3D Eikonal Solvers, Part II: Anisotropic Traveltimes

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Abstract

In anisotropic media the direction of energy propagation is not in general tangent to the wavefront normal. On the other hand, finite difference eikonal solvers compute the solution based on the traveltime gradient and the wavefront normal. Local convexity of the wavefronts in transverse isotropic (TI) media is studied for the eikonal solver to determine the correct upwind direction of the energy propagation.

A second-order essentially non-oscillatory (ENO) scheme is applied to the computation of traveltimes in TI media. To overcome instability inherited in the standard expanding-box schemes, the ENO scheme is implemented incorporating a down-n-out marching and a post sweeping iteration. Various numerical techniques such as the maximum angle condition, the average normal velocity, and a cache-based implementation are adopted to further improve accuracy and efficiency of the algorithm. It is numerically verified that the resulting algorithm is stable, second-order accurate, and efficient. The algorithm has been successfully tested for the computation of traveltimes in heterogeneous synthetic TI models having large contrasts.

1. Introduction

Asymptotic ray theory for general anisotropic media has been studied by Cerveny [2], Daley and Hron [6], and Musgrave [14] among others. These results have shown that substitution of the assumed solution into the equations of particle motion yields an eigenvalue problem. The resulting eigenvalues are the normal velocities of three different types of wavefronts, namely, a compressional (quasi-\( P \)) wavefront, a shear vertical (quasi-SV) wavefront, and a shear horizontal (quasi-SH) wavefront. The corresponding eigenvectors give the direction of the displacement vectors associated with each of the three wavefront types.

In an isotropic medium, the direction of the compressional \( P \) eigenvector is tangent to the wavefront normal and the shear SV and SH eigenvectors are respectively

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normal and binormal to the $P$ eigenvector. But in anisotropic media, the $P$ eigenvector is not in general tangent to the wavefront normal, hence the prefix “quasi”. The direction of wave propagation therefore may differ from the wavefront normal.

The simplest anisotropy of broad geophysical applicability is transverse isotropy (TI) for which materials have the same property value when measured in a plane but a different value when measured perpendicular to the plane. The symmetry axis of the TI medium is a line normal to the plane.

Seismic techniques incorporating high frequency asymptotic representation of the 3D elastic Green’s function require efficient solution methods for the computation of traveltimes. The use of ray tracing followed by interpolation of traveltimes is a popular and robust method for computing diffraction trajectories for most of small or moderate velocity contrasts. For a comprehensive treatment of ray theory, see e.g. [3, 4, 5, 9]. An alternative to the method is to compute traveltimes by solving directly the eikonal equation on a regular grid by finite difference (FD) schemes; see [7, 8, 10, 11, 13, 19, 20, 22, 23, 24].

For eikonal solvers, upwind schemes are requisite to sharply resolve discontinuities in the travertime derivatives, while central differences improve the accuracy of the computed traveltime. A second-order essentially non-oscillatory (ENO) scheme [15, 16] implemented with the expanding-box strategy satisfies these requirements. However, the expanding-box scheme may be unstable for turning rays. Recently, Kim and Cook [13] suggested a second-order ENO eikonal solver for isotropic traveltimes of which stability is enforced by a dynamic down-n-out (DNO) marching and a post sweeping (PS) iteration. Local mesh refinement is adopted to overcome the accuracy degradation at singularities such as source points. The resulting algorithm, called ENO-DNO-PS, has been successfully tested for the first-arrival traveltimes in synthetic and real velocity models. In [11], I introduced strategies such as the maximum angle condition, the average normal velocity, and a cache-based implementation for ENO-DNO-PS to be able to further improve accuracy and efficiency.

This article is the second part of the series: 3D eikonal solvers, following [11] and to be followed by [12]. In this paper we apply ENO-DNO-PS for the traveltimes computation in TI media of either vertical or generally inclined symmetry axes. Because the direction of energy propagation in anisotropic media is not in general tangent to the wavefront normal, and because FD eikonal solvers compute the solution based on the traveltimes gradients and wavefront normal, a special care is required in determining the upwind direction. Local convexity of the wavefronts in TI media is adopted to determine the correct upwind direction of the energy propagation.

The paper is organized as follows. The next section reviews accuracy-efficiency techniques such as the maximum angle condition, the average normal velocity, and a cache-based implementation. The second-order ENO schemes for the TI traveltimes
are discussed in the following section. Then, numerical experiments are presented to show accuracy and efficiency of the anisotropic ENO-DNO-PS. Conclusions are made in the last section. In Appendix Appendix A, the TI eikonal equations are revised to better fit with the FD eikonal solver. Appendix Appendix B includes the local convexity of TI wavefronts.

2. Accuracy-efficiency techniques

In this section, we briefly review the maximum angle condition, the average normal slowness, and a cache-based implementation studied in [11].

**Maximum angle condition:** The TI eikonal equation (A.3) in evolutionary form in depth reads

\[ \tau_z = \sqrt{s^2 - \tau_x^2 - \tau_y^2}, \tag{2.1} \]

where \( s := 1/v \) is the slowness. Here the subscript \( K \) appeared in (A.3) is dropped for a simple presentation. To accurately compute the traveltimes along rays whose velocity vectors make angle with the vertical not larger than a prescribed angle \( \theta_{\text{max}} \), \( 0 < \theta_{\text{max}} < \pi/2 \), the maximum angle condition (MAC) can be imposed as follows:

\[ \tau_z = \sqrt{\max(s^2 \cos^2 \theta_{\text{max}}, s^2 - \tau_x^2 - \tau_y^2)}. \tag{2.2} \]

MAC was first introduced by Gray and May [10] in a slightly different form; a smooth function in the place of “max” was incorporated to smoothly approximate traveltimes of rays propagating near or larger than the maximum angle. MAC as given in (2.2) fits well with ENO-DNO-PS, because the PS iteration applies other directional marchings to take care of rays having large angles with the vertical.

**Average normal slowness:** The eikonal solver can improve its accuracy by incorporating the average normal slowness which is obtained by integrating the slowness along the wavefront normal direction. However, in realistic media, it is difficult to accurately compute the average normal slowness without raypath information. A reasonable approximation can be obtained as follows. In a \((z+)\)-directional marching, let the traveltimes on the \((k+1)\)-th level be to be computed from those on the \(k\)-th level. Then, an approximate average slowness can be formulated as

\[ \hat{s}_{i,j}^k = \frac{1}{\Delta z} \int_{x_{i,j}^k}^{x_{i,j}^{k+1}} \frac{1}{v(x)} \, d\sigma = \begin{cases} \frac{s_{i,j}^k}{\Delta z}, & \text{if } v_{i,j}^{k+1} = v_{i,j}^k, \\ \log \left( \frac{v_{i,j}^{k+1}/v_{i,j}^k}{v_{i,j}^{k+1} - v_{i,j}^k} \right), & \text{else}, \end{cases} \tag{2.3} \]

where the velocity has been provided at grid points. For other directional marchings, one can easily obtain the corresponding formulae.

It has been observed that the practical accuracy of the computed traveltime depends strongly on the ability of the numerical code to compute an average slowness
close to the average normal slowness; no matter how accurate the basic numerical schemes are, the solution incorporating the point slowness would be first-order accurate in general media.

Remark. As an alternative to (2.3), one may consider the following. Let $\phi$ and $\psi$ be the ray angles from the vertical in the $x$- and $y$-directions, respectively. Then they can be approximated by

$$
\phi = \tan^{-1}\left(\frac{D_x\tau}{D_z\tau}\right), \quad \psi = \tan^{-1}\left(\frac{D_y\tau}{D_z\tau}\right),
$$

where $(D_x\tau, D_y\tau, D_z\tau)$ denotes an appropriate FD scheme for $\nabla \tau$ at the point $x^k_{i,j}$. However, the corresponding average velocity hardly improves the traveltime accuracy over (2.3), seemingly due to the limitation that $D_z\tau$ should be computed by one-sided FD schemes utilizing traveltimes on the $k$-th and earlier levels.

Cache-based implementation: In traveltime computation, the array allocated for storing the traveltime is often too large to reside in the cache. Since the traveltimes on only a plane are accessed (a few times) in a directional marching of box-expanding FD eikonal solvers, one can copy the data on the plane to a temporary buffer, which is often small enough to be in the cache. Such cache-based implementation can speed up the traveltime computation, because the processor can access the data on the cache immediately instead of searching over the main memory.

3. The TI eikonal solver

The TI eikonal equations of the vertical or generally inclined symmetry axes are revised in Appendix A to better fit with FD schemes. The quasi-SH eikonal equation is a quadratic polynomial of each component of the traveltime gradient, while quasi-$P$ and quasi-$SV$ eikonal equations are quartic polynomials of the form (e.g., for the down-going wavefronts)

$$
a_4\tau_{K,z}^4 + a_3\tau_{K,z}^3 + a_2\tau_{K,z}^2 + a_1\tau_{K,z} + a_0 = 0, \\
K = P, \quad SV,
$$

where $a_4, \ell = 0, \ldots, 4$, are again quartic polynomials of $\tau_{K,x}$ and $\tau_{K,y}$ and they are also functions of elastic parameters.

Appendix B discusses local convexity of TI wavefronts, which implies that the wavefront normal determines the upwind directions of the energy propagation for FD eikonal solvers. In this section, we will focus on FD schemes for quasi-$P$ and quasi-$SV$ eikonal equations. The quasi-SH eikonal equation is much easier to solve.

Let $x^k_{i,j} = (x_i, y_j, z_k)$ be grid points of the uniform cell size ($\Delta x, \Delta y, \Delta z$); denote $f^k_{i,j} = f(x_i, y_j, z_k)$ for a function $f$ of independent variables $(x, y, z)$. Define the
forward (+) and backward (−) FD operators for \( \tau_{K,x} \):

\[
D_{x}^{\pm} \tau_{K,x}^{k} = \pm \frac{\tau_{K,x}^{k+1} - \tau_{K,x}^{k}}{\Delta x}.
\]

(3.2)

Then, the second-order ENO differences for \( \tau_{K,x} \) read

\[
\tau_{K,x} \approx D_{x}^{\pm,2} \tau_{K} = D_{x}^{\pm} \tau_{K} \mp \frac{1}{2} \Delta x \cdot m(D_{x}^{-2} \tau_{K,x}^{i,j} - S, 0),
\]

(3.3)

where

\[
m(a, b) = \begin{cases} 
0, & \text{if } ab \leq 0, \\
|a|, & \text{if } |a| \leq |b| \text{ and } ab > 0, \\
|b|, & \text{if } |a| > |b| \text{ and } ab > 0.
\end{cases}
\]

(4.4)

Now, define the second-order upwind ENO approximation for \( \tau_{K,x} \):

\[
\tilde{D}_{x}^{i,j} \tau_{K,x}^{k} = \text{mod}_{\text{max}} \left( \max(D_{x}^{-2} \tau_{K,x}^{i,j} - S, 0), \min(D_{x}^{-2} \tau_{K,x}^{i,j} - S, 0) \right) + S,
\]

(5.5)

where “mod_{max}” returns the larger value in modulus and \( S = \tan \phi = d_{1}/d_{3} \), where \( D = (d_{1}(x), d_{2}(x), d_{3}(x)) \) is the symmetry axis at \( x \). (For the isotropic and VTI traveltimes, \( S \equiv 0 \).) Such upwind ENO differences are essentially central second-order [11].

After approximating \( \tau_{K,x} \) and \( \tau_{K,y} \) by the second-order upwind ENO scheme (3.5) and then evaluating the coefficients \( a_{t}, \ell = 0, \ldots, 4 \), at each grid point, one can solve (3.1) for \( \tau_{K,z} \) by utilizing an analytic formula [1] or an iterative algorithm [17, 18]. It has four solutions, as indicated in Appendix Appendix A. For down-going wavefronts, two of them are positive and the other two are negative. The positive solutions correspond to the wave propagation to the \((z+)\)-direction and the negative ones to the \((z-)\)-direction. The larger ones in modulus are related to quasi-SV, while the smaller ones in modulus represent quasi-\( P \). However, the numerical solutions are often complex-valued with small imaginary parts when the ray direction is near to the horizon. Let \( \zeta_{i,j,m}^{k}, m = 1, \ldots, 4 \), be the four solutions of (3.1) at the point \( x_{i,j}^{k} \) such that

\[
\Re \zeta_{i,j,1}^{k} \leq \Re \zeta_{i,j,2}^{k} \leq \Re \zeta_{i,j,3}^{k} \leq \Re \zeta_{i,j,4}^{k}.
\]

(6.6)

Then, we can formulate the down marching of the second-order Runge-Kutta ENO algorithm for (3.1) as

\[
\tau_{K,i,j}^{k+1/2} = \tau_{K,i,j}^{k} + \Delta z_{\text{CFL}} \cdot H[\tau_{K,i,j}^{k}],
\]

\[
\tau_{K,i,j}^{k+1} = \frac{1}{2} \left( \tau_{K,i,j}^{k} + \tau_{K,i,j}^{k+1/2} + \Delta z_{\text{CFL}} \cdot H[\tau_{K,i,j}^{k+1/2}] \right),
\]

(7.7)
where

\[ H[\tau^k_{K,i,j}] = \begin{cases} 
\max \left( \cos \frac{\theta_{\max}}{\alpha_0}, \Re \xi_{i,j,3}^k \right), & \text{if } K = P, \\
\max \left( \cos \frac{\theta_{\max}}{\beta_0}, \Re \xi_{i,j,4}^k \right), & \text{if } K = SV.
\end{cases} \tag{3.8} \]

Here \( \Delta z_{CFL} \leq \Delta z \) is to be determined to satisfy the CFL stability condition or the maximum angle condition. It is recommended to march each step of length \( \Delta z \) by advancing a few (often, 2-4) substeps of length \( \Delta z_{CFL} \) \([13]\); set \( \theta_{\max} = \cos^{-1} \frac{1}{4} \approx 75.5^\circ \) for the DNO marching and \( \theta_{\max} = \cos^{-1} \frac{1}{2} = 60^\circ \) for the PS iteration.

For the \((z-)\)-directional marching, \( k + 1/2 \) and \( k + 1 \) in (3.7) should be replaced by \( k - 1/2 \) and \( k - 1 \), respectively, and (3.8) by

\[ H[\tau^k_{K,i,j}] = \begin{cases} 
\max \left( \cos \frac{\theta_{\max}}{\alpha_0}, -\Re \xi_{i,j,2}^k \right), & \text{if } K = P, \\
\max \left( \cos \frac{\theta_{\max}}{\beta_0}, -\Re \xi_{i,j,1}^k \right), & \text{if } K = SV.
\end{cases} \tag{3.9} \]

4. Numerical experiments

ENO-DNO-PS incorporating the accuracy-efficiency techniques such as the maximum angle condition, the average normal slowness, and the cache-based implementation was implemented for the traveltime computation in TI media in 3D. The main driver routines were written in C++ and the core computation routines were in F77. The computation is carried out on a Gateway Solo, a 266 MHz laptop having 128M memory and a Linux operating system. The elapsed time (CPU) is measured in second for the user time.

4.1. ENO-DNO-PS: Isotropy and VTI

In this subsection, we address accuracy and efficiency of the algorithm (3.7) for isotropic and VTI traveltimes. The code for the VTI traveltimes can be easily obtained by modifying the one for the isotropic traveltimes. The main steps are (a) to add the required arrays for elastic parameters and (b) to replace the isotropic dispersion relation by (3.8)-(3.9). The subroutines for the second-order upwind ENO formula (3.5) and the accuracy-efficiency techniques can be utilized without modification.

Set the domain \( \Omega = (0,6000 \text{ m})^3 \). Point sources are imposed inside or on the surface of the domain. The domain is partitioned into uniform cells: \( h = \Delta x = \Delta y = \Delta z \). The numerical error is measured as \( E(\tau^h) = \| \tau^h - \tau_{\text{analytic}} \|_\infty \), where \( \tau^h \) denotes the computed traveltime with a grid size \( h \).

For constant TI media, one can compute traveltimes by solving e.g. the ray-trace equations (6) in \([5]\) by implementing a Newton-type iterative algorithm. As the VTI
<table>
<thead>
<tr>
<th>$h$</th>
<th>Isotropy $E(\tau^h)$ CPU</th>
<th>VTI-$P$ $E(\tau^h)$ CPU</th>
<th>VTI-$SV$ $E(\tau^h)$ CPU</th>
<th>VTI-$SH$ $E(\tau^h)$ CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.7e-3 0.21</td>
<td>1.2e-3 0.36</td>
<td>3.7e-3 0.36</td>
<td>2.1e-3 0.24</td>
</tr>
<tr>
<td>100</td>
<td>4.6e-4 1.6</td>
<td>3.4e-4 2.6</td>
<td>9.1e-4 2.7</td>
<td>5.9e-4 1.8</td>
</tr>
<tr>
<td>50</td>
<td>1.1e-4 12.5</td>
<td>9.2e-5 21.0</td>
<td>2.2e-4 21.9</td>
<td>1.5e-4 14.2</td>
</tr>
</tbody>
</table>

Table 1: The numerical results for ENO-DNO-PS for isotropic and VTI traveltimes in Mesaverde Sandstone. The source is located at $x_s = (3000, 3000, 0)$.

velocity model, we choose Mesaverde Sandstone [21], i.e., set the elastic parameters in (A.5) as

$$
\alpha_0 = 4349, \quad \beta_0 = 2571, \quad \epsilon = 0.091, \\
\delta = 0.148, \quad \gamma = 0.105.
$$

(4.1)

In Table 1, we present the numerical results for ENO-DNO-PS for isotropic and VTI traveltimes in Mesaverde Sandstone. For the isotropic traveltime, we choose $v(x) = \alpha_0$. Halving the grid size reduces the error by a factor of four; both isotropic and VTI traveltimes show a second-order convergence in accuracy. The elapsed times for VTI $P$- and $SV$-traveltimes are about 60% larger than that of the isotropic traveltime, while the VTI $SH$-traveltimes can be computed with 15% more cost. Note that the number of grid points is 226,981 when $h = 100m$. ENO-DNO-PS on a 266 MHz laptop solves such a problem within 2-3 seconds for isotropic and VTI traveltimes, with the numerical error far less than one millisecond. It is so accurate and efficient! It has been numerically verified for real isotropic models from the Gulf of Mexico that ENO-DNO-PS is accurate enough for the grid size of 150m; see [13].

Fig. 1 depicts the computed traveltimes in Mesaverde Sandstone for the VTI $P$-wavefront (left) and the VTI $SV$-wavefront (right). Clearly, the $P$-wavefront travels faster than the $SV$-wavefront. The $P$-wavefront is apparently faster in the horizontal direction than in the vertical.

4.2. ENO-DNO-PS: ITI

The code for inclined transverse isotropic (ITI) traveltimes can be obtained from the VTI code with a moderate modification, including a quartic polynomial solver and a subroutine for the evaluation of the coefficients in (A.16)-(A.18).

Table 2 contains the numerical results for ENO-DNO-PS for ITI traveltimes in Mesaverde Sandstone. The symmetry axis is set as $\phi = 30^\circ$ and $\psi = -15^\circ$. (See Appendix Appendix A.) Other algorithm parameters are chosen the same as in Table 1. As one can see from the table, the solution is accurate enough even for the grid size of
Figure 1: The computed traveltimes in Mesaverde Sandstone for the VTI *P*-wavefront (left) and the VTI *SV*-wavefront (right). The grid size $h = 100\text{m}$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>ITI-<em>P</em></th>
<th>ITI-<em>SV</em></th>
<th>ITI-<em>SH</em></th>
</tr>
</thead>
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<tr>
<td></td>
<td>$E(\tau^h)$</td>
<td>CPU</td>
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<td>126.9</td>
<td>1.3e-3</td>
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</table>

Table 2: The numerical results for ENO-DNO-PS for ITI traveltimes in Mesaverde Sandstone. Set $\phi = 30^\circ$ and $\psi = -15^\circ$. The source is located at $\mathbf{x}_s = (3000, 3000, 0)$.

200m. The CPUs for ITI *P*- and *SV*-traveltimes become about 50 times larger than those for VTI traveltimes, while the cost of ITI *SH*-traveltimes is about twice that of VTI *SH*-traveltimes. The high computation costs for ITI *P*- and *SV*-traveltimes seem coming from the evaluation of the complicated coefficients in (A.16)-(A.18) and the iterative quartic polynomial solver, the Laguerre’s method, adopted from [18]. By applying the analytic formula for the quartic polynomials [1], one can improve the efficiency (but not so much).

Figures 2 and 3 respectively contain the computed traveltimes and the numerical errors in millisecond, for the ITI *P*-wavefront (left) and the ITI *SV*-wavefront (right). The error is calculated as $1000 \cdot (\tau^h - \tau_{\text{analytic}})$. The grid size $h = 100\text{m}$. Note that the accuracy near the source is well controlled due to the introduction of the locally uniform mesh refinement (LUMR) initialization technique [13].
4.3. A synthetic VTI model

For reasonable tests of ENO-DNO-PS for realistic VTI models, we choose the domain $\Omega = (0,9000m)^3$ and the following model

$$
\alpha_0(\mathbf{x}) = \begin{cases} 
4500 \text{ m/s}, & \text{if } \mathbf{x} \in [2250,4500]^3, \\
1500 + 0.5z \text{ m/s}, & \text{else},
\end{cases}
$$

$$
\beta_0(\mathbf{x}) = \frac{2}{3}\alpha_0(\mathbf{x}) \text{ m/s},
$$

$$
\varepsilon(\mathbf{x}) \equiv 0.2, \quad \delta(\mathbf{x}) \equiv 0.1, \quad \gamma(\mathbf{x}) \equiv 0.2.
$$

Fig. 4 presents the computed traveltimes superposed on the synthetic model (4.2) for the VTI P-wavefront (left) and the VTI SV-wavefront (right). The point source is located at $\mathbf{x}_s = (2000,4500,0)$ and the grid size $h = 150m$. The anisotropic ENO-DNO-PS takes 7.3 and 7.9 seconds respectively for VTI P-wavefront and VTI SV-wavefront. The PS iteration seems already converged in one iteration; the second iteration did not improve the solution. Such a fast convergence of the PS iteration has been observed for isotropic traveltimes in various synthetic and real models [13]. Here we have experienced the same for TI traveltimes. As a matter of fact, in principle, there is no limitation for well-designed expanding-box isotropic eikonal solvers to be applicable for TI traveltimes. As one can see from the figure, the headwaves are clearly verified for both traveltimes.
Figure 3: The numerical errors (in millisecond) in Mesaverde Sandstone for the ITI $P$-wavefront (left) and the ITI $SV$-wavefront (right). The symmetry axis is set as $\phi = 30^\circ$ and $\psi = -15^\circ$. The grid size $h = 100$ m.

5. Conclusions

We have implemented second-order expanding-box ENO schemes for the computation of TI traveltimes, for which the stability is enforced by the DNO marching and the PS iteration. To further improve accuracy and efficiency of the resulting algorithm, ENO-DNO-PS, we have adopted techniques such as the maximum angle condition, the average normal slowness, and the cache-based implementation. The local convexity of TI wavefronts is considered to determine the correct upwind direction of the energy propagation, without reckoning the ray directions. The anisotropic ENO-DNO-PS has been tested to be accurate and efficient for various heterogeneous models having large contrasts. The computed traveltimes are accurate enough for the grid size of 100 m.

This article is the second part of the 3D eikonal solver series. ENO-DNO-PS was first designed for the first-arrivals in isotropic media (Part I); it will be applied to the computation of amplitudes (Part III) and most-energetic traveltimes (Part IV).

Acknowledgment

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Figure 4: The computed traveltimes superposed on the synthetic model (4.2) for the VTI P-wavefront (left) and the VTI SV-wavefront (right). The grid size $h = 150\text{m}$.

References


Appendix A. The TI eikonal equations

A.1. Vertically transverse isotropic media

We first derive the eikonal equations in a general framework, for a completeness of the paper. Suppose that a wavefront is given by \( t = \tau(x_0, x) \) for a fixed source at \( x_0 \) and that the wavefront reaches the point \( x + \Delta x \) at the time \( t + \Delta t \). Let \( \tau(x) = \tau(x_0, x) \). Then we have \( \Delta t = \tau(x + \Delta x) - \tau(x) = \nabla \tau \cdot \Delta x + O(|\Delta x|^2) \). Thus for the infinitesimal displacement \( dx \), the time increment \( dt \) satisfies \( dt = \nabla \tau \cdot dx \), and therefore

\[
\nabla \tau \cdot \mathbf{v} = 1, \quad \mathbf{v} = \frac{dx}{dt}.
\]

(A.1)

When \( dx \) is parallel to \( \nabla \tau \), i.e., when \( dx \) is normal to the wavefront, one can see

\[
|\nabla \tau| = \frac{1}{v},
\]

(A.2)

where \( v = |\mathbf{v}| \). The above equation is called the eikonal equation, which governs the propagation of wavefronts (in either isotropic or anisotropic media), provided that \( v \) is the wavefront normal velocity (i.e., the phase velocity).

The vertically transverse isotropic (VTI) eikonal equations read

\[
|\nabla \tau_K|^2 = \frac{1}{v_K}, \quad K = P, \ SV, \ SH,
\]

(A.3)

where \( v_K \) is the phase velocity of the quasi-\( K \) wave. The angle-dependent phase velocities \( v_K = v_K(\theta) \) are known as follows (see e.g. Thomsen [21]):

\[
\begin{align*}
    v_P^2(\theta) &= \alpha_0^2 \left[ 1 + \epsilon \sin^2 \theta + D^* (\theta) \right], \\
    v_{SV}^2(\theta) &= \beta_0^2 \left[ 1 + \frac{\alpha_0^2}{\beta_0^2} \epsilon \sin^2 \theta - \frac{\alpha_0^2}{\beta_0^2} D^* (\theta) \right], \\
    v_{SH}^2(\theta) &= \beta_0^2 \left[ 1 + 2 \gamma \sin^2 \theta \right],
\end{align*}
\]

(A.4)
where $\theta$ be the angle between the wavefront normal and the vertical,

$$D^*(\theta) = \frac{Q}{2} \left\{ \left[ 1 + \frac{4\delta - \varepsilon}{Q} \right] \sin^2 \theta \cos^2 \theta + \frac{4(Q + \varepsilon) \varepsilon}{Q^2} \sin^4 \theta \right\}^{1/2} - 1,$$

$$\alpha_0 = \sqrt{C_{33}/\rho},$$
$$\beta_0 = \sqrt{C_{44}/\rho},$$
$$\varepsilon = (C_{11} - C_{33})/(2C_{33}),$$
$$\delta = (C_{13} + C_{44})^2 - (C_{33} - C_{44})^2,$$
$$\gamma = (C_{66} - C_{44})/(2C_{44}),$$
$$Q = 1 - \beta_0^2/\alpha_0^2,$$

and $\rho$ is the density of the medium. Note that

$$\sin^2 \theta = \frac{\tau_{K,x}^2 + \tau_{K,y}^2}{|\nabla \tau_K|^2}, \quad \cos^2 \theta = \frac{\tau_{K,z}^2}{|\nabla \tau_K|^2}. \quad \text{(A.6)}$$

Using (A.3)-(A.6), one can derive the following eikonal equations for VTI media:

$$(1 + 2\gamma)\tau_{S,H,x}^2 + (1 + 2\gamma)\tau_{S,H,y}^2 + \tau_{S,H,z}^2 = \frac{1}{\beta_0^2}, \quad \text{(A.7)}$$

where

$$B = \frac{1}{\alpha_0^2} + \frac{1}{\beta_0^2} - 2(1 + \delta + (\varepsilon - \delta)\alpha_0^2) u_K^2,$$
$$C = \left( (1 + 2\varepsilon) u_K^2 - \frac{1}{\alpha_0^2} \right) \cdot \left( u_K^2 - \frac{1}{\beta_0^2} \right),$$
$$u_K^2 = \tau_{K,x}^2 + \tau_{K,y}^2, \quad K = P, \ SV. \quad \text{(A.8)}$$

The first equation of (A.7) can also be formulated for $\tau_{K,x}$ or $\tau_{K,y}$ which is again a quartic polynomial of even order terms. It has four solutions. For a physical situation of vertically moving wavefronts, two of them are positive and the other two are negative. Positive solutions correspond to the propagation of waves to the $(z+)$-direction and negative ones to the $(z-)\text{-direction}$. The larger ones in modulus are related to the slower type of wave (quasi-$SV$), while the smaller ones in modulus represent the faster type of wave propagation (quasi-$P$). We explicitly write it in the evolution equations

$$\tau_{P,z} = \sqrt{(B - \sqrt{B^2 - 4C})/2},$$
$$\tau_{SV,z} = \sqrt{(B + \sqrt{B^2 - 4C})/2},$$
$$\tau_{SH,z} = \sqrt{\frac{1}{\alpha_0^2} - (1 + 2\gamma)\tau_{S,H,x}^2 - (1 + 2\gamma)\tau_{S,H,y}^2}, \quad \text{(A.9)}$$

where the waves propagate to the $(z+)$-direction.
A.2. Generally inclined transverse isotropic media

In this section, we reformulate the eikonal equations in the generally inclined transverse isotropic (ITI) media such that they are convenient to be utilized for FD eikonal solvers. Let \( D(\mathbf{x}) = (d_1(\mathbf{x}), d_2(\mathbf{x}), d_3(\mathbf{x})) \), \( |D(\mathbf{x})| = 1, \ d_3(\mathbf{x}) \geq 0 \), be the symmetric axis at a point \( \mathbf{x} \) in an ITI medium. We may consider the local coordinate system \( (x', y', z') \) in which \( D \) is acting like the vertical. Let \( \phi \) and \( \psi \) be the inclination angles of the inclined symmetric axis \( D \) in the \( x' \)- and \( y' \)-directions, respectively, and \( \theta' \) be the angle between the wavefront normal and \( D \). Then the VTI equation (A.3) can be modified for the ITI medium as follows:

\[
|\nabla_{x'} \tau_K|^2 = \frac{1}{v_{K}^2(\theta')}, \quad K = P, \ SV, \ SH,
\]

where \( \nabla_{x'} \tau_K = (\tau_{K,x'}, \tau_{K,y'}, \tau_{K,z'}) \) is the gradient of \( \tau_K \) with respect to the local coordinate system \( (x', y', z') \) and \( v_{K}^2(\theta') \) are defined as in (A.4), with \( \theta \) replaced by \( \theta' \). Similarly as in the VTI cases, \( \theta' \) satisfies

\[
\sin^2 \theta' = \frac{\tau_{K,x'}^2 + \tau_{K,y'}^2}{|\nabla_{x'} \tau_K|^2}, \quad \cos^2 \theta' = \frac{\tau_{K,z'}^2}{|\nabla_{x'} \tau_K|^2}.
\]

Note that \( \alpha_0 \) and \( \beta_0 \) are now normal velocities for the inclined quasi-\( P \) and quasi-\( SV \) waves, respectively.

Let \( T : X \rightarrow X' \) denote the coordinate change from \( X \)-coordinates to \( X' \)-coordinates. Then, it is not difficult to see

\[
T = [T_{ij}] = \begin{bmatrix}
\frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\
\frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\
\frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}
\end{bmatrix}.
\]

Define \( (x_1, x_2, x_3) = (x, y, z) \) and \( (x'_1, x'_2, x'_3) = (x', y', z') \), and let \( \tau_{x_i} \) be the \( x_i \)-derivative of \( \tau \). Since

\[
\tau_{x'_i} = \sum_{j=1}^{3} \tau_{x_j} \frac{\partial x_j}{\partial x'_i}, \quad i = 1, 2, 3,
\]

we can see

\[
\nabla_{x'} \tau_K = T \nabla_{x} \tau_K.
\]

Then, from (A.4)-(A.5) and (A.10)-(A.14), the eikonal equations for the quasi-\( P \) and quasi-\( SV \) waves in an ITI medium read

\[
a_{j,4} \tau_{K,x_j} + a_{j,3} \tau_{K,x_j}^3 + a_{j,2} \tau_{K,x_j}^2 + a_{j,1} \tau_{K,x_j} + a_{j,0} = 0, \quad K = P, \ SV,
\]

(A.15)
where

\[ a_{j,4} = B_1(T_{1j}^2 + T_{2j}^2)^2 + B_2T_{3j}^2(T_{1j}^2 + T_{2j}^2) + T_{3j}^4, \]

\[ a_{j,3} = 4B_1(T_{1j}^2 + T_{2j}^2)(R_{1j}T_{1j} + R_{2j}T_{2j}) + 4R_{3j}T_{3j}^2 \]

\[ + 2B_2[R_{3j}T_{3j}(T_{1j}^2 + T_{2j}^2) + T_{3j}^2(R_{1j}T_{1j} + R_{2j}T_{2j})], \]

\[ a_{j,2} = B_1 \{ 4(R_{1j}T_{1j} + R_{2j}T_{2j})^2 \]

\[ + 2(T_{1j}^2 + T_{2j}^2)(R_{1j}^2 + R_{2j}^2) + B_2[R_{3j}^2(T_{1j}^2 + T_{2j}^2) \]

\[ + T_{3j}^2(R_{1j}^2 + R_{2j}^2) + 4R_{3j}T_{3j}(R_{1j}T_{1j} + R_{2j}T_{2j}) \]

\[ + 6R_{3j}^2T_{3j}^2 - B_3(T_{1j}^2 + T_{2j}^2) - C_1T_{3j}^2 \}, \]  

(A.16)

\[ a_{j,1} = 4B_1(R_{1j}T_{1j} + R_{2j}T_{2j})(R_{1j}^2 + R_{2j}^2) + 4R_{3j}^2T_{3j}^2 \]

\[ + 2B_2[R_{3j}^2(R_{1j}T_{1j} + R_{2j}T_{2j}) + R_{3j}T_{3j}(R_{1j}^2 + R_{2j}^2)] \]

\[ - 2B_3(R_{1j}T_{1j} + R_{2j}T_{2j}) - 2C_1R_{3j}^2T_{3j}, \]

\[ a_{j,0} = B_1(R_{1j}^2 + R_{2j}^2)^2 + B_2R_{3j}^2(R_{1j}^2 + R_{2j}^2) + R_{1j}^4 \]

\[ - B_3(R_{1j}^2 + R_{2j}^2) - C_1R_{3j}^2 + C_2, \]

with

\[ R_{ij} = \sum_{k \neq j} T_{ik} \tau_{K_xK_y}, \]  

(A.17)

and

\[ B_1 = 1 + 2\varepsilon, \]

\[ B_2 = 2[1 + \delta + (\varepsilon - \delta)\alpha_0^2/\beta_0^2], \]

\[ B_3 = 1/\alpha_0^2 + (1 + 2\varepsilon)/\beta_0^2, \]

\[ C_1 = 1/\alpha_0^2 + 1/\beta_0^2, \]

\[ C_2 = 1/(\alpha_0^2\beta_0^2). \]  

(A.18)

Note that the transformation \( T \) can be expressed as a composite rotations \( R_{yz}R_{xz} \):

\[ T = \begin{bmatrix}
\cos \phi & 0 & \sin \phi \\
-\sin \phi \sin \psi & \cos \psi & \cos \phi \sin \psi \\
-\sin \phi \cos \psi & -\sin \psi & \cos \phi \cos \psi
\end{bmatrix}, \]  

(A.19)

where

\[ \cos \phi = \frac{d_3}{\sqrt{d_1^2 + d_2^2}}, \quad \sin \phi = \frac{d_1}{\sqrt{d_1^2 + d_2^2}}, \]

\[ \cos \psi = \frac{d_3}{\sqrt{d_2^2 + d_3^2}}, \quad \sin \psi = \frac{d_2}{\sqrt{d_2^2 + d_3^2}}. \]  

(A.20)

For quasi-SH waves in an ITI model, we can formulate the eikonal equation similarly:

\[ a_{j,2}T_{SH,xj}^2 + a_{j,1}T_{SH,xj} + a_{j,0} = 0, \]  

(A.21)

where
Figure 5: Local convexity of the wavefronts in TI media. It implies that VTI waves cannot propagate to the \((x+)-\)direction when the wavefront is facing the \((x-)-\)direction, and vice versa, in locally constant TI media. That is, the wavefront normal determines the upwind directions of energy propagation) for FD eikonal solvers.

\[
\begin{align*}
a_{j,2} & = G(T_{1j}^2 + T_{2j}^2) + T_{3j}^2, \\
a_{j,1} & = 2[G(R_{1j}T_{1j} + R_{2j}T_{2j}) + R_{3j}T_{3j}], \\
a_{j,0} & = G(R_{1j}^2 + R_{2j}^2) + R_{3j}^2 - 1/\beta_0^2,
\end{align*}
\]

with \(G = 1 + 2\gamma\) and \(R_{ij}\) being defined as in (A.17). When \(D = (0, 0, 1)^T\), (A.15) and (A.21) become (A.7).

Appendix B. Local convexity of the TI wavefronts

In this section, we will show that the TI wavefronts are locally convex. We first consider the quasi-\(P\) waves in a locally constant VTI medium. Then, it is known that the rays are locally straight and the straight rays determine the wavefronts. (We can show it using the scale invariance of the eikonal equation and Fermat’s principle.) Consider a wavefront, choose two different points \(P_1\) and \(P_2\) on the same wavefront, and draw two rays starting from the source \(S\) to the points, as in Fig. 5. Let \(\theta_r\), \(r = 1, 2\), be the (counterclockwise) angle between the vertical and the ray pointing \(P_r\). Then, it suffices to show that the wavefront passing \(P_1\) and \(P_2\) cannot touch the interior of \(\triangle SP_1P_2\), the triangle of vertices \(S, P_1,\) and \(P_2\).

Pick a point \(M\) on the ray \(SP_1\) and let \(N\) be the intersection of \(P_1P_2\) and the line that begins at \(M\) and runs parallel to the ray \(SP_2\). Since \(P_1\) and \(P_2\) are on the same wavefront, we have to get

\[
\tau(S, P_1) = \tau(S, P_2).
\]

The scale invariance of the eikonal equation implies

\[
\tau(M, P_1) = \tau(M, N),
\]
and therefore
\[ \tau(S, P_1) = \tau(S, M) + \tau(M, N), \]  
(B.3)
for every \( M \in \overline{SP_1} \) and the corresponding intersection \( N \). It implies that for every point \( N \) on \( P_1P_2 \), the (smallest) traveltime from the source to the point \( N \) is not larger than \( \tau(S, P_1) \), i.e.,
\[ \tau(S, N) \leq \tau(S, P_1) = \tau(S, P_2), \quad \forall N \in \overline{P_1P_2}. \]  
(B.4)
This shows that the wavefront passing \( P_1 \) and \( P_2 \) cannot touch the interior of \( \triangle SP_1P_2 \), which completes the proof for the local convexity of VTI quasi-\( P \) wavefronts.

**Remark.** Choose \( \theta_2 = -\theta_1 \). Then, one can see that there is no such case that VTI quasi-\( P \) waves propagate to the \((x+)\)-direction with the wavefront facing the \((x-)\)-direction, and vice versa.

The above argument is applicable to the VTI quasi-\( SH \) waves in the same way. It is known that the quasi-\( SV \) waves may develop cusps in some strong anisotropic media. An example can be found in Green River Shale (about 20% anisotropy for every shear parameter). However, such a strong anisotropy is very rare in realistic media. Under the no-cusp assumption, we can apply Fermat’s principle again to conclude the local convexity of quasi-\( SV \) wavefronts. Since the TI waves can be computed by adopting local coordinate changes in which the inclined symmetric axis is acting like the vertical, the local convexity of their wavefronts follows with minor modifications.

Since most FD eikonal solvers proceed the solution based on the current level of traveltimes (more precisely, the wavefront normal and curvature), finding the accurate direction of energy propagation is very crucial to satisfy causality. In fact, we do not have to find accurate ray directions for eikonal solvers; only the requirement is to find the correct upwind directions. The interesting implication of the local convexity of the TI wavefronts is that the wavefront normal determines the upwind directions of the energy propagation for FD eikonal solvers. Thus FD eikonal solvers can compute accurate anisotropic traveltimes without reckoning ray directions.

The local convexity of TI wavefronts does not imply that the TI wavefronts are always convex. They can develop shocks in heterogeneous media, as shown in Figure 4. By the Huygens’s principle, each point on the advancing wavefront can be regarded as the source of a secondary wave and that a later wavefront is the envelope tangent to all the secondary waves. The eikonal solvers compute the first-arrival traveltimes by chasing the earliest arrival among the secondary wavefronts coming to the grid point under consideration.