Limits of Solutions to the Generalized Ginzburg-Landau Functional

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§1 Introduction

Let $\Omega \subset R^n$, $n \geq 2$, be a bounded C^1 domain. Let $N \subset R^k$ be a Riemannian manifold without boundary. For $1 < p \leq n$ and $\epsilon > 0$, we consider the generalized Ginzburg-Landau functional

$$I_\epsilon^p(u) = \int_\Omega (\frac{1}{p}|Du|^p + \frac{1}{\epsilon^p} F(u))\, dx, \tag{1.1}$$

over $W^{1,p}(\Omega, R^k)$. Here $F(p) = \lambda(\mathrm{d}^2(p, N))$ and $\lambda : R_+ \to R_+$ is smooth with $\lambda(t) = t$ for $t \leq \delta_N^2$, $\lambda(t) = 4\delta_N^2$ for $t \geq 4\delta_N^2$, $\lambda' \geq 0$, $\lambda'' \geq 0$, $d(p, N)$ denotes the distance from p to N, and $\delta_N > 0$ is chosen so that $d(p, N)$ is smooth for p with $d(p, N) \leq 2\delta_N$. Suppose that $u_\epsilon \in W^{1,p}(\Omega, R^k)$ is a critical point of I_ϵ^p. Then it satisfies the Euler-Lagrange equations:

$$\mathrm{div}(|Du|^{p-2} Du) + \frac{1}{\epsilon^p} f(u) = 0. \tag{1.2}$$

Here $f(u) = -(DF)(u)$ is the negative gradient of F w.r.t. u. Note that weak solutions to (1.2) are smooth for $p = 2$. It is also well-known that, for $p \neq 2$, weak solutions u_ϵ to (1.2) are in $C^{1,\alpha}$, for some $\alpha \in (0, 1)$ (see also [Tp]). The functional I_ϵ^2 has been used by Chen-Struwe [CS], where they proved the negative gradient flow of I_ϵ^2 converges to a global weak solution of the heat equations of harmonic maps from Riemannian manifolds without boundary into N. Later, Chen-Lin [CL] used the same approach to extend the main result of [CS] to the case that the domain is a Riemannian manifold with boundary. For $p = 2$, two key properties for solutions u_ϵ to (1.2) are: the energy monotonicity inequality, the gradient estimates under small energy hypothesis (see also [CS] [CL]).

Note also that if $N = S^{k-1} \subset R^k$ (the unit $(k-1)$-sphere in R^k), we replace F by $\frac{1}{4}(1-|u|^2)^2$ so that I_ϵ^2 becomes the usual Ginzburg-Landau functional, which has been widely

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used by people to model the superconductivity theory ($k = n = 2$). Recent years there have been many important research works done in this direction, the reader can consult with [BBH] (references therein). In particular, the limit behavior for energy minimizers of I_ϵ^2 under fixed Dirichlet boundary data $g \in \mathrm{Lip}(\Omega, S^1)$, as $\epsilon \downarrow 0$, was established by [BBH]. Later, Han-Li [HaL] and independently Hong [Hm] extended one of the main results of [BBH] to energy minimizing maps of I_ϵ^n for $n \geq 3$ and $k = n$.

On the other hand, it is also a very interesting question to look at the limit for solutions to (1.2), with uniformly bounded energies. In [LW], we show that for $p = 2$, if N doesn't support harmonic S^2 (nonconstant harmonic maps from S^2 to N), then any H^1-weak limit of solutions to (1.2) is a strong limit and hence a stationary harmonic map whose singular set is of Hausdorff codimension at least 4. The ideas in [LW], originated by Lin [Lf] [Lf1] in his study of blowing up of stationary harmonic maps, are to study the defect measures associated with the weak convergence and show that stratifications of nontrivial defect measures can generate at least one harmonic S^2.

The purpose of this note is to extend the result of [LW] to I_ϵ^p, for any $p \in (1, n]$. Our results are divided into two cases: non-integers p's; integers p's.

To state the results, let us first recall the definition of stationary p-harmonic maps.

A map $v \in W^{1,p}(\Omega, N)$ is called a p-harmonic map if

$$\mathrm{div}(|Dv|^{p-2}Dv) + |Dv|^{p-2}A(v)(Dv, Dv) = 0, \tag{1.3}$$

in the sense of distribution. Here $A(\cdot)(\cdot,\cdot)$ denotes the 2nd fundamental form of N in R^k.

A p-harmonic map $v \in W^{1,p}(\Omega, N)$ is called a stationary p-harmonic map if, for any $X \in C_0^1(\Omega, R^n)$,

$$\frac{d}{dt}|_{t=0} \int_\Omega |Du_t|^p \, dx = 0, \tag{1.4}$$

where $u_t(x) = u(x + tX(x))$.

One direct consequence of (1.4) is the energy monotonicity inequality (cf. Price [Pp]):

$$\begin{aligned} &R^{p-n}\int_{B_R(a)} |Dv|^p - r^{p-n}\int_{B_r(a)} |Dv|^p \\ &\geq p\int_{B_R(a)\setminus B_r(a)} |x-a|^{p-n}|Du|^{p-2}|\frac{\partial u}{\partial |x-a|}|^2, \end{aligned} \tag{1.5}$$

for any $a \in \Omega$, $0 < r \leq R < \mathrm{dist}(a, \partial\Omega)$.

A nonconstant C^1 p-harmonic map from S^l to N is called a p-harmonic S^l, here $2 \le l \le n$.

Now we state our theorems.

Theorem A. *Assume that $p \in (1,n)$ is not an integer. Let $\epsilon_i \downarrow 0$ and $u_{\epsilon_i} \in W^{1,p}(\Omega, R^k)$ be solutions to (1.2), with $I^p_{\epsilon_i}(u_{\epsilon_i}) \le C < \infty$. Assume $u_{\epsilon_i} \to u$ weakly in $W^{1,p}(\Omega, R^k)$. Then*

$$e_{\epsilon_i}(u_{\epsilon_i})\,dx \equiv (\frac{1}{p}|Du_{\epsilon_i}|^p + \frac{1}{\epsilon_i^p}F(u_{\epsilon_i}))\,dx \to \frac{1}{p}|Du|^p\,dx$$

as Radon measures. In particular, $u_{\epsilon_i} \to u$ strongly in $W^{1,p}(\Omega, R^k)$.

This strong convergence, combines with Federer dimension reduction argument (cf. [SU] [HL] [Ls] [Fm] for minimizing p-harmonic map cases), yields

Corollary B. *Under the same assumptions as theorem A. The map u obtained is a stationary p-harmonic map. Moreover, there exists a closed $\Sigma \subset \Omega$, with Hausdorff dimension at most $n-[p]-1$, such that $u \in C^{1,\alpha}(\Omega \setminus \Sigma, N)$. If, in addition, N has no p-harmonic S^l for $[p] \le l \le n-1$, then $u \in C^{1,\alpha}(\Omega, N)$ for some $\alpha \in (0,1)$, and $u_{\epsilon_i} \to u$ in C^1 norm. Here $[p]$ denotes the largest integer part of p.*

It is well-known that any p-energy minimizing map is a stationary p-harmonic map. Theorem A indicates that for non-integer p's, the Ginzburg-Landau approximation may be useful to construct stationary (possibly non-minimizing) p-harmonic maps.

Theorem C. *For $p = 2, 3, \cdots, n$. Let $\epsilon_i \downarrow 0$ and $u_{\epsilon_i} \in W^{1,p}(\Omega, R^k)$ be solutions to (1.2), with $I^p_{\epsilon_i}(u_{\epsilon_i}) \le C < \infty$. Assume $u_{\epsilon_i} \to u$ weakly in $W^{1,p}(\Omega, R^k)$. Suppose that there is no p-harmonic S^p in N. Then*

$$e_{\epsilon_i}(u_{\epsilon_i})\,dx \equiv (\frac{1}{p}|Du_{\epsilon_i}|^p + \frac{1}{\epsilon_i^p}F(u_{\epsilon_i}))\,dx \to \frac{1}{p}|Du|^p\,dx$$

as Radon measures. In particular, $u_{\epsilon_i} \to u$ strongly in $W^{1,p}(\Omega, R^k)$.

Similar to Corollary B, we have

Corollary D. *Under the same assumptions as theorem C. The map u obtained is a stationary p-harmonic map. Moreover, there exists a closed $\Sigma \subset \Omega$, with Hausdorff dimension at most $n-p-2$, such that $u \in C^{1,\alpha}(\Omega \setminus \Sigma, N)$ for some $\alpha \in (0,1)$. If, in addition, N has no p-harmonic S^l for $p+1 \le l \le n-1$, then $u \in C^{1,\alpha}(\Omega, N)$ and $u_{\epsilon_i} \to u$ in C^1 norm.*

To obtain these theorems, we first study solutions u_ϵ to (1.2) and use the Pohozaev identity to show (see §2 below) that u_ϵ satisfies an energy monotonicity inequality, similar to (1.5). Note that the argument for the Bochner inequality seems to need the requirement that $p \geq 2$. To prove the gradient estimates for u_ϵ under the small energy hypothesis, we take a compactness method, which seems to be new and may have its own interest, to show that for any $q > n$, $\frac{1}{\epsilon^p} f(u_\epsilon)$ is uniformly bounded in L^q, provided that u_ϵ is uniformly bounded in C^1. For non-integer p's, we use a result due to Marstrand [Mm] to show that the concentration set, along the convergence process, has $(n-p)$ dimensional Hausdorff measure zero. For integer p's, we follow the idea in [LW] with several necessary and nontrivial modifications to show that if the concentration set has positive $(n-p)$-dimensional measure, then we can blow up a p-harmonic S^p in N.

As a by-product of the estimates for (1.2) and the replacement ideas by Chen-Lin [CL1], we are able to give an alternative proof to the partial regularity result for p-energy minimizing maps. It is well-known, modular all other steps, that the following theorem is the key to the partial regularity result to p-energy minimizing maps.

Theorem E ([SU] [HL] [Ls] [Fm]). *There exist $\epsilon_0 > 0$, $0 < \alpha_0 < 1$ such that if $u \in W^{1,p}(B_1, N)$ is a p-energy minimizing map and $\int_{B_1} |Du|^p \leq \epsilon_0^p$, then $u \in C^{\alpha_0}(B_{\frac{1}{2}}, N)$, and*

$$\|u\|_{C^{\alpha_0}(B_{\frac{1}{2}})} \leq C(\epsilon_0, n, N). \tag{1.6}$$

In contrast with theorem C, when $p = 2, 3, \cdots, n$ and N does support harmonic S^p's, we may expect that the strong convergence from u_{ϵ_i} to u may fails. Therefore, by the analysis of §4 below, there exist a closed subset $\Sigma \subset \Omega$, with $H^{n-p}(\Sigma) > 0$ and $H^{n-p}(\Sigma \cap K) < \infty$ for any compact $K \subset\subset \Omega$, and a nonnegative Radon measure ν on Ω, with $\mathrm{spt}(\nu) \subset \Sigma$ and $\nu(\Omega) > 0$, such that

$$e_{\epsilon_i}(u_{\epsilon_i})(x)\, dx \to \frac{1}{p}|Du|^p(x)\, dx + \nu \equiv \mu, \tag{1.7}$$

as convergence of Radon measures. Moreover, for H^{n-p} a.e $x \in \Sigma$,

$$0 < \Theta^{n-p}(\nu, x) \equiv \lim_{r \downarrow 0} r^{p-n} \nu(B_r(x)) < \infty. \tag{1.8}$$

Hence, we can apply the argument of [Lf], on his proof of the rectifiablity result for the concentration set for stationary harmonic maps, with slight modifications, to show that Σ is a $(n-p)$-rectifiable set (One can also follow the abstract approach by Preiss [Pd] to reach the same conclusion). Here, in §5 below, we shall adopt a different and conceptually easier approach, namely the generalized varifold approach which is a natural extension of the classical varifold concept of Almgren [Af] and Allard [Aw] and recently adopted by Ambrosio-Soner [AS] in their study of the dynamics of Ginzburg-Landau equations with complex values and Lin [Lf2] in the study of mapping problems. Roughly, we associate each u_{ϵ_i} with a $(n-p)$-generalized varifold $V_{u_{\epsilon_i}}$ on Ω by letting

$$V_{u_{\epsilon_i}}(x, A) = \delta_{A(u_{\epsilon_i})(x)} e_{\epsilon_i}(u_{\epsilon_i})(x)\, dx,\ \forall (x, A) \in \Omega \times A_{n-p}.$$

Here A_{n-p} is a class of $n \times n$ matrices defined by

$$A_{n-p} = \{A \in R^{n\times n} : A \text{ is symmetric}, \text{trace}(A) = n-p, -(n-p)I_n \leq A \leq I_n\}$$

where I_n is the identity matrix of order n, and $A(u_{\epsilon_i}) \in A_{n-p}$ is defined by

$$\begin{aligned} A(u_{\epsilon_i})(x) &= I_n - p\frac{Du_{\epsilon_i} \otimes Du_{\epsilon_i}}{|Du_{\epsilon_i}|^2}(x), \text{ if } Du_{\epsilon_i} \neq 0, \\ &= I_{n-p}, \qquad\qquad\qquad \text{otherwise.} \end{aligned} \tag{1.9}$$

Then we will show that the first variation of $V(u_{\epsilon_i})$, $\delta V(u_{\epsilon_i})$, converges to zero. Hence, we can assume that there exists a $(n-p)$-generalized varifold V on Ω such that $V_{u_{\epsilon_i}} \to V$ weakly in $\Omega \times A_{n-p}$ and $\delta V = 0$ (i.e., V is stationary). By extending the Allard's rectifiablity theorem from the classical varifolds to the generalized varifolds, we will show that VL$\{x \in \Omega : 0 < \Theta^{n-p}(\|V\|, x) < \infty\}$ is a $(n-p)$-rectifiable classical varifold, here the Radon measure $\|V\| = \mu$ is the weight of V on Ω. In particular, $\Sigma = \{x \in \Omega : 0 < \Theta^{n-p}(\|V\|, x) < \infty\}$ is a $(n-p)$-rectifiable set. In fact, we have

Theorem F. *Under the same notations as above. We have that Σ is a $(n-p)$-rectifiable set. Moreover, V is stationary and $V = V_u + V(\Sigma, \Theta^{n-p}(\nu, \cdot))$. Here*

$$V(\Sigma, \Theta^{n-p}(\nu, \cdot))(x) = \delta_{T_x\Sigma}\Theta^{n-p}(\nu, x) H^{n-p} L\Sigma, \tag{1.10}$$

denotes the $(n-p)$-rectifiable varifold associated with Σ and $\Theta^{n-p}(\nu, \cdot)$. In particular, we have the stationarity for the pair (u, ν) as follows. For any $X \in C_0^1(\Omega, R^n)$,

$$\begin{aligned} &\int_\Omega \frac{1}{p}|Du|^p div(X) - \sum_{1\leq ij\leq n} |Du|^{p-2} u_i u_j X_j^i \\ &= -\int_\Sigma div_\Sigma(X)\Theta^{n-p}(\nu, \cdot)\, dH^{n-p}. \end{aligned} \tag{1.11}$$

The paper is written as follows. In §2, we provide the necessary estimates for u_ϵ. In §3, we prove theorem E. In §4, we prove theorems A, C, corollaries B, D. In §5, we introduce the generalized varifold concept and proved theorem F.

§2 Priori Estimates for solutions to (1.2)

In this section, we derive the monotonicity inequality and ϵ_0-gradient estimates for solutions to (1.2).

Lemma 2.1. *Let $u_\epsilon \in W^{1,p}(\Omega, R^k)$ solve (1.2). Then we have, for any $x \in \Omega$ and $0 < r \leq R < d(x, \partial\Omega)$,*

$$
\begin{aligned}
R^{p-n} \int_{B_R(x)} e_\epsilon(u_\epsilon) - r^{p-n} \int_{B_r(x)} e_\epsilon(u_\epsilon) \\
\geq \int_{B_R(x)\setminus B_r(x)} |y-x|^{p-n} |Du_\epsilon|^{p-2} |\frac{\partial u_\epsilon}{\partial |y-x|}|^2. \\
+ p \int_r^R t^{p-n-1} \int_{B_t} \frac{1}{\epsilon^p} F(u_\epsilon).
\end{aligned}
\tag{2.1}
$$

Proof. Without loss of generality, we may assume $x = 0 \in \Omega$, and $0 < R < d(0, \partial\Omega)$. By the smoothness Lemma (see, e.g. [Tp] [Mc]), we know that $|Du_\epsilon|^{\frac{p-2}{2}} |D^2 u_\epsilon| \in L^2(\Omega, R^k)$. Note also that $u_\epsilon \in C^1(\Omega, R^k)$. Hence we have $|Du_\epsilon|^p \in W^{1,2}(\Omega, R^k)$. Now multiplying (1.2) by $x \cdot Du_\epsilon$ and integrating it over B_R and using the integration by parts, we have

$$
\begin{aligned}
0 = & - \int_{B_R} |Du_\epsilon|^p + n \int_{B_R} (\frac{1}{p}|Du_\epsilon|^p + \frac{1}{\epsilon^p} F(u_\epsilon)) \\
& - R \int_{\partial B_R} (\frac{1}{p} |Du_\epsilon|^{\frac{p}{2}} + \frac{1}{\epsilon^p} F(u_\epsilon)) + R \int_{\partial B_R} |Du_\epsilon|^{p-2} |\frac{\partial u_\epsilon}{\partial r}|^2
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{d}{dR}(R^{p-n} \int_{B_R} e_\epsilon(u_\epsilon)) = & R^{p-n} \int_{\partial B_R} |Du_\epsilon|^{p-2} |\frac{\partial u_\epsilon}{\partial r}|^2 \\
& + R^{p-n-1} \int_{B_R} \frac{p}{\epsilon^p} F(u_\epsilon).
\end{aligned}
$$

Integrating it, we obtain (2.1). ∎

Lemma 2.2. *Let $u_\epsilon \in W^{1,p}(\Omega, R^k)$ solve (1.2). Then, for any $x \in \Omega$ and $0 < R < d(x, \partial\Omega)$, we have*

$$
\int_{B_{\frac{R}{2}}(x)} \frac{1}{\epsilon^p} \lambda'(d^2(u_\epsilon, N)) d(u_\epsilon, N) \leq C(R^{n-p} + \int_{B_R(x)} |Du_\epsilon|^p).
\tag{2.2}
$$

Proof. Since $f(u) = -DF(u)$, $f(u_\epsilon) = -2\lambda'(d^2(u_\epsilon, N))d(u_\epsilon, N)\frac{d}{du_\epsilon}d(u_\epsilon, N)$. Note that for $d(u_\epsilon, N) < 2\delta_N$ there exists a smooth unit-vector field ν in R^k such that $\frac{d}{du_\epsilon}d(u_\epsilon, N) = \nu(u_\epsilon)$. Let $\phi \in C_0^\infty(B_R(x))$ be such that $\phi(y) = 1$ for $y \in B_{\frac{R}{2}}(x)$, and $|D\phi| \leq \frac{2}{R}$. Multiplying (1.2) by $\nu(u_\epsilon)\phi^2$ and integrating it over $B_R(x)$, we obtain

$$\begin{aligned} &2\int_{B_R(x)} \frac{1}{\epsilon^p}\lambda'(d^2(u_\epsilon, N))d(u_\epsilon, N)\phi^2 \\ &= -\int_{B_R(x)} |Du_\epsilon|^{p-2}Du_\epsilon \cdot D(\nu(u_\epsilon)\phi^2). \end{aligned}$$

Note that

$$\begin{aligned} &|\int_{B_R(x)} |Du_\epsilon|^{p-2}Du_\epsilon \cdot D(\nu(u_\epsilon)\phi^2)| \\ &= |\int_{B_R(x)} (|Du_\epsilon|^p\nu'(u_\epsilon)\phi^2 + |Du_\epsilon|^{p-2}Du_\epsilon D\phi^2\nu(u_\epsilon))| \\ &\leq C\int_{B_R(x)} |Du_\epsilon|^p + \frac{C}{R}\int_{B_R(x)} |Du_\epsilon|^{p-1} \\ &\leq C(R^{n-p} + \int_{B_R(x)} |Du_\epsilon|^p). \end{aligned}$$

Here we have used the fact that $|D\nu| \leq C$ and the Schwartz inequality at the last step. This certainly implies (2.2). ■

Now we are ready to show the gradient estimates under the small energy hypothesis.

Lemma 2.3. *There exist $\epsilon_0 > 0$, $0 < \alpha_0 < 1$, $C_0 > 0$ such that if $u_\epsilon \in W^{1,p}(\Omega, R^k)$ solves (1.2), and for $a \in \Omega$, $0 < r < d(a, \partial\Omega)$,*

$$r^{p-n}\int_{B_r(a)} e_\epsilon(u_\epsilon) \leq \epsilon_0^p,$$

then

$$r^p \sup_{B_{\frac{r}{2}}(a)} e_\epsilon(u_\epsilon) \leq C_0\epsilon_0^p, \tag{2.3}$$

and

$$r^{\alpha_0}\|Du_\epsilon\|_{C^{\alpha_0}(B_{\frac{r}{2}}(a))} \leq C\epsilon_0. \tag{2.4}$$

Proof. For $a \in \Omega$ and $0 < r < d(a, \partial\Omega)$. Similar to [Sr]. Let $r_1 > 0$ and $x_1 \in B_{r_1}(a)$ be such that

$$\max_{0 \leq s \leq r} (r-s)^p \max_{B_s(a)} e_\epsilon(u_\epsilon) = (r-r_1)^p \max_{B_{r_1}(a)} e_\epsilon(u_\epsilon) = (r-r_1)^p e_\epsilon(u_\epsilon)(x_1).$$

Denote $\bar{e}^p = \max_{B_{r_1}(a)} e_\epsilon(u_\epsilon)$. Then we know

$$\begin{aligned}\max_{B_{\frac{r-r_1}{2}}(x_1)} e_\epsilon(u_\epsilon) &\leq \max_{B_{\frac{r+r_1}{2}}(a)} e_\epsilon(u_\epsilon) \\ &\leq \frac{(r-r_1)^p \max_{B_{r_1}(a)} e_\epsilon(u_\epsilon)}{(r-\frac{r+r_1}{2})^p} \leq 2^p \max_{B_{r_1}(a)} e_\epsilon(u_\epsilon) = 2^p \bar{e}^p.\end{aligned}$$

Let $r_2 = \frac{(r-r_1)\bar{e}}{2}$ and define $v_\epsilon(x) = u_\epsilon(x_1 + \frac{x}{\bar{e}})$. Then v_ϵ satisfies

$$\max_{B_{r_2}(0)} e_{\bar{\epsilon}}(v_\epsilon) \leq 2^p, \;\; e_{\bar{\epsilon}}(v_\epsilon)(0) = 1, \tag{2.5}$$

where $\bar{\epsilon} = \bar{e}\epsilon$, and

$$\operatorname{div}(|Dv_\epsilon|^{p-2} Dv_\epsilon) + \frac{1}{\bar{\epsilon}^p} f(v_\epsilon) = 0, \quad \text{in } B_{r_2}. \tag{2.6}$$

Now we claim $r_2 \leq 1$. It is clear that $r_2 \leq 1$ implies (2.3). Suppose, otherwise, that $r_2 > 1$. Then we claim that there exists a universal constant $C > 0$ such that

$$1 \leq C \int_{B_1} e_{\bar{\epsilon}}(v_\epsilon). \tag{2.7}$$

We prove (2.7) by contradiction. Suppose that (2.7) is not true. Then there exist $\bar{\epsilon}_i \to 0$ and a sequence of $\{v_{\epsilon_i}\} \subset W^{1,p}(\Omega, R^k)$ satisfying (2.5), (2.6), but

$$\int_{B_1} e_{\bar{\epsilon}_i}(v_{\epsilon_i}) = \eta_i \to 0. \tag{2.8}$$

Claim. *For any positive integer q. There exist $\delta_q \in (0,1)$, $C_q > 0$, such that*

$$\int_{B_{\delta_q}} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^q \leq C_q, \;\; \forall i. \tag{2.9}$$

Once (2.9) is proven, we see that

$$\int_{B_{\delta_q}} (\frac{1}{\bar{\epsilon}_i^p} |f(v_{\epsilon_i})|)^q \leq 2^q \int_{B_{\delta_q}} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^q \leq C_q$$

so that the result of [Tp] implies that, by choosing $q > n$, there exists $\alpha_q \in (0,1)$ such that

$$\|v_{\epsilon_i}\|_{C^{1,\alpha_q}(B_{\frac{\delta_q}{2}})} \le C_q, \ \forall i. \tag{2.10}$$

Therefore we can assume that $v_{\epsilon_i} \to v_0$ in $C^1(B_{\frac{\delta_q}{2}})$. Moreover, (2.8) implies that $v_0 \in N$ is a constant. But this contradicts with $|Dv_0|(0) = 1$, which follows from (2.5) and the C^1-convergence.

Now, we return to prove (2.9).

Proof of Claim. Lemma 2.2 implies that (2.9) is true for $q = 1$. By the induction on q, we may assume that (2.9) is true for $q = k \ge 2$. Then, by the Fubini's theorem, there exists $\delta_{k+1} \in (\frac{\delta_k}{2}, \delta_k)$ such that

$$\int_{\partial B_{\delta_{k+1}}} (\frac{1}{\bar{\epsilon_i}^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^k \le C_{k+1}. \tag{2.11}$$

Multiplying (2.6) by $(\frac{\lambda'(d^2(v_{\epsilon_i},N))d(v_{\epsilon_i},N)}{\bar{\epsilon}_i^p})^k \nu(v_{\epsilon_i})$ and integrating it over $B_{\delta_{k+1}}$, we have

$$\begin{aligned}
&\int_{B_{\delta_{k+1}}} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^{k+1} \\
&\le - \int_{B_{\delta_{k+1}}} |Dv_{\epsilon_i}|^{p-2} Dv_{\epsilon_i} \cdot D((\frac{\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)}{\bar{\epsilon}_i^p})^k \nu(v_{\epsilon_i})) \\
&+ \int_{\partial B_{\delta_{k+1}}} |Dv_{\epsilon_i}|^{p-2} \frac{\partial v_{\epsilon_i}}{\partial r} \cdot (\frac{\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)}{\bar{\epsilon}_i^p})^k \nu(v_{\epsilon_i}) \\
&= I + II
\end{aligned}$$

Note that, by (2.5) and (2.11), we have

$$\begin{aligned}
|II| &\le \int_{\partial B_{\delta_{k+1}}} |Dv_{\epsilon_i}|^{p-1} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^k \\
&\le \max_{B_1} |Dv_{\epsilon_i}|^{p-1} \int_{\partial B_{\delta_{k+1}}} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^k \\
&\le 2^{p-1} C_{k+1}
\end{aligned}$$

For I, we have

$$\begin{aligned}
I = &- \int_{B_{\delta_{k+1}}} |Dv_{\epsilon_i}|^{p-2} Dv_{\epsilon_i} \cdot D(\nu(v_{\epsilon_i}))(\frac{\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)}{\bar{\epsilon}_i^p})^k \\
&- \frac{k}{\bar{\epsilon}_i^p} \int_{B_{\delta_{k+1}}} [(\frac{\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)}{\bar{\epsilon}_i^p})^{k-1} \\
&\cdot |Dv_{\epsilon_i}|^{p-2} Dv_{\epsilon_i} \cdot D(\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)) \nu(v_{\epsilon_i})] \\
&= I_1 + I_2
\end{aligned}$$

It is easy to see from (2.5) and (2.11) that

$$|I_1| \le C \max_{B_1} |Dv_{\epsilon_i}|^p \int_{B_{\delta_k}} (\frac{1}{\bar{\epsilon}_i^p} \lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N))^k \le 2^p C_k$$

Using the fact that $\lambda' \ge 0, \lambda'' \ge 0$, we also have

$$\begin{aligned} & Dv_{\epsilon_i} \cdot D(\lambda'(d^2(v_{\epsilon_i}, N)) d(v_{\epsilon_i}, N)) \nu(v_{\epsilon_i}) \\ & = \lambda'(d^2(v_{\epsilon_i}, N)) |Dv_{\epsilon_i} \cdot \nu(v_{\epsilon_i})|^2 \\ & + 2\lambda''(d^2(v_{\epsilon_i}, N)) d^2(v_{\epsilon_i}, N) |Dv_{\epsilon_i} \cdot \nu(v_{\epsilon_i})|^2 \ge 0, \end{aligned}$$

so that

$$I_2 \le 0.$$

Putting all these inequalities together, we prove that (2.9) holds for $q = k+1$. Therefore (2.9) and (2.7) are proved. Now we want to show that (2.7) is impossible. In fact, by using the monotonicity inequality (2.1) for v_ϵ, we have

$$\begin{aligned} \int_{B_1} e_{\bar{\epsilon}}(v_\epsilon) & \le r_2^{p-n} \int_{B_{r_2}} \frac{1}{p} |Dv_\epsilon|^p + \frac{1}{(\bar{\epsilon}\epsilon)^p} F(v_\epsilon) \\ & = (\frac{r - r_1}{2})^{p-n} \int_{B_{\frac{r-r_1}{2}}(x_1)} \frac{1}{p} |Du_\epsilon|^p + \frac{1}{\epsilon^p} F(u_\epsilon) \\ & \le C r^{p-n} \int_{B_r(a)} e_\epsilon(u_\epsilon) \le C \epsilon_0^p \end{aligned}$$

Hence (2.7) implies that $1 \le C\epsilon_0^p$, which is impossible if we choose ϵ_0 sufficiently small. This finish the proof of (2.3). Once (2.3) is proven, we can apply the Claim again to show that (2.4) also holds true. ■

§3 Proof of Theorem E

In this section, we give an alternative proof of Theorem E by using the estimates of §2. The idea was originally due to Chen-Lin [CL1], where theorem E is proved for the case $p = 2$.

It is well-known that iterations of the following energy decay result and Morrey Lemma [Mc] imply Theorem E (see, e.g. [SU] [HL] [Ls] [Fm]).

Lemma 3.1. *There exist $\epsilon_0 > 0$, $\theta_0 \in (0,1)$, such that if $u \in W^{1,p}(B_2, N)$ is a p-energy minimizing map, with $\int_{B_2} |Du|^p \le \epsilon_0^p$, then*

$$\theta_0^{p-n} \int_{B_{\theta_0}} |Du|^p \le \frac{1}{2} \int_{B_1} |Du|^p. \tag{3.1}$$

Proof. For $\epsilon > 0$, let $v_\epsilon \in W^{1,p}(B_2 \setminus B_1, R^k)$ satisfy

$$\int_{B_2\setminus B_1} \frac{1}{p}|Dv_\epsilon|^p + \frac{1}{\epsilon^p}F(v_\epsilon) = \inf\{\int_{B_2\setminus B_1} \frac{1}{p}|Dv|^p + \frac{1}{\epsilon^p}F(v)\}$$

among $v \in W^{1,p}(B_2 \setminus B_1, R^k)\}$, with $v|_{\partial(B_2\setminus B_1)} = u|_{\partial(B_2\setminus B_1)}$. The existence of v_ϵ is standard. Note that v_ϵ satisfies (1.2) in $B_2 \setminus B_1$, and we have, by comparing it to u,

$$\int_{B_2\setminus B_1} \frac{1}{p}|Dv_\epsilon|^p + \frac{1}{\epsilon^p}F(v_\epsilon) \le \int_{B_2\setminus B_1} \frac{1}{p}|Du|^p \le \epsilon_0^p. \tag{3.2}$$

Here we have used the fact that $F(u) = 0$. Lemma 2.1 implies that

$$r^{p-n}\int_{B_r(a)} \frac{1}{p}|Dv_\epsilon|^p + \frac{1}{\epsilon^p}F(v_\epsilon) \le \epsilon_0^p, \tag{3.3}$$

for any a with $|a| = \frac{3}{2}$, and $0 < r < \frac{1}{4}$. Therefore, by choosing ϵ_0 sufficiently small, we can apply (2.3) of Lemma 2.3 to conclude that

$$\|Dv_\epsilon\|_{C^{\alpha_0}(\{\frac{11}{8}\le|x|\le\frac{13}{8}\})} \le C\epsilon_0, \tag{3.4}$$

for some $0 < \alpha_0 < 1$. By the lower semi-continuity, we may assume that $v_\epsilon \to v_*$ weakly in $W^{1,p}(B_2 \setminus B_1, R^k)$. It is clear that $v_* : B_2 \setminus B_1 \to N$, $v_* = u$ on $\partial(B_2 \setminus B_1)$. Moreover,

$$\int_{B_2\setminus B_1} |Dv_*|^p \le \int_{B_2\setminus B_1} |Du|^p.$$

This implies that $v_* : B_2 \setminus B_1 \to N$ is also a p-energy minimizing map. Moreover, (3.4) implies that

$$\max_{|x|=\frac{3}{2}} |Dv_*| \le C\epsilon_0. \tag{3.5}$$

Now, we define $u_* : B_2 \to N$ by

$$\begin{aligned} u_* &= u, \text{ for } |x| \le 1 \\ &= v_*, \text{ for } 1 \le |x| \le 2 \end{aligned}$$

It is clear that u_* is a p-energy minimizing map and

$$\int_{B_2} |Du_*|^p = \int_{B_2} |Du|^p$$

Moreover, (3.5) yields

$$\operatorname{osc}_{\partial B_{\frac{3}{2}}} |u_*| \leq C\epsilon_0. \tag{3.6}$$

Now we have

Claim. *If $\epsilon_0 > 0$ is sufficiently small, then*

$$osc_{B_{\frac{3}{2}}} |u_*| \leq C\epsilon_0. \tag{3.7}$$

Proof of Claim. To prove this claim, we recall a fact (see, Jost [Jj]) that there exist a $\delta_0 = \delta_0(N) > 0$ and a Lipschitz map $\Phi_0 : N \to B(p, \delta_0)$, which satisfies

$$\Phi_0(q) = q \ \forall q \in B(p, \delta_0), |D\phi|(q) < 1 \ \forall q \in N \setminus B(p, \delta_0). \tag{3.8}$$

Here $B(p, \delta_0) \subset N$ denotes a geodesic ball, centered at p, with radius δ_0. Note that if we let $p = u_*(a_0)$ for some $a_0 \in \partial B_{\frac{3}{2}}$, then $u_*(\partial B_{\frac{3}{2}}) \leq B(p, 2C\epsilon_0) \subset N$. Therefore, if we choose $\epsilon_0 = \frac{\delta_0}{2C}$ then we must have $u_*(B_{\frac{3}{2}}) \subset B(p, \delta_0)$. For, otherwise, we have $\Phi_0(u_*) \neq u_*$ and $\Phi_0(u_*)|_{\partial B_{\frac{3}{2}}} = u_*|_{\partial B_{\frac{3}{2}}}$, but

$$\int_{B_{\frac{3}{2}}} |D(\Phi_0(u_*))|^p < \int_{B_{\frac{3}{2}}} |Du_*|^p,$$

which contradicts with the minimizing property of u_*. This finishes the proof of (3.7). Note that (3.7) implies, in particular, $\operatorname{osc}_{B_1} u \leq C\epsilon_0$.

Now we sketch how (3.7) implies (3.1). To do it, let $v \in W^{1,p}(B_1, R^k)$ be given by

$$\begin{aligned} \operatorname{div}(|Dv|^{p-2}Dv) &= 0, \quad \text{in } B_1 \\ v &= u, \quad \text{on } \partial B_1 \end{aligned} \tag{3.9}$$

Hence we have (see [Uk] for $p \geq 2$, and [Tp] for $1 < p < 2$)

$$\|Dv\|_{L^\infty(B_{\frac{1}{2}})} \leq C\|Du\|_{L^p(B_1)}. \tag{3.10}$$

In particular, for $0 < \theta < \frac{1}{2}$,

$$\theta^{p-n} \int_{B_\theta} |Dv|^p \leq C\theta^p \int_{B_1} |Du|^p. \tag{3.11}$$

Multiplying (1.3) and (3.9) by $u-v$ and integrating over B_1, we obtain

$$\begin{aligned}
&\int_{B_1}(|Du|^{p-2}Du-|Dv|^{p-2}Dv)\cdot(Du-Dv)\\
&=\int_{B_1}|Du|^{p-2}A(u)(Du,Du)\cdot(u-v)\\
&\le C\max_{B_1}|u-v|\int_{B_1}|Du|^p\\
&\le C\operatorname{osc}_{B_1}u\int_{B_1}|Du|^p\le C\epsilon_0\int_{B_1}|Du|^p.
\end{aligned}$$

For $p\ge 2$, and $a,b\in R^k$, we have

$$|a-b|^p\le 2^{p-1}(|a|^{p-2}a-|b|^{p-2}b)\cdot(a-b). \tag{3.12}$$

Hence we have, for $p\ge 2$,

$$\int_{B_1}|Du-Dv|^p\le C\epsilon_0\int_{B_1}|Du|^p. \tag{3.13}$$

For $1<p<2$, and $a,b\in R^k$ (see [HLM]), we have

$$(p-1)(|a|+|b|)^{p-2}|a-b|^2\le(|a|^{p-2}a-|b|^{p-2}b)\cdot(a-b). \tag{3.14}$$

Applying (3.14) and the Schwartz inequality, we have

$$\begin{aligned}
&\int_{B_1}|Du-Dv|^p\\
&\le C\Big(\int_{B_1}|Du|^p+|Dv|^p\Big)^{\frac{2-p}{2}}\\
&\cdot\Big(\int_{B_1}(|Du|^{p-2}Du-|Dv|^{p-2}Dv)\cdot(Du-Dv)\Big)^{\frac{p}{2}}\\
&\le C\Big(\int_{B_1}|Du|^p\Big)^{\frac{2-p}{2}}\Big(C\epsilon_0\int_{B_1}|Du|^p\Big)^{\frac{p}{2}}\\
&\le C\epsilon_0^{\frac{p}{2}}\int_{B_1}|Du|^p.
\end{aligned} \tag{3.15}$$

Note that

$$\theta^{p-n}\int_{B_\theta}|Du|^p\le 2^p\theta^{p-n}\int_{B_\theta}|Dv|^p+2^p\theta^{p-n}\int_{B_\theta}|Du-Dv|^p$$

Putting (3.11), (3.13), (3.15) together, we then have, for all $p>1$,

$$\theta^{p-n}\int_{B_\theta}|Du|^p\le C\theta^p\int_{B_1}|Du|^p+C\sqrt{\epsilon_0}\theta^{p-n}\int_{B_1}|Du|^p. \tag{3.16}$$

Choosing $\theta = \theta_0 = (4C)^{-\frac{1}{p}}$ and $\epsilon_0 = (\frac{\theta_0^{n-p}}{4C})^2$, we get

$$\theta_0^{p-n} \int_{B_{\theta_0}} |Du|^p \le \frac{1}{2} \int_{B_1} |Du|^p$$

This finishes the proof of Lemma 3.1. ∎

§4 Proof of Theorems A, C, Corollaries B, D

In this section, we will prove the conclusions of theorem A, C, and corollaries B, D. The arguments here are similar to [Lf] [LW].

First we recall a theorem due to Marstrand [Mm] (see also [Lf1] for a simple proof).

Theorem 4.1([Mm]). *Let s be a positive number. Suppose that μ a Radon measure on R^n such that the density $\Theta^s(\mu, a)$ exists and is positive and finite in a set of positive μ measure. Then s is an integer.*

Now, let $\{u_{\epsilon_i}\}$ solve (1.2) and $u_{\epsilon_i} \to u$ weakly in $W^{1,p}(\Omega, R^k)$. Since $I^p_{\epsilon_i}(u_{\epsilon_i})$ is bounded, we may assume that there exists a nonnegative Radon measure ν on Ω such that

$$e_{\epsilon_i}(u_{\epsilon_i})\, dx \to \frac{1}{p}|Du|^p\, dx + \nu, \quad \text{as } \epsilon_i \to 0,$$

as convergence of Radon measures on Ω. Denote $\mu = \frac{1}{p}|Du|^p\, dx + \nu$. The monotonicity inequality (2.1), for u_{ϵ_i}, implies

$$r^{p-n}\mu(B_r(x)) \le R^{p-n}\mu(B_R(x)), \tag{4.1}$$

for any $x \in \Omega$, $0 < r \le R < d(x, \partial\Omega)$. Hence

$$\Theta^{n-p}(\mu, x) = \lim_{r \downarrow 0} r^{p-n}\mu(B_r(x))$$

exists for any $x \in \Omega$ and is upper semi-continuous in Ω. Define

$$\Sigma = \cap_{r>0}\{x \in \Omega | \liminf_{i\to\infty} r^{p-n} \int_{B_r(x)} e_{\epsilon_i}(u_{\epsilon_i}) \ge \epsilon_0^p\}$$

where ϵ_0 is the same constant as in Lemma 2.3. Then (2.1) implies that $\Sigma \subset \Omega$ is a closed set and

$$H^{n-p}(\Sigma \cap K) \le C(K) < \infty,$$

for any compact $K \subset\subset \Omega$. Moreover, Lemma 2.3 implies

$$u_{\epsilon_i} \to u, \quad \text{in } C^1_{\text{loc}} \cap W^{1,p}_{\text{loc}}(\Omega \setminus \Sigma, R^k).$$

Let $\mathrm{sing}(u)$ denote the discontinuous set of u, and $\mathrm{spt}(\nu)$ denote the support of ν. Note that $\mathrm{sing}(u) \subset \Sigma$ and $u \in C^{1,\alpha_0}_{\mathrm{loc}}(\Omega \setminus \Sigma, N)$ is a p-harmonic map in Ω. In fact, we have

Claim 1. *$sing(u) \cup spt(\nu) = \Sigma$.*

Let $B_R \subset\subset \Omega \setminus \Sigma$ be any ball. Taking $\epsilon_i \downarrow 0$, (2.1) implies that for $0 < r \leq R$,

$$
\begin{aligned}
&R^{p-n} \int_{B_R} (\frac{1}{p}|Du|^p + d\nu) - r^{p-n} \int_{B_r} (\frac{1}{p}|Du|^p + d\nu) \\
&= \int_{B_R \setminus B_r} |x|^{p-n} |Du|^{p-2} |\frac{\partial u}{\partial r}|^2 + p \int_r^R t^{p-n-1} \nu(B_t)\, dt. \qquad (4.2)
\end{aligned}
$$

Here we have used the fact that

$$
\frac{1}{p}|Du_{\epsilon_i}|^p\, dx \to \frac{1}{p}|Du|^p\, dx, \ \frac{1}{\epsilon_i^p} F(u_{\epsilon_i})\, dx \to \nu
$$

as convergence of Radon measures in $\Omega \setminus \Sigma$. On the other hand, since u is a C^1 p-harmonic map on $\Omega \setminus \Sigma$, we know that u also satisfies the monotonicity inequality (1.5) on B_R. Therefore, subtracting (4.2) by (1.5), we have

$$
R^{p-n} \nu(B_R) - r^{p-n} \nu(B_r) = p \int_r^R t^{p-n-1} \nu(B_t)\, dt,
$$

or

$$
\frac{d}{dr}(r^{-n} \nu(B_r)) = 0, \ \forall 0 < r \leq R. \qquad (4.3)
$$

In particular, $\Theta^n(\nu, z) = 0$ for all $z \in \Omega \setminus \Sigma$. Hence $\nu(B_R) = 0$. This implies that $\mathrm{spt}(\nu) \subset \Sigma$. Therefore, $\mathrm{sing}(u) \cup \mathrm{spt}(\nu) \subset \Sigma$. On the other hand, if $x_0 \notin \mathrm{sing}(u) \cup \mathrm{spt}(\nu)$, then there exists $r_0 > 0$ such that $\nu(B_{r_0}(x_0)) = 0$ and

$$
r_0^{p-n} \int_{B_{r_0}(x_0)} |Du|^p \leq \frac{\epsilon_0^p}{2}.
$$

Hence $x_0 \notin \Sigma$. This finishes the proof of claim 1.

Claim 2. *For any $a \in \Sigma$,*

$$
\epsilon_0^p \leq \Theta^{n-p}(\mu, a) < \infty. \qquad (4.3)
$$

For H^{n-p} a.e. $a \in \Sigma$,

$$
\Theta^{n-p}(\nu, a) = \Theta^{n-p}(\mu, a). \qquad (4.4)
$$

(4.3) follows from the definition of Σ and (2.1). To see (4.4), recall a theorem, due to Federer-Ziemer [FZ], that for H^{n-p} a.e. $x \in \Omega$, we have

$$\lim_{r\to 0} r^{p-n} \int_{B_r(x)} |Du|^p \, dx = 0. \tag{4.5}$$

Hence, for H^{n-p} a.e. $a \in \Sigma$, we have $\Theta^{n-p}(\nu, a) = \Theta^{n-p}(\mu, a)$.

Claim 3.

$$e_{\epsilon_i}(u_{\epsilon_i})\, dx \not\rightharpoonup \frac{1}{p}|Du|^p \, dx \Leftrightarrow \nu(\Omega) > 0, \;\; H^{n-p}(\Sigma) > 0.$$

To see this, we first note that for any compact $K \subset\subset \Omega$, and $r_0 = \frac{1}{2}\mathrm{dist}(K, \partial\Omega)$, we have

$$\Theta^{n-p}(\mu, a) \le r_0^{p-n}\mu(B_{r_0}(a)) = \lim_{\epsilon_i \to 0} r_0^{p-n} \int_{B_{r_0}(a)} e_{\epsilon_i}(u_{\epsilon_i}) \le r_0^{p-n} E,$$

for $a \in \Sigma \cap K$, here $E = \sup_i I^p_{\epsilon_i}(u_{\epsilon_i}) < \infty$. Hence the comparison Lemma implies

$$\epsilon_0^p H^{n-p}(\Sigma \cap K) \le \mu(\Sigma \cap K) \le r_0^{p-n} E H^{n-p}(\Sigma \cap K).$$

Therefore, $\mu(\Sigma) = 0 \Leftrightarrow H^{n-p}(\Sigma) = 0$. This, combines with the fact $0 \le \nu \le \mu$, gives the claim.

Proposition 4.2 *Under the same notations as above. We have*

$$\lim_{i\to\infty} \int_{\Omega\cap K} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}) = 0,$$

for any compact subset $K \subset\subset \Omega$.

Proof. If follows from the proof of claim 1 that for any $\beta > 0$ sufficiently small

$$\lim_{i\to\infty} \int_{(\Omega\cap K)\setminus\Sigma_\beta} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}) = 0,$$

where $\Sigma_\beta = \{x \in \Omega : \mathrm{dist}(x, \Sigma) \le \beta\}$. Therefore, it suffices to show that

$$\lim_{i\to\infty} \int_{K\cap\Sigma_\beta} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}) = 0(\beta),$$

where $\lim_{\beta\to 0} 0(\beta) = 0$. The monotonicity inequality (2.1) implies, for any $a \in \Sigma$,

$$R^{p-n}\mu(B_R(a)) - r^{p-n}\mu(B_r(a)) \ge p \lim_{i\to\infty} \int_r^R t^{p-n-1} \int_{B_t(a)} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}).$$

In particular $\Theta^{n-p}(\mu, a) = \lim_{r\downarrow 0} r^{p-n}\mu(B_r(a))$ exists and there exists $\beta_0 > 0$ such that for any $\beta < \beta_0$

$$\int_0^\beta t^{p-n-1} \lim_{i\to\infty} \int_{B_t(a)} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}) = 0(\beta).$$

This, combines with the Fubini's theorem, implies

$$\lim_{i\to\infty} \int_{B_\beta(a)} \frac{1}{\epsilon_i^p} F(u_{\epsilon_i}) = 0(\beta),$$

so that a simple covering argument yields the desired inequality. ■

Now we can give the proof of theorem A.

Proof of Theorem A

Assume that $e_{\epsilon_i}(u_{\epsilon_i})\,dx \not\rightharpoonup \frac{1}{p}|Du|^p\,dx$. Then we know $H^{n-p}(\Sigma) > 0$ and $\nu(\Sigma) > 0$. Claim 2 implies that for H^{n-p} (or ν) a.e. $a \in \Sigma$, $0 < \Theta^{n-p}(\nu, a) < \infty$. Hence we can apply Theorem 4.1 to ν and Σ to conclude that $n-p$ is integer. But this contradicts with the fact that $p \in (1, n)$ is not an integer. Hence the conclusion of theorem A is true. ■

Proof of Theorem C

Here we follow [Lf] [LW] closely. We will show that if $H^{n-p}(\Sigma) > 0$ then we can blow up u_{ϵ_i} to get a non-constant p-harmonic map from S^p to N. Denote $\bar\Sigma = \{a \in \Sigma : \Theta^{n-p}(\nu, a) = \Theta^{n-p}(\mu, a)\}$. Then we know $H^{n-p}(\bar\Sigma) = H^{n-p}(\Sigma) > 0$. For $\bar\Sigma$, we need

Lemma 4.3 (The geometric Lemma 2.4 of [Lf]) *There exists $E \subset \bar\Sigma$, with $H^{n-p}(E) > 0$, such that for any $x \in E$ and $r_i \downarrow 0$ there are $\{x_i^j\}_{j=1}^{n-p} \subset \bar\Sigma \cap B_{r_i}(x)$ satisfying*

$$|x_i^j - x_i^k| \geq \delta r_i, \;\; 0 \leq j < k \leq n-p, \tag{4.6}$$

$$d(x_i^j - x_i^0, span\{x_i^1 - x_i^0, \cdots, x_i^{j-1} - x_i^0\}) \geq \delta r_i, \quad 2 \leq j \leq n-p, \tag{4.7}$$

for some uniform $\delta > 0$, here $x_i^0 = x$.

Now we pick up a $x_0 \in E$, with

$$\limsup_{r\downarrow 0} r^{p-n} H^{n-p}(\bar\Sigma \cap B_r(x_0)) > 0, \;\; \lim_{r\downarrow 0} r^{p-n} \int_{B_r(x_0)} |Du|^p = 0,$$

and $\Theta^{n-p}(\mu, \cdot)$ is H^{n-p}-approximately continuous at x_0. Define the rescaling measures μ_i by

$$\mu_i(A) = r_i^{p-n}\mu(x_0 + r_i A), \; \forall \text{ Borel } A \subset R^m.$$

Then we have

$$2^{n-p}\epsilon_0^p \le \mu_i(B_2) \le C(x_0, \Sigma) < \infty$$

Hence we can assume that $\mu_i \to \mu_*$ for some nonnegative Radon measure μ_*. By the diagonal process, we can also assume that, after passing to subsequences,

$$e_{\epsilon_i}(u_{\epsilon_i})\,dx \to \mu_*,$$

and $u_{\epsilon_i} \to c_0$ weakly in $W^{1,p}$ for some constant $c_0 \in N$. Let $\Sigma_* = \operatorname{spt}(\mu_*)$. Then we have

$$H^{n-p}(\Sigma_*) > 0,\ \Theta^{n-p}(\mu_*, x) = \Theta^{n-p}(\mu, x_0) \text{ for any } x \in \Sigma_*.$$

Moreover, by the choice of x_0 and Lemma 4.3, we know that for $\delta > 0$ given by Lemma 4.3, there exist $(n-p)$ points $\{\xi^j\}_{j=1}^{n-p} \subset \Sigma_* \setminus B_\delta$ such that

$$|\xi^j - \xi^k| \ge \delta, \quad 1 \le j \le k \le n-p,$$

$$d(\xi^j, \operatorname{span}(\xi^1, \cdots, \xi^{j-1})) \ge \delta,\ 2 \le j \le n-p.$$

It is clear that $\{\xi^j\}_{j=1}^{n-p}$ spans a $(n-p)$-dimensional subspace in R^n, denoted as R^{n-p}. Write $R^n = R^p \times R^{n-p}$. Moreover, repeating (2.1) around $0, \xi^1, \cdots, \xi^{n-p}$, we know that there exists $\xi_0 \in \Sigma_*$ such that

$$\int_{B_{\frac{\delta}{4}}(\xi_0)} |Du_{\epsilon_i}|^{p-2}|D_T u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^p}F(u_{\epsilon_i}) \to 0,\ \text{as } i \to \infty,$$

here T denote vectors in $R^{n-p} \equiv \{(0,, \cdots, 0, x_{p+1}, \cdots, x_n)\}$. In particular,

$$\int_{B^p_{\frac{\delta}{4}}(0)\times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \sum_{j=p+1}^{n} |Du_{\epsilon_i}|^{p-2}|\frac{\partial u_{\epsilon_i}}{\partial x_j}|^2 + \frac{1}{\epsilon_i^p}F(u_{\epsilon_i}) \to 0. \tag{4.8}$$

Therefore, by the weak L^1 estimates of the Hardy-Littlewood maximal functions, there exists a $A_i \subset B^{n-p}_{\frac{\delta}{4}}(\xi_0)$, with $H^{n-p}(A_i) > 0$, such that for $p_i \in A_i$,

$$\sup_{0<r<\frac{\delta}{4}} r^{p-n} \int_{B_r^{n-p}(p_i)} f_i \to 0, \quad \text{as } i \to \infty. \tag{4.9}$$

Here

$$f_i = \int_{B^p_{\frac{\delta}{4}}(0)} \sum_{j=p+1}^{n} |Du_{\epsilon_i}|^{p-2}|\frac{\partial u_{\epsilon_i}}{\partial x_j}|^2 + \frac{1}{\epsilon_i^p}F(u_{\epsilon_i}).$$

Now we have

Claim 4. $\mu_*(x_1, \cdots, x_p, x_{p+1}, \cdots, x_n) = \Theta^{n-p}(\mu, x_0) H^{n-p} L(\{0\} \times R^{n-p})$.

To see this, let $\phi \in C_0^\infty(B^p_{\frac{\delta}{4}}(0))$ and $\eta \in C_0^\infty(B^{n-p}_{\frac{\delta}{4}}(\xi_0))$. We have, for $p+1 \le j \le n$,

$$
\begin{aligned}
&\int_{B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \Big(\int_{B^p_{\frac{\delta}{4}}(0)} e_{\epsilon_i}(u_{\epsilon_i}) \phi^2 \Big) \frac{\partial \eta}{\partial x_j} \\
&= - \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \Big(|Du_{\epsilon_i}|^{p-2} Du_{\epsilon_i} \cdot D \frac{\partial u_{\epsilon_i}}{\partial x_j} \\
&\qquad - \frac{1}{\epsilon_i^p} f(u_{\epsilon_i}) \Big) \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2 \eta \\
&= \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \operatorname{div}(|Du_{\epsilon_i}|^{p-2} Du_{\epsilon_i} \phi^2 \eta) \frac{\partial u_{\epsilon_i}}{\partial x_j} \\
&\qquad + \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \frac{1}{\epsilon_i^p} f(u_{\epsilon_i}) \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2 \eta \\
&= \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} \Big(\operatorname{div}(|Du_{\epsilon_i}|^{p-2} Du_{\epsilon_i}) + \frac{1}{\epsilon_i^p} f(u_{\epsilon_i}) \Big) \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2 \eta \\
&\qquad + \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} |Du_{\epsilon_i}|^{p-2} Du_{\epsilon_i} \cdot D(\phi^2 \eta) \frac{\partial u_{\epsilon_i}}{\partial x_j} \\
&= 2 \sum_{1 \le l \le p} \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_l} \frac{\partial \phi}{\partial x_l} \phi \eta \frac{\partial u_{\epsilon_i}}{\partial x_j} \\
&\qquad + \sum_{p+1 \le k \le n} \int_{B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)} |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_k} \frac{\partial \eta}{\partial x_k} \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2 \\
&= I + II
\end{aligned}
$$

Here we have used the smoothness Lemma (see [Tp] [Mc]) that guarantees $|Du_\epsilon|^{p-2} Du_\epsilon \in W^{1,2}(\Omega, R^k)$ so that we can use the integrations by parts in the above derivation. This implies, for $p+1 \le j \le n$,

$$
\begin{aligned}
\frac{\partial}{\partial x_j} \Big(\int_{B^p_{\frac{\delta}{4}}(0)} e_{\epsilon_i}(u_{\epsilon_i}) \phi^2 \Big) &= -2 \sum_{1 \le l \le p} \int_{B^p_{\frac{\delta}{4}}(0)} |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_l} \frac{\partial \phi}{\partial x_l} \phi \frac{\partial u_{\epsilon_i}}{\partial x_j} \\
&\quad + \sum_{p+1 \le k \le n} \frac{\partial}{\partial x_k} \int_{B^p_{\frac{\delta}{4}}(0)} |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_k} \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2,
\end{aligned} \tag{4.10}
$$

in the sense of distribution.

By using the bound of $I^p_{\epsilon_i}(u_{\epsilon_i})$ and (4.8), we can see that $I, II \to 0$ as $i \to \infty$. This implies that for $p+1 \le j \le n$,

$$\frac{\partial}{\partial x_j} \int_{B^p_{\frac{\delta}{4}}(0)} e_{\epsilon_i}(u_{\epsilon_i})\phi^2 \to 0, \text{ in } \mathcal{D}'(B^{n-p}_{\frac{\delta}{4}}(\xi_0)). \tag{4.9}$$

Hence $\mu_*(x) = g(x_1, \cdots, x_p)\, dx_{p+1} \cdots dx_n$ for some nonnegative Radon measure g in R^p. Moreover, we know that $\mathrm{spt}(g)$ consists of at most finite many points, and for $(x_1, \cdots, x_p) \in \mathrm{spt}(g)$, we have $g(x_1, \cdots, x_p) = \Theta^{n-p}(\mu, x_0)$. Therefore, we obtain the claim.

Claim 4 implies that

$$e_{\epsilon_i}(u_{\epsilon_i})\, dx \to 0, \text{ locally in } (B^p_{\frac{\delta}{4}}(0) \times B^{n-p}_{\frac{\delta}{4}}(\xi_0)) \setminus (\{0\} \times R^{n-p}).$$

Now we begin the blowing-up process as follows. Choose $p_i \in A_i$, with $|p_i - \xi_0| \le \frac{\delta}{8}$. There exist $x_i \in B^p_{\frac{\delta}{4}}(0)$ and $\delta_i \downarrow 0$ such that

$$\begin{aligned} \int_{B^p_{\delta_i}(x_i)} e_{\epsilon_i}(u_{\epsilon_i})(x, p_i)\, dx &= \max\{\int_{B^p_{\delta_i}(a)} e_{\epsilon_i}(u_{\epsilon_i})(x, p_i)\, dx \mid a \in B^p_{\frac{\delta}{4}}(0)\} \\ &= \frac{\epsilon_0^p}{C(n)} \end{aligned} \tag{4.11}$$

for some large $C(n)$ to be chosen later. Define the maps $v_i(x) = u_{\epsilon_i}((x_i, p_i) + \delta_i x)$. Then

$$\mathrm{div}(|Dv_i|^{p-2} Dv_i) + \frac{1}{\bar{\epsilon}_i^p} f(v_i) = 0, \quad \text{in } B^n_{\frac{\delta}{8\delta_i}}(0), \tag{4.12}$$

here $\bar{\epsilon}_i = \frac{\epsilon_i}{\delta_i} \to 0$.

$$\sup_{0<r<\frac{\delta}{8\delta_i}} r^{p-n} \int_{B^{n-p}_r(0)} \int_{B^p_{\frac{\delta}{8\delta_i}}(0)} \sum_{j=p+1}^{n} |Dv_i|^{p-2} |\frac{\partial v_i}{\partial x_j}|^2 + \frac{1}{\bar{\epsilon}_i^p} F(v_i) \to 0. \tag{4.13}$$

$$\begin{aligned} \int_{B^p_1(0)} \frac{1}{p}|Dv_i|^p + \frac{1}{\bar{\epsilon}_i} F(v_i)(x,0)\, dx &= \frac{\epsilon_0^p}{C(n)} \\ &\ge \int_{B^p_1(a)} \frac{1}{p}|Dv_i|^p + \frac{1}{\bar{\epsilon}_i} F(v_i)(x,0)\, dx, \end{aligned} \tag{4.14}$$

for all $a \in \Omega_i$, here $\Omega_i = \delta_i^{-1}(B^p_{\frac{\delta}{4}}(0) \setminus \{x_i\})$. Moreover, we claim that if we choose $C(n)$ sufficiently large, then we have

$$2^{p-n} \int_{B^{n-p}_2(0) \times B^p_2(a)} \frac{1}{p}|Dv_i|^p + \frac{1}{\bar{\epsilon}_i} F(v_i) \le \epsilon_0^p, \ \forall a \in \Omega_i. \tag{4.15}$$

To see (4.15), we need to recall the strong constancy Lemma of Allard ([Aw1] page 3-5).

Lemma 4.4. *For $a \in R^m$ and $0 < r < \infty$. Let $u \in L^1(B_r(a))$ be nonnegative and $\int_{B_r(a)} |u| \leq Mr^m$. Let $f_j, X_j^i \in L^1(B_r(a))$ for $1 \leq ij \leq m$, and $0 \leq \delta < \infty$ be such that*

$$\int_{B_r(a)} |X_j^i| + r|f_j| \leq \delta r^m$$

Assume that $D_j u = div(X_j) + f_j$ for $1 \leq j \leq m$, here $X_j = (X_j^1, \cdots, X_j^m)$. Then there exists a $\delta_1 = \delta_1(M, \epsilon) > 0$ such that if $\delta \leq \delta_1$ then there exists a $0 \leq c < \infty$ such that

$$r^{-m} | \int_{B_r(a)} u\phi - c \int_{B_r(a)} \phi | \leq \epsilon \sup_{B_r(a)} |\phi|, \ \forall \phi \in C^1(B_{\frac{r}{2}}(a)).$$

Applying Lemma 4.4 to our setting with $a = p_i$, $r \in (\frac{\delta}{8}, \frac{\delta}{4})$, m replaced by $n - p$, $u = \int_{B^p_{\delta_i}(x_i)} e_{\epsilon_i}(u_{\epsilon_i})\phi^2 \in L^1(B_r^{n-p}(a))$,

$$f_l = -2 \int_{B^p_{\frac{\delta}{4}}(x_i)} \sum_{k=1}^p |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_k} \frac{\partial u_{\epsilon_i}}{\partial x_l} \frac{\partial \phi}{\partial x_k} \phi,$$

$X_l^j = \int_{B^p_{\delta_i}(x_i)} |Du_{\epsilon_i}|^{p-2} \frac{\partial u_{\epsilon_i}}{\partial x_l} \frac{\partial u_{\epsilon_i}}{\partial x_j} \phi^2$ for $p + 1 \leq l, j \leq n$. Then one can check all conditions of Lemma 4.4 are satisfied. Hence for any small $\beta > 0$ there exists $0 \leq c < \infty$ such that for $i >> 1$,

$$\begin{aligned} & r^{p-n} | \int_{B_r^{n-p}(p_i) \times B^p_{\frac{\delta}{4}}(x_i)} \eta \phi^2 e_{\epsilon_i}(u_{\epsilon_i}) - c \int_{B_r^{n-p}(p_i)} \eta | \\ & \leq \beta \|\eta\|_{L^\infty(B_r^{n-p}(x_i))}, \end{aligned} \tag{4.16}$$

for all $\frac{\delta}{8} < r < \frac{\delta}{4}$ and $\eta \in C_0^\infty(B^{n-p}_{\frac{r}{2}}(x_i))$. Taking (4.14) into account, we see that $c \leq \frac{2\epsilon_0^p}{C(n)}$. Hence, if we choose $\delta > 0$ sufficiently small and $C(n)$ sufficiently large, then we have

$$(2\delta_i)^{p-n} \int_{B^{n-p}_{\delta_i}(p_i) \times B^p_{\delta_i}(a)} e_{\epsilon_i}(u_{\epsilon_i}) \leq \frac{\epsilon_0^p}{2}, \ \forall a \in \Omega_i. \tag{4.17}$$

Rescaling (4.17), we obtain (4.15). Applying Lemma 2.3, we now conclude that

$$v_i \to v, \ \text{ locally in } C^1(R^p \times B_1^{n-p}(0), R^k).$$

Moreover, (4.13) implies $v(x_1, \cdots, x_p, x_{p+1}, \cdots, x_n) = v(x_1, \cdots, x_p)$. Note that $\frac{\epsilon_i}{\delta_i} \to 0$. Hence $v : R^p \to N$ is a C^1 p-harmonic map. (4.14) implies that

$$0 < \int_{R^p} |Dv|^p < \infty.$$

Therefore, using the conformal invariance of p-energy in R^p, the conformal equivalence of $S^p \setminus \{\text{north pole}\}$ and R^p, and the removablity result of isolated singularity for p-harmonic map on S^p (due to [SaU] for $p = 2$, and [MY] for $p \geq 3$), v yields a non-constant p-harmonic S^p in N. But this contradicts with the assumption on N. The proof of Theorem C is complete. ■

Now we give proofs to the two corollaries.

Proof of Corollary B

¿From Theorem A, we know that $u \in C^{1,\alpha}(\Omega \setminus \Sigma, N)$ for some $\alpha \in (0,1)$, here Σ is defined at the front of this section and $H^{m-p}(\Sigma) = 0$. Moreover, the strong convergence from u_{ϵ_i} to u guarantees that u is a stationary p-harmonic map. To prove that Σ is of Hausdorff dimension at most $n - [p] - 1$, we need to apply Federer dimension reduction argument (see, e.g. [SU] [HL]). Note that the key point to apply the Federer dimension reduction argument is to show that for any $a \in \Sigma$ and $r_i \to 0$, there exists a stationary p-harmonic map ϕ such that $v_i(x) = u(a + r_i x) \to \phi$ strongly in $W^{1,p}(B_1, N)$ (after passing to subsequences). Note that, by (1.5), we know that v_i is bounded in $W^{1,p}(B_1, N)$ so that there exists a p-harmonic map $\phi \in W^{1,p}(B_1, N)$ such that $v_i \to \phi$ weakly in $W^{1,p}(B_1, N)$. To see the strong convergence, we note that for u_{ϵ_i} given by theorem A and for each $i > 0$, the same proof as theorem A implies that $u_{\epsilon_j}(a + r_i \cdot) \to v_i$, as $j \to \infty$, strongly in $W^{1,p}(B_1, R^k)$. Therefore, by the standard diagonal process, we know that there exist subsequences $j(i) \to \infty$ such that $u_{\epsilon_{j(i)}} \to \phi$ strongly in $W^{1,p}(B_1, R^k)$. In particular, $v_{j(i)} \to \phi$ strongly in $W^{1,p}(B_1, N)$. Now, we can follow exactly the argument of [SU] or [HL] to show that Σ is of Hausdorff dimension $n - [p] - 1$. Furthermore, if N doesn't support any p-harmonic S^l for $[p] \leq l \leq n-1$, then $\Sigma = \emptyset$. Hence $u_{\epsilon_i} \to u$ in $C^1(\Omega, R^k)$.
■

Proof of Corollary D

Since N doesn't support any p-harmonic S^p, theorem C implies that $u_{\epsilon_i} \to u$ strongly in $W^{1,p}(\Omega, R^k)$ and u is a stationary p-harmonic map. The same argument as in the proof of corollary B implies that Σ is at most of Hausdorff dimension $n - p - 1$. Moreover, applying again the assumption that N doesn't have any p-harmonic S^p, we have Σ is at most of Hausdorff dimension $n-p-2$. The nonexistence of any p-harmonic S^l for $p+1 \le n-1$ then yields $\Sigma = \emptyset$. Hence the rest of corollary D follows. ■

§5. Generalized Varifolds and the Proof of Theorem F

In this section, we assume that the readers are familiar with the theory of classical varifolds by Allard [Aw] and Almgren [Af](see also Simon [Sl] for details). We recall the basic definition and properties of generalized varifolds (see also Almgren [Af], Ambrosio-Soner [AS], or Lin [Lf2] Lin-Wang [LW2]) and then apply the generalized varifold to prove the theorem F.

Definition 5.1. For $1 \le l \le n$ and a bounded C^1 domain $\Omega \subset R^n$. A l-dimensional generalized varifold V is a nonnegative Radon measure on $\Omega \times A_l$, where

$$A_l = \{A \in R^{n\times n} : A \text{ is symmetric, trace}(A) = l, -lI_n \le A \le I_n\},$$

here I_n denotes the identity matrix of order n. The class of all generalized l-varifolds is denoted by $V_l^*(\Omega)$. For $V \in V_l^*(\Omega)$, we let $\|V\|$ denote the weight of V, which is just $\pi_\#(V)$, here $\pi(x,A) = x : \Omega \times A_l \to \Omega$. Since $G_l(n)$, the standard Grassmann manifolds of l-dimensional unoriented planes in R^n, is contained in A_l, we have that $V_l(\Omega) \subset V_l^*(\Omega)$, here $V_l(\Omega)$ denotes the class of l-dimensional classical variflods (see [Sl] for the definition).

Definition 5.2. For any given $V \in V_l^*(\Omega)$, the first variation of V, δV, is a distribution on $C_0^1(\Omega, R^n)$ defined by

$$\delta V(X) = -\int_{\Omega\times A_l} DX(x) : A\, dV(x,A),\ \forall X \in C_0^1(\Omega, R^n). \tag{5.1}$$

Here $A : B = \sum_{ij} A_{ij}B_{ij}$ denotes the scalar product of $A, B \in R^{n\times n}$. V is stationary if $\delta V = 0$. $V \in V_l^*(\Omega)$ is said to have its first variation as a Radon measure, if

$$\begin{aligned}\|\delta V\|(G) &= \sup\{|\delta V(X)| : X \in C_0^1(\Omega, R^n), \|X\|_{L^\infty} \le 1, \mathrm{spt}(X) \subset G\} \\ &\le C(G) < \infty,\ \forall G \subset\subset \Omega.\end{aligned} \tag{5.2}$$

Note also that the convergence of $V \in V_l^*(\Omega)$ is understood as weak convergence of Radon measures on $\Omega \times A_l$. Moreover, if $V_i \to V$, then $\delta V_i \to \delta V_i$ as convergence of distributions. Furthermore, if $\sup_i \|\delta V_i\|(A) < \infty$ for $A \subset \Omega$, then

$$\|\delta V\|(A) \le \liminf_{i\to\infty} \|\delta V_i\|(A) < \infty. \tag{5.3}$$

Recall also that a set $E \subset R^n$ is called l-rectifiable if except a H^l measure zero subset E can be covered by countably many l-dimen sional C^1 submanifolds in R^n. A $V \in V_l^*(\Omega)$ is said to be a l-rectifiable varifold if there exist a l-rectifiable set $E \subset \Omega$ and a locally H^l integrable and positive function θ such that $V = \delta_{T_xE}\theta H^l \mathrm{L} E$, here T_xE denotes the tangent plane of E at x and δ_{T_xE} denotes the Dirac mass at T_xE.

Now we apply the generalized varifold concept to study the limiting behaviors of solutions to (1.2) for integer power p's.

For $p = 2, \cdots, n-1, n$, $\epsilon > 0$, and any map $u_\epsilon \in W^{1,p}(\Omega, R^k)$ solving (1.2), we associate a generalized $(n-p)$-varifold $V_{u_\epsilon} \in V_{n-p}^*(\Omega)$ as follows. $V_{u_\epsilon}(x) = \delta_{A(u_\epsilon)(x)} e_\epsilon(u_\epsilon)(x)\,dx$, where

$$
\begin{aligned}
A(u_\epsilon)(x) &= I_n - p\frac{Du_\epsilon \otimes Du_\epsilon}{|Du_\epsilon|^2}(x), \quad \text{if } |Du|(x) \neq 0, \\
&= I_{n-p}, \qquad\qquad\qquad\qquad \text{if } |Du_\epsilon|(x) = 0.
\end{aligned}
\tag{5.4}
$$

It is clear that $A(u_\epsilon) \in A_{n-p}$ so that $V_{u_\epsilon} \in V_{n-p}^*(\Omega)$. Moreover,

$$
\|V_{u_\epsilon}\|(B) = \int_B e_\epsilon(u_\epsilon)(x)\,dx, \forall B \subset \Omega,
$$

and, for $X \in C_0^1(\Omega, R^n)$,

$$
\begin{aligned}
\delta V_{u_\epsilon}(X) = &-\int_\Omega ((\frac{1}{p}|Du_\epsilon|^p + \frac{1}{\epsilon^p}F(u_\epsilon))\mathrm{div}(X) - p\sum_{ij}|Du_\epsilon|^{p-2}u_{\epsilon,i}u_{\epsilon,j}X_j^i) \\
&+ p\int_\Omega DX(x) : \frac{Du_\epsilon \otimes Du_\epsilon}{|Du_\epsilon|^2}\frac{1}{\epsilon^p}F(u_\epsilon)(x)\,dx.
\end{aligned}
\tag{5.5}
$$

Lemma 5.1. *Under the same notations as above. We have*

$$
\delta V_{u_{\epsilon_i}}(X) = p\sum_{jl}\int_\Omega \frac{u_{\epsilon,j}u_{\epsilon,l}X_j^l}{|Du_\epsilon|^2}\frac{F(u_\epsilon)}{\epsilon^p}.
\tag{5.6}
$$

In particular

$$
|\delta V_{u_\epsilon}(X)| \leq p\int_\Omega |DX|\frac{F(u_\epsilon)}{\epsilon^p}.
\tag{5.7}
$$

Proof. (5.6) follows from (5.5) and th following Pohozaev identity:

$$
\int_\Omega (\frac{1}{p}|Du_\epsilon|^p + \frac{1}{\epsilon^p}F(u_\epsilon))\mathrm{div}(X) = \sum_{jl}\int_\Omega |Du_\epsilon|^{p-2}u_{\epsilon,j}u_{\epsilon,l}X_j^l,
\tag{5.8}
$$

for any $X \in C_0^1(\Omega, R^n)$. (5.7) follows easily from (5.6). ■

For $\epsilon_i \to 0$, we let $u_i \equiv u_{\epsilon_i} \in W^{1,p}(\Omega, R^k)$ solve (1.2) and assume that $u_i \to u$ weakly in $W^{1,p}(\Omega, R^k)$. Then we know that $\|V_{u_i}\|(\Omega)$ is bounded. Hence, we may assume that there exist a nonnegative Radon measure ν on Ω and a generalized $(n-p)$-varifold $V \in V^*_{n-p}(\Omega)$ such that

$$e_{\epsilon_i}(u_i)(x)\, dx \to \frac{1}{p}|Du|^p(x)\, dx + \nu \equiv \mu,$$

and

$$V_{u_i} \to V.$$

Moreover, $\|V\| = \mu$, Proposition 4.2 and Lemma 5.1 imply that $\delta V = 0$ (i.e., V is stationary). It also follows from the monotonicity inequality (2.1) that

$$r^{p-n}\|V\|(B_r(x)) \le R^{p-n}\|V\|(B_R(x)), \tag{5.9}$$

for all $x \in \Omega$ and $0 < r \le R < \mathrm{dist}(x, \partial\Omega)$. In particular,

$$\Theta^{n-p}(\|V\|, x) = \lim_{r \downarrow 0} r^{p-n}\|V\|(B_r(x))$$

exists for all $x \in \Omega$ and is upper semicontinuous. Furthermore, it is easy to check that the concentration set Σ defined by §4 is

$$\Sigma = \{x \in \Omega : \epsilon_0^p \le \Theta^{n-p}(\|V\|, x) < \infty\}$$

and $u_i \to u$ in $C^1_{\mathrm{loc}}(\Omega \setminus \Sigma, R^k)$. In fact, Lemma 2.3 also implies

$$\Sigma = \{x \in \Omega : 0 < \Theta^{n-p}(\|V\|, x) < \infty\}$$

Now we can extend the proof of Allard's rectifiability theorem (see [Aw]) to show that

Lemma 5.2. *Under the same notations as above. We have that $V \llcorner \Sigma$ is a $(n-p)$-rectifable varifold. In particular, Σ is a $(n-p)$-rectifiable set.*

Proof. This is theorem 4.9 of Lin-Wang [2]. For the convenience of readers, we sketch it here. It suffices to prove that for H^{n-p} a.e. $a \in \Sigma$ the tangent measure of V is flat and unique. More precisely, we need to show that for H^{n-p} a.e. $x_0 \in \Sigma$ there exists a $(n-p)$-dimensional plane $T_0 \in G_{n-p}(n)$ and a positive constant θ_0 such that

$$\mathcal{D}_{x_0,r}(V) \to \delta_{T_0}\theta_0 H^{n-p} \llcorner T_0, \text{ as } r \to 0. \tag{5.10}$$

Here $\mathcal{D}_{x_0,r}(V)$ denotes the rescaling of V with center at x_0 and scale r.

To prove (5.10), first observe that we can write $V = V_x\|V\|$, where V_x is a measurable function from Ω to the space of probability measures on A_{n-p}. Hence, for H^{n-p} a.e. $x_0 \in \Sigma$,

$$\Theta^{n-p}(\|V\|,\cdot) \text{ and } V_x \text{ is } H^{n-p} \text{ approximately continuous at } x_0,$$

and

$$\Theta^{n-p,*}(\Sigma, x_0) = \limsup_{r\downarrow 0} r^{p-n} H^{n-p}(\Sigma \cap B_r(x_0)) \geq 2^{p-n}.$$

Hence, by Lemma 4.3, we can assure that for any $r \to 0$ there exists a subsequence $r_i \to 0$ and a $(n-p)$-dimensional plane T, which may depend on subsequences, such that

$$\mathcal{D}_{x_0,r_i}(V) \to V_{x_0}\Theta^{n-p}(\|V\|, x_0)H^{n-p}\llcorner T.$$

On the other hand, since V is stationary, $\mathcal{D}_{x_0,r_i}(V)$ is also stationary, therefore we have

$$\delta(V_{x_0}H^{n-p}\llcorner T) = 0,$$

so that the constancy theorem for varifolds (see, Simon [Sl]) implies that $V_{x_0} = \delta_T$. This implies that T is unique and independent of the choice of subsequences. This gives (5.10). Therefore, $V\llcorner\Sigma$ is $(n-p)$-rectifiable and Σ is a $(n-p)$-rectifiable set. ∎

Completion of Proof of Theorem F

It follows from Lemma 5.2 and (4.4) that

$$V\llcorner\Sigma = \delta_{T_x\Sigma}\Theta^{n-p}(\nu, x)H^{n-p}\llcorner\Sigma.$$

This implies (1.10). On the other hand, since $u_i \to u$ in $C^1_{\text{loc}}(\Omega\setminus\Sigma, R^k)$, we have $V\llcorner(\Omega\setminus\Sigma) = V_u$. It is easy to see now that (1.11) follows from (1.10), the stationarity of V, and the formula of V. ∎

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