G)$ and no other pole on the line $\Re(s)=\delta(G)$ as desired.

The Schwartz kernel of the scattering operator may be recovered from that of the resolvent by the formula ([17], eq. (4.12))

$$\mathcal{K}_{\mathcal{S}_X(s)}(b,b') = (2s-n)\frac{\Gamma(s-n/2)}{\Gamma(n/2-s)}\left[\rho(m)\rho(m')\right]^{-s}\mathcal{K}_{R_X(s)}(m,m';s)\Big|_{\substack{m=b\\ m'=b'}} \tag{3}$$

for $b\neq b'$. Thus, roughly and informally, the scattering operator has the meromorphy of $\Gamma(s-n/2+1)R_X(s)$ for $\Re(s)<n/2$. In fact, the residues of $\mathcal{S}_X(s)$ at its poles are known to be finite-rank smooth operators, and the same is true of the resolvent kernel. It follows from this observation that the set of poles of $\mathcal{S}_X(s)$ is contained in the set of poles of $\Gamma(s-n/2+1)R_X(s)$. For a particular pole of $R_X(s)$ to give rise to a pole of $\mathcal{S}_X(s)$, the residue of the right-hand side of (3) must be nonzero.

Theorem 2 of [14] asserts that $\mathcal{K}_{R_X(s)}(m,m')$ has rank-one residue of the form $c(G)F(m)F(m')$ at $s=\delta(G)$ for a function $F\in C^\infty(X)$ and a nonzero constant $c(G)$. Indeed, it follows from Theorem 2 of [14] that $F(\pi(w))=\tilde{F}(w)$, where $\pi:\mathbf{H}^{n+1}\to X$ is the canonical projection and $\tilde{F}$ is the smooth, G-automorphic function on $\mathbf{H}^{n+1}$ given by

$$\tilde{F}(w)=\int_{\partial_\infty\mathbf{H}^{n+1}}P(w,\mathbf{x}')^{\delta(G)}d\mu(\mathbf{x}'). \tag{4}$$

Here the integration goes over $\partial_\infty\mathbf{H}^{n+1}$ (e.g. $\mathbf{R}^n\cup\{\infty\}$ in the upper half-space model), μ is a measure on $\partial_\infty\mathbf{H}^{n+1}$ supported on the limit set of the discrete group G, and $P(w,\mathbf{x}')$ is the non-Euclidean Poisson kernel. In order to prove that $\mathcal{S}_X(s)$ also has a pole at $s=\delta(G)$, we need to show that $F(m)=\rho(m)^{\delta(G)}G(m)$ for a function $G\in C^\infty(\bar{X})$ with $G|_{\partial\bar{X}}\neq 0$.

To do so, we will compute the asymptotic behavior of $\tilde{F}(w)$ in the upper half-space model, using the fact that $\tilde{F}$ is a generalized eigenfunction of the hyperbolic Laplacian Δ_0. In the upper half-space model, $w=(\mathbf{x},y)$ with $\mathbf{x}\in\mathbf{R}^n$ and $y>0$, and the Poisson kernel takes the form

$$P(w,\mathbf{x}')=\frac{y}{|\mathbf{x}-\mathbf{x}'|^2+y^2}.$$

Since G is convex co-compact, there is a finite-sided fundamental domain $\mathcal{F}$ for the action of G whose closure in the Euclidean topology of $\mathbf{R}^{n+1}$ does not touch the limit set. The fundamental domain $\mathcal{F}$ can thus be covered in a neighborhood of infinity by hemispheres isometric to the hemisphere

$$H=\left\{(\mathbf{x},y)\in\mathbf{R}^{n+1}_+:|\mathbf{x}|^2+y^2<1\right\}$$

which have the same property. In such a hemispherical neighborhood the coordinate function y is a determining function for $\partial_\infty\mathbf{H}^{n+1}$ which projects to a determining function for $\partial\bar{X}$. Thus it suffices to show that in each such neighborhood, $\tilde{F}(w)=y^{\delta(G)}\tilde{G}(w)$ where $\tilde{G}$ is the restriction of a smooth function on $\mathbf{R}^{n+1}$ with $\tilde{G}(\mathbf{x},0)\neq 0$.

Using the representation formula (4) and the fact that μ is supported in the limit set, it is easy to see that for any hemispherical neighborhood H of infinity in $\mathbf{H}^{n+1}$ that does not touch the limit set of G, $\tilde{F}(w)=y^{\delta(G)}\tilde{G}(w)$ where $\tilde{G}$ is the restriction to H of a function in $C^\infty(\mathbf{R}^{n+1})$; in particular, $\tilde{G}$ has a Taylor series in y about $y = 0$ to all orders. We wish to show that, in some hemispherical neighborhood H, $\tilde{G}(x, 0) \neq 0$. Supposing this not to be the case, we will show that $\tilde{F}$ is square-integrable in any hemispherical neighborhood of infinity, so that F is a square-integrable eigenfunction of Δ_X, a contradiction, since Δ_X has no such eigenfunctions if $\delta(G) < n/2$.

Since $\tilde{F}$ is a generalized eigenfunction of Δ_0 it has an asymptotic expansion of the form

$$\tilde{F}(w) \sim \sum_{j=0}^{\infty} a_j(\mathbf{x})\, y^{\delta(G)+2j}$$

where, from the differential equation

$$(\Delta_0 - s(n-s))\tilde{F} = 0$$

with $s = \delta(G)$ and

$$\Delta_0 = -(y\partial_y)^2 + n(y\partial_y) - y^2\Delta_{\mathbf{x}},$$

the coefficients $a_j(x)$ obey the recurrence

$$a_{j+1} = \frac{1}{[(s+2j+2)(n-(s+2j+2)) - s(n-s)]}\Delta_{\mathbf{x}} a_j$$

provided $\delta(G) \neq n/2 - k$ for some positive integer k. If $\delta(G) = n/2 - k$ for some positive integer k, it remains true that

$$\tilde{F}(w) \sim \sum_{j=0}^{k-1} a_j(\mathbf{x})\, y^{\delta(G)+2j} + \mathcal{O}\left(y^{\delta(G)+2k}\right)$$

with the same recurrence. In either case, if $a_0 = 0$ we then conclude that $\tilde{F}$ is square-integrable with respect to hyperbolic volume measure $y^{-(n+1)} d\mathbf{x}\, dy$ in a neighborhood of infinity in H. If $a_0 = 0$ for all hemispherical neighborhoods, we conclude that $\tilde{F}$ projects to a square-integrable eigenfunction on X, a contradiction.

It follows from these observations that F is a generalized eigenfunction of Δ_X having the form $F(m) = \rho(m)^{\delta(G)} G(m)$ where $G \in C^\infty(\bar{X})$ and $g = G|_{\partial\bar{X}} \neq 0$. It now follows from (3) that the scattering operator has a first-order pole at $s = \delta(G)$ with rank-one residue

$$c(G)\, g(b) g(b').$$

It is now clear that $\nu^-_{\delta(G)} = 1$, so that $\nu_{\delta(G)} = -1$. Since $\Gamma(s - n/2 + 1) R_X(s)$ has no other poles on the line $\Re(s) = \delta(G)$ by Patterson's theorem, and the poles of the scattering operator are among those of this operator, we have proved:

Lemma 2.2. *Let $X = G\backslash\mathbf{H}^{n+1}$ be a convex co-compact hyperbolic manifold and suppose that $\delta(G) < n/2$. Then the first pole of $Z_X'(s)/Z_X(s)$ occurs at $s = \delta(G)$ and the residue is one. Moreover, there are no other poles of $Z_X'(s)/Z_X(s)$ on the line $\Re(s) = \delta(G)$.*

This completes the proof of Proposition 2.1.

3. Asymptotics of the Length Spectrum

To study the zeta function near its first pole we make a convenient factorization for $\Re(s)$ large which remains valid near $s = \delta(G)$. Write

$$Z_X(s) = Z_{0,X}(s)\, Z_{1,X}(s)$$

where

$$Z_{0,X}(s) = \prod_{\{\gamma\}} (1 - \exp(-s\ell(\gamma))$$

is Ruelle's zeta function, and

$$Z_{1,X}(s) = \prod_{\{\gamma\}} \prod_{\substack{k_1,\cdots,k_n \geq 0 \\ k_1+\cdots+k_n \geq 1}} \left[1 - \alpha_1(\gamma)^{k_1} \cdots \alpha_n(\gamma)^{k_n} \exp(-(s+|k|)\,\ell(\gamma))\right]$$

is holomorphic in the strip $\Re(s) > \delta(G) - 1$ using the Poincaré series (2). It follows from Proposition 2.1 that, in all cases, $Z_{1,X}'(s)/Z_{1,X}(s)$ has a simple pole with residue 1 at $s = \delta(G)$ and no other pole on the line $\Re(s) = \delta(G)$. We can now apply the Wiener-Ikehara Tauberian theorem as used, for example, in ([12], section 10) to recover Theorem 1.1.

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