

STRONG TYPE ESTIMATES FOR HOMOGENEOUS BESOV CAPACITIES

In Memory of Bjorn Dahlberg

D.R. Adams and J. Xiao

INTRODUCTION

This article is a direct outcome of communication between the authors regarding the weak type estimates for homogeneous Besov capacities with end-point parameters. Working with homogeneous Besov capacity $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q})$ associated with the Euclidean space \mathbb{R}^n , $n \geq 1$, we show

Theorem 1. *Let $f \in \dot{\Lambda}_\alpha^{p,q}$. Then there is a constant C depending on α, p, q, n such that*

(i)

$$\int_0^\infty \left(\text{cap}(\{x \in \mathbb{R}^n : |f(x)| \geq t\}; \dot{\Lambda}_\alpha^{p,q}) \right)^{q/p} dt^q \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^q, \quad 1 < p \leq q < \infty;$$

(ii)

$$\int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : |f(x)| \geq t\}; \dot{\Lambda}_\alpha^{p,q}) dt^p \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p, \quad 1 < q \leq p < \infty.$$

It is worth remarking that the case $q = p$ is due to Maz'ya [Ma] and the case $1 \leq p \leq q < \infty$, $0 < \alpha < 1$ is essentially due to Wu [W] where he uses a different definition of the capacity.

1991 *Mathematics Subject Classification*: Primary 42A16, 46E30, 47B38.

Key words and phrases: Homogeneous Besov capacities, strong type estimates, Carleson-like measures, tents.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

The relevant definitions as well as some basic results will be arranged in Section 1 from which the weak type estimate:

$$\text{cap}(\{x : |f(x)| \geq t\}; \dot{\Lambda}_\alpha^{p,q}) \leq \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^q t^{-q}$$

holds trivially for all $f \in \dot{\Lambda}_\alpha^{p,q}$. Whereas the proof of the main theorem is presented in Section 2. For some applications, Section 3 provides a discussion of two topics of general interest: the embedding from the homogeneous Besov spaces into the Lorentz spaces with respect to different Borel measures and the predual spaces of those signed Borel measures on either \mathbb{R}^n or \mathbb{R}_+^{n+1} (the upper half space of \mathbb{R}^{n+1}).

It is our pleasure to thank Stephen Michael Elery for the technical typing support.

1. BACKGROUND

From now on, we always assume that $p, q \in (1, \infty)$, and $\alpha \in (0, \infty)$. Let $\dot{\Lambda}_\alpha^{p,q}$ be the homogeneous Besov space which is the completion of all functions $f \in C_0^\infty(\mathbb{R}^n)$ with the semi-norm:

$$\|f\|_{\dot{\Lambda}_\alpha^{p,q}} = \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h^k f\|_p^q}{|h|^{n+\alpha q}} dh \right)^{1/q} < \infty,$$

where $\|\cdot\|_p$ means the usual L^p -norm, k is an integer greater than α , and Δ_h^k denotes the k -th difference: $\Delta_h^k = \Delta_h^1 \Delta_h^{k-1}$; $\Delta_h^1 f(x) = f(x+h) - f(x)$. It is well known that this definition is independent of k (cf. [St, p.153]). Whenever $p = q$, the homogeneous Besov spaces arise naturally as the traces (restriction to \mathbb{R}^n) of the Riesz potential on \mathbb{R}^{n+1} .

The Riesz potential of order $\beta \in (0, 2n)$ is defined by

$$I_\beta^{(2n)} * f(x) = \int_{\mathbb{R}^{2n}} |x-y|^{\beta-2n} f(y) dy.$$

Of course, it is natural to write $I_\beta^{(2n)}(x) = |x|^{\beta-2n}$ for the Riesz kernel. From [Ad1, Theorem 5.1], we have the following characterization: if $u(x)$ and $I_\beta^{(2n)} * |f|(x, 0)$ are both in $L_{loc}^1(\mathbb{R}^n)$, with

$$(1) \quad f(x, t) = |t|^{-\beta} \Delta_t^k u(x), \quad 0 < \alpha < k,$$

then

$$(2) \quad u(x) = C I_\beta^{(2n)} * f(x, 0), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Here and henceforth the letter C denotes various constants which may differ from one formula to the next even within a single string of estimates.

Note that if $u \in \dot{\Lambda}_\alpha^{p,q}$, then the function $f(\cdot, \cdot)$ in (1) belongs to the mixed norm space $L^{p,q}(\mathbb{R}^{2n})$, i.e.,

$$\|f\|_{p,q} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dx \right)^{q/p} dy \right)^{1/q} < \infty.$$

With small changes of Stein's argument in [St1, p. 192-193], we note the homogeneous version of the extension/restriction theorem in [Ad1] (see Theorem 5.2). For $\beta \in (0, 2n)$, let $\dot{\mathcal{L}}_\beta^{p,q} = I_\beta^{(2n)} * L^{p,q}(\mathbb{R}^{2n})$ with norm

$$\|I_\beta^{(2n)} * f\|_{\beta,p,q} = \|f\|_{p,q}.$$

Then we have

Theorem A. *There is a linear extension operator $\mathcal{E} : \dot{\Lambda}_\alpha^{p,q} \rightarrow \dot{\mathcal{L}}_{\alpha+n/q}^{p,q}$ and a linear restriction operator $\mathcal{R} : \dot{\mathcal{L}}_{\alpha+n/q}^{p,q} \rightarrow \dot{\Lambda}_\alpha^{p,q}$ such that $\mathcal{R}\mathcal{E}$ is the identity, and*

- (i) $\|\mathcal{E}u\|_{\alpha+n/q,p,q} \leq C\|u\|_{\dot{\Lambda}_\alpha^{p,q}}$ for $u \in \dot{\Lambda}_\alpha^{p,q}$;
- (ii) $\|\mathcal{R}v\|_{\dot{\Lambda}_\alpha^{p,q}} \leq C\|v\|_{\alpha+n/q,p,q}$ for $v \in \dot{\mathcal{L}}_{\alpha+n/q}^{p,q}$.

The homogeneous $\dot{\Lambda}_\alpha^{p,q}$ -capacity of any compact set $K \subset \mathbb{R}^n$ is defined by

$$\text{cap}(K; \dot{\Lambda}_\alpha^{p,q}) = \inf\{\|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p : f \in C_0^\infty(\mathbb{R}^n) \text{ and } f \geq 1 \text{ on } K\}.$$

We extend this definition to any subset $E \subset \mathbb{R}^n$ by

$$\text{cap}(E; \dot{\Lambda}_\alpha^{p,q}) = \sup_{K \subset E} \text{cap}(K; \dot{\Lambda}_\alpha^{p,q}),$$

where the supremum is taken over all compact subsets K of E . Observe that the above capacity is different from Wu's one in [Wu] where he considers the q -th power of the homogeneous Besov norm for $\alpha \in (0, 1)$. It is worth pointing out that the main reason for our choice is that the Hausdorff dimension of $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q})$ is $n - \alpha p$. By this we mean the number $d = n - \alpha p$ such that

$$H^{d+\epsilon}(\cdot) \ll \text{cap}(K; \dot{\Lambda}_\alpha^{p,q}) \ll H^{d-\epsilon}(\cdot)$$

for all $\epsilon \in (0, d)$. Here $H^\beta(\cdot)$, $\beta \in (0, n]$, now stands for the classical Hausdorff capacity (content): for compact set $K \subset \mathbb{R}^n$,

$$H^\beta(K) = \inf \sum_j r_j^\beta$$

where the infimum is over all countable coverings of K by balls B_j with radius r_j . The partial ordering \ll between two capacities $\text{cap}_k(\cdot)$, $k = 1, 2$, $\text{cap}_1 \ll \text{cap}_2$ means that $\text{cap}_1(K) = 0$ whenever $\text{cap}_2(K) = 0$; see also [Ad2].

Because of Theorem A, we introduce also the capacities associated with the spaces $\dot{\mathcal{L}}_\beta^{p,q}$: for any compact set $K \subset \mathbb{R}^{2n}$,

$$\text{cap}(K; \dot{\mathcal{L}}_\beta^{p,q}) = \inf\{\|v\|_{\beta,p,q}^p : v \in C_0^\infty(\mathbb{R}^{2n}) \text{ and } v \geq 1 \text{ on } K\}.$$

This definition, together with the extension/restriction and Theorem A, implies

Corollary A. *On compact subsets of \mathbb{R}^n ,*

$$\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}) \sim \text{cap}(\cdot; \dot{\mathcal{L}}_{\alpha+n/q}^{p,q}).$$

For this conclusion, see also [Ad1], Proposition 5.1. Here \sim means the comparability between two quantities, i.e., the ratio of the two set functions on compact sets is bounded above and below by finite positive constants independent of the sets. Following [Me], we easily see that this comparability holds for all open sets in \mathbb{R}^n .

The Riesz capacity of order $\alpha > 0$ and degree (p, q) is defined by

$$R_{\beta,p,q}^{(2n)}(E) = \inf \{ \|f\|_{p,q}^p : f \in L^{p,q}(\mathbb{R}^{2n}), f \geq 0, \text{ and } I_\beta^{(2n)} * f \geq 1 \text{ on } E \}$$

for any set $E \subset \mathbb{R}^{2n}$.

Theorem B. *The following properties of the Riesz capacities hold:*

- (i) *Monotonicity:* $R_{\beta,p,q}^{(2n)}(E_1) \leq R_{\beta,p,q}^{(2n)}(E_2)$, $E_1 \subseteq E_2$;
- (ii) *Subadditivity:* $R_{\beta,p,q}^{(2n)}(\cup_j E_j) \leq \sum_j R_{\beta,p,q}^{(2n)}(E_j)$ for $p \leq q$, and

$$\left(R_{\beta,p,q}^{(2n)}(\cup_j E_j) \right)^{1/p} \leq \sum_j \left(R_{\beta,p,q}^{(2n)}(E_j) \right)^{1/p} \text{ for all } p, q > 1.$$

- (iii) $R_{\beta,p,q}^{(2n)}(\emptyset) = 0$, $\emptyset = \text{empty set}$;
- (iv) $R_{\beta,p,q}^{(2n)}(E) = 0$ if and only if there is a nonnegative $f \in L^{p,q}(\mathbb{R}^{2n})$ such that $I_{\beta,p,q}^{(2n)} * f(x) = \infty$ on E .
- (v) $R_{\beta,p,q}^{(2n)}(\cdot)$ is an outer capacity, i.e., $R_{\beta,p,q}^{(2n)}(E) = \inf_{O \supseteq E} R_{\beta,p,q}^{(2n)}(O)$ where O is open set in \mathbb{R}^{2n} . Moreover, every analytic set E in \mathbb{R}^{2n} is capacitable, i.e., $R_{\beta,p,q}^{(2n)}(E) = \sup_{K \subseteq E} R_{\beta,p,q}^{(2n)}(K)$ where K is compact set in \mathbb{R}^{2n} .
- (vi) $R_{\beta,p,q}^{(2n)}(K) = \text{cap}(K, \dot{\mathcal{L}}_\beta^{p,q})$ for all compact sets in \mathbb{R}^{2n} .

For an account of this theorem, we refer the reader to [Ad1, Chapter V].

2. PROOF OF THEOREM 1

We first note that by Theorem A above, any $u \in \dot{\Lambda}_\alpha^{p,q}$ can be written as

$$u(x) = I_{\alpha+n/q}^{(2n)} * f(x, 0) = \mathcal{R}\mathcal{E}u(x),$$

where $f \in L^{p,q}(\mathbb{R}^{2n})$.

Accordingly, we apply Corollary A and Theorem B (vi) to obtain

$$\begin{aligned} \text{cap}\left(\{x : u(x) \geq t\}; \dot{\Lambda}_\alpha^{p,q}\right) &= \text{cap}\left(\{x : I_{\alpha+n/q}^{(2n)} * f(x, 0) \geq t\}; \dot{\Lambda}_\alpha^{p,q}\right) \\ &\sim \text{cap}\left(\{x \in \mathbb{R}^n : I_{\alpha+n/q}^{(2n)} * f(x, 0) \geq t\}; \dot{\mathcal{L}}_{\alpha+n/q}^{p,q}\right) \\ &\leq R_{\alpha+n/q,p,q}^{(2n)}\left(\{(x, y) \in \mathbb{R}^{2n} : I_{\alpha+n/q}^{(2n)} * f(x, y) \geq t\}\right). \end{aligned}$$

Finally, it follows from Theorem A once again that

$$\|f\|_{p,q} \equiv \|\mathcal{E}u\|_{\dot{\mathcal{L}}_{p,q,\alpha+n/q}} \leq C\|u\|_{\dot{\Lambda}_\alpha^{p,q}}.$$

So, we will verify that if $\beta = \alpha + n/q \in (0, 2n)$ then

$$(3) \quad \int_0^\infty \left(R_{\beta,p,q}^{(2n)}(\{x \in \mathbb{R}^{2n} : I_\beta^{(2n)} * f(x) \geq t\})\right)^{q/p} dt^q \leq C\|f\|_{p,q}^q, \quad 1 < p \leq q < \infty;$$

and

$$(4) \quad \int_0^\infty R_{\beta,p,q}^{(2n)}(\{x \in \mathbb{R}^{2n} : I_\beta^{(2n)} * f(x) \geq t\}) dt^p \leq C\|f\|_{p,q}^p, \quad 1 < q \leq p < \infty.$$

To begin with, write

$$\begin{aligned} &\int_0^\infty \left(R_{\beta,p,q}^{(2n)}(\{x \in \mathbb{R}^{2n} : I_\beta^{(2n)} * f(x) \geq t\})\right)^{q/p} dt^q \\ &\leq C \sum_{j=\infty}^\infty 2^{jq} \left(R_{\beta,p,q}^{2n}(\{x \in \mathbb{R}^{2n} : I_\beta^{(2n)} * f(x) \geq 2^j\})\right)^{q/p} \\ &\leq C \sum_{j=\infty}^\infty 2^{jq} \left(R_{\beta,p,q}^{2n}(\{x \in \mathbb{R}^{2n} : \phi_j(I_\beta^{(2n)} * f)(x) \geq 2^j\})\right)^{q/p}, \end{aligned}$$

where ϕ is a $C^\infty(\mathbb{R})$ function satisfying

$$\phi(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1, \end{cases}$$

and $\phi_j(t) = 2^j \phi(2^{-j}t - 1)$.

Now, for the reader's convenience, recall the definition of T_β – the Strichartz type operator defined in Dahlberg [Da, Theorem 3]. Given an integer $m = 2n$, $m > \beta > 0$, let k be the largest integer less than β . If $\gamma = (\gamma_1, \dots, \gamma_m)$ is a multi-index of non-negative integers, then we write $|\gamma| = \sum \gamma_j$, $\gamma! = \gamma_1! \cdots \gamma_m!$, and $D^\gamma = \partial^{|\gamma|} / \partial x_1^{\gamma_1} \cdots \partial x_m^{\gamma_m}$. For

$x \in \mathbb{R}^m$, set $x^\gamma = \prod x_j^{\gamma_j}$ and $q_\gamma(x) = D^\gamma |x|^{\beta-m}/(\gamma!)$. Moreover, for $r \geq 0$ and $x, y \in \mathbb{R}^m$, put

$$P_{r,y}(x) = |x + ry|^{\beta-m} - \sum_{|\gamma| < k} r^\gamma y^\gamma q_\gamma(x).$$

It is clear that whenever $f \in C_0^\infty(\mathbb{R}^m)$ we have

$$P_{r,y} * f(x) = I_\beta^{(m)} * f(x + ry) - Q_x(ry),$$

where

$$Q_x(z) = \sum_{|\gamma| \leq k} z^\gamma D^\gamma I_\beta^{(m)} * f(x) (\gamma!)^{-1}.$$

and hence, for $s \geq 1$ we define

$$T_\beta^s f(x) = \left(\int_0^\infty \left(\int_{|y| < 1} |P_{r,y} * f(x)|^s dy \right)^{2/s} r^{-1-2\beta} dr \right)^{1/2},$$

and put $T_\beta = T_\beta^1$.

Since the proof of [Da, Theorem 3] uses the Banach space valued singular integral theory which ultimately rests on Theorem 3 of the paper of Benedek-Calderon-Panzone [BeCaPa], we can replace the L^p -norms there with the mixed $L^{p,q}$ -norms, and ultimately get $\|T_\beta^s f\|_{p,q} \sim \|f\|_{p,q}$ when $s = 1$. For $s > 1$, but sufficiently close to 1, [Da] also yields the mixed norm estimate $\|T_\beta^s f\|_{p,q} \leq c \|f\|_{p,q}$, $1 < p, q < \infty$. Furthermore, Lemma 3 and hence Theorem 2 in [Da] are valid for $L^{p,q}$ -spaces. Consequently, there is a function $g \in L^{p,q}(\mathbb{R}^{2n})$ such that

$$\phi_j(I_\beta^{(2n)} * f) = I_\beta^{(2n)} * g_j$$

with $\|g_j\|_{p,q} \leq C \|f\|_{p,q}$.

In the sequel, we show

$$\sum_j \|g_j\|_{p,q}^q \leq C \|f\|_{p,q}^q, \quad 1 < p \leq q < \infty;$$

and

$$\sum_j \|g_j\|_{p,q}^p \leq C \|f\|_{p,q}^p, \quad 1 < q \leq p < \infty.$$

We do this by proceeding as in [Da], and prove

$$(5) \quad \sum_j \|T_\beta g_j\|_{p,q}^q \leq C \|f\|_{p,q}^q, \quad 1 < p \leq q < \infty.$$

and

$$(6) \quad \sum_j \|T_\beta g_j\|_{p,q}^p \leq C \|f\|_{p,q}^p, \quad 1 < q \leq p < \infty.$$

In order to prove (5) we argue that for $q \geq p$,

$$\sum_j \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (T_\beta g_j)^p dx \right)^{q/p} dy \leq \int_{\mathbb{R}^n} \left(\sum_j \int_{\mathbb{R}^n} (T_\beta g_j)^p dx \right)^{q/p} dy.$$

Now we again refer to Dahlberg [Da]

$$\sum_j (T_\beta g_j(z))^p \leq C \left(Mf(z) + T_\beta^s f(z) \right)^p, \quad a.e.,$$

For all $s > 1$, note that the point z is in \mathbb{R}^{2n} , and $M = M^{(2n)}$ stands for the usual Hardy-Littlewood maximal operator:

$$Mf(x) = M^{(2n)}f(x) = \sup_{x \in B} |B|^{-1} \int_B |f(z)| dz,$$

where the supremum is taken over all balls containing x , and $|B|$ denotes the volume of B .

Thus we have that if $z = (x, y) \in \mathbb{R}^{2n}$, $x, y \in \mathbb{R}^n$ then

$$\sum_j \|T_\beta g_j\|_{p,q}^q \leq C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(Mf(z) + T_\beta^s f \right)^p dx \right)^{q/p} dy \leq C \|f\|_{p,q}^q$$

due to the mixed norm estimates of Fefferman-Stein [FeSt] for the maximal function (recall that $M^{(2n)}f \leq M^{(n)}M^{(n)}f$ - the iterated maximal functions). Also one needs here mixed norm estimates for $T_\beta^s f$ for $s > 1$ (sufficiently close to 1) as it uses again the full strength of the Banach space valued singular integral theory.

To see (6), we write $a_j = \int_{\mathbb{R}^n} (T_\beta g_j)^p dy$ to get

$$\begin{aligned} \sum_j \|T_\beta g_j\|_{p,q}^p &= \sum_j \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (T_\beta g_j)^p dx \right)^{q/p} dy \right)^{p/q} \\ &= \sum_j \left(\int_{\mathbb{R}^n} a_j^{q/p} dy \right)^{p/q}. \end{aligned}$$

Taking the q/p -th power here, we are looking at

$$\left(\sum_j \left(\int_{\mathbb{R}^n} a_j^{q/p} dy \right)^{p/q} \right)^{q/p} = \left\| \left\{ \int_{\mathbb{R}^n} a_j^{q/p} dy \right\} \right\|_{\ell^{p/q}}.$$

But if $p/q \geq 1$, then we apply Minkowski's inequality and the above is less than or equal to

$$\begin{aligned} \int_{\mathbb{R}^n} \left\| \left\{ a_j^{q/p} dy \right\} \right\|_{\ell^{p/q}} dy &= \int_{\mathbb{R}^n} \left(\sum_j a_j \right)^{q/p} dy \\ &\leq C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (Mf + T_\beta^s f)^p dx \right)^{q/p} dy \\ &\leq C \|f\|_{p,q}^q, \end{aligned}$$

so that

$$\sum_j \|T_\beta g_j\|_{p,q}^p \leq C \|f\|_{p,q}^p.$$

Using the same techniques, it is now easy to see the proof of

Corollary 1. *If $f \in \dot{\Lambda}_\alpha^{p,q}$, with Mf the maximal function of f , then*

- (i) $\int_0^\infty (\text{cap}(\{x \in \mathbb{R}^n : Mf(x) \geq t\}; \dot{\Lambda}_\alpha^{p,q})^{q/p} dt^q \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^q, \quad 1 < p \leq q < \infty;$
- (ii) $\int_0^\infty (\text{cap}(\{x \in \mathbb{R}^n : Mf(x) \geq t\}; \dot{\Lambda}_\alpha^{p,q}) dt^p \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p, \quad 1 < q \leq p < \infty.$

Note that in [Proposition 5.2, Ad1] we also have

$$\text{cap}(\{x \in \mathbb{R}^n : Mf(x) \geq t\}; \dot{\Lambda}_\alpha^{p,q}) \leq Ct^{-p} \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p$$

for all $t > 0$ and any $p, q > 1$.

3. SOME APPLICATIONS

3.1. To begin with, we consider the embedding theorems for the homogeneous Besov spaces.

For $r, s \in (0, \infty)$ and μ , a nonnegative Borel measure on \mathbb{R}^n , let $L(r, s)(\mu)$ be the Lorentz space of those f satisfying

$$\|f\|_{L(r,s)(\mu)} = \left(\int_0^\infty \mu(\{x : |f(x)| \geq t\})^{s/r} dt^s \right)^{1/s} < \infty.$$

Theorem 2. *Let μ be a non-negative Borel measure on \mathbb{R}^n .*

- (i) *If $1 < p \leq q < \infty$ then $\|f\|_{L(p,q)(\mu)} \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}$ holds for all $f \in \dot{\Lambda}_\alpha^{p,q}$ if and only if $\mu(K) \leq C \text{cap}(K; \dot{\Lambda}_\alpha^{p,q})$ holds for all compact sets K .*

- (ii) If $1 < q \leq p < \infty$ then $\|f\|_{L(q,p)(\mu)} \leq C\|f\|_{\dot{\Lambda}_\alpha^{p,q}}$ holds for all $f \in \dot{\Lambda}_\alpha^{p,q}$ if and only if $\mu(K) \leq C(\text{cap}(K; \dot{\Lambda}_\alpha^{p,q}))^{q/p}$ holds for all compact sets K .

Proof. We will only prove (i) since the proof of (ii) is similar. The sufficiency is a straightforward consequence of Theorem 1. Regarding the necessity, suppose that $\|f\|_{L(p,q)(\mu)} \leq C\|f\|_{\dot{\Lambda}_\alpha^{p,q}}$ holds for all $f \in \dot{\Lambda}_\alpha^{p,q}$. Fix a compact set $K \subset \mathbb{R}^n$. Then for any $\epsilon > 0$ there is an $f \in \dot{\Lambda}_\alpha^{p,q}$ such that $f \geq 1$ on K and

$$\text{cap}(K; \dot{\Lambda}_\alpha^{p,q}) + \epsilon \geq \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p.$$

Note that by Theorem 1,

$$\begin{aligned} \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^q &\geq C \int_0^\infty \mu(\{x : |f(x)| \geq t\})^{q/p} dt^q \\ &\geq C \int_0^1 \mu(\{x : |f(x)| \geq t\})^{q/p} dt^q \\ &\geq C\mu(K)^{q/p}. \end{aligned}$$

So $\mu(K) \leq C\text{cap}(K; \dot{\Lambda}_\alpha^{p,q})$.

Remarks. (i) Theorem 2 may be used to give a full description of the space of the pointwise multipliers of $\dot{\Lambda}_\alpha^{p,q}$; see also [Steg, Theorem B] for some special cases.

(ii) From [AdBa] it follows that if $\|f\|_{p,q}$ is finite then $I_\beta^{(2n)} * |f|(\cdot, 0)$ is in the Lorentz space $L(r, q)$ with respect to Lebesgue measure, where $\beta = n(1/p - 1/r) + n/q$. In particular,

$$\left(\int_0^\infty |\{x \in \mathbb{R}^n : I_\beta^{(2n)} * |f|(x, 0)| \geq \lambda\}|^{q/r} dt^q \right)^{p/q} \leq C\|f\|_{p,q}^p.$$

Consequently, if $I_\beta^{(2n)} * |f|(x, 0) \geq 1$ for x in the compact set K , then

$$|K|^{p/r} \leq C\|f\|_{p,q}^p.$$

This, together with $\beta = \alpha + n/q$ and $r = p^* = np/(n - \alpha p)$, implies

$$(7) \quad |K|^{p/r} \leq C\text{cap}(K, \dot{\Lambda}_\alpha^{p,q}).$$

Putting this into Theorem 1 gives that as $1 < p \leq q$,

$$\int_0^\infty \left(|\{x \in \mathbb{R}^n : |F(x)| \geq \lambda\}|^{1-\alpha p/n} \right)^{q/p} dt^q \leq C\|f\|_{p,q}^q.$$

which is the Sobolev type embeddings for the Besov spaces (cf [He]):

$$\|f\|_{L(p^*,q)} \leq C\|f\|_{\dot{\Lambda}_\alpha^{p,q}},$$

in case of $1 < p \leq q < \infty$. On the other hand, for all $p, q < 1$, the result of Herz [He] always implies (7). Hence one is lead to double that estimate (ii) of Theorem 1 is sharp for $1 < q < p < \infty$. This point is left open.

3.2. Denote by $L^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ the class of all signed Borel measures μ on \mathbb{R}^n obeying

$$\|\mu\|_{L_{p,q,\alpha}^\infty} = \sup_{K \subset \mathbb{R}^n} \frac{|\mu|(K)}{\text{cap}(K; \dot{\Lambda}_\alpha^{p,q})} < \infty$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$. Also, let $L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ consist of all $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q})$ quasi-continuous functions F on \mathbb{R}^n (cf. [Ad4]), for which

$$\|F\|_{L_{p,q,\alpha}^1} = \int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : |F(x)| \geq \lambda\}; \dot{\Lambda}_\alpha^{p,q}) d\lambda < \infty.$$

Theorem 3. *The pairing*

$$\langle F, \mu \rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} F(x) d\mu(x)$$

realizes the dual of $L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ as equivalent to the space $L^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$.

Proof. It is not hard to see that each $\mu \in L^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ induces a bounded linear functional on $L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$. In fact, for such a μ and for any $f \in L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$, we have

$$\begin{aligned} |\langle F, \mu \rangle_{\mathbb{R}^n}| &\leq \int_0^\infty |\mu|(\{x \in \mathbb{R}^n : |F(x)| \geq \lambda\}) d\lambda \\ &\leq C \|\mu\|_{L_{p,q,\alpha}^\infty} \int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : |F(x)| \geq \lambda\}; \dot{\Lambda}_\alpha^{p,q}) d\lambda \\ &\leq C \|\mu\|_{L_{p,q,\alpha}^\infty} \|F\|_{L_{p,q,\alpha}^1}. \end{aligned}$$

For the converse, since $C_0(\mathbb{R}^n)$ (the class of continuous functions with compact support on \mathbb{R}^n) is contained in $L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$, every bounded linear functional L on $L^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ with finite operator norm $\|L\|$ is given by

$$\int_{\mathbb{R}^n} F(y) d\mu(y), \quad F \in C_0(\mathbb{R}^n),$$

for some signed Borel measure μ on \mathbb{R}^n . However for any $G \in C_0(\mathbb{R}^n)$, one has:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Gd|\mu| \right| &\leq \int_{\mathbb{R}^n} |G|d|\mu| \\ &= \sup \left\{ \int_{\mathbb{R}^n} \Psi d\mu : \Psi \in C_0(\mathbb{R}^n), |\Psi| \leq |G| \right\} \\ &\leq \|L\| \sup \left\{ \|\Psi\|_{L^1_{p,q,\alpha}} : \Psi \in C_0(\mathbb{R}^n), |\Psi| \leq |G| \right\} \\ &\leq \|L\| \cdot \|G\|_{L^1_{p,q,\alpha}}. \end{aligned}$$

Thus, by the limit argument, we obtain

$$|\mu|(K) \leq C \|L\| \text{cap}(K; \dot{\Lambda}_\alpha^{p,q}).$$

Therefore we complete the proof.

Remarks. For the Hausdorff capacity setting of Theorem 3, see also [Ad2].

3.3. Next, we study Carleson-like measures on the upper half space. Suppose that \mathbb{R}_+^{n+1} is the upper half space: $\mathbb{R}^n \times (0, \infty)$. The tent space $T(O)$ based on an open set $O \subset \mathbb{R}^n$ is given by

$$T(O) = \{(x, t) : B(x, t) \subset O\},$$

where $B(x, t)$ stands for the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$.

In what follows, let $P_t(x) = c_n(1 + |x|^2)^{-(n+1)/2}$ be the Poisson kernel, where c_n is the constant such that $\int_{\mathbb{R}^n} P_t(x)dx = 1$ for all $t > 0$.

Theorem 4. *Let μ be a non-negative Borel measure on \mathbb{R}_+^{n+1} .*

(i) *If $1 < p \leq q < \infty$ then*

$$\int_0^\infty (\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : |P_t * f(x)| > \lambda\}))^{q/p} dt^q \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^q$$

holds for all $f \in \dot{\Lambda}_\alpha^{p,q}$ if and only if $\mu(T(O)) \leq C \text{cap}(O; \dot{\Lambda}_\alpha^{p,q})$ holds for every open set $O \subset \mathbb{R}^n$.

(ii) *If $1 < q \leq p < \infty$ then*

$$\int_0^\infty (\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : |P_t * f(x)| > \lambda\}))^{p/q} dt^p \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p$$

holds for all $f \in \dot{\Lambda}_\alpha^{p,q}$ if and only if $\mu(T(O)) \leq C (\text{cap}(O; \dot{\Lambda}_\alpha^{p,q}))^{q/p}$ holds for every open set $O \subset \mathbb{R}^n$.

Proof. We give a proof of (ii) and leave that of (i) for the interested reader.

For $x \in \mathbb{R}^n$ let $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |y - x| < 1\}$ be the cone at x . With this, the nontangential function $N(F)$ of a measurable function $F(\cdot, \cdot)$ on \mathbb{R}_+^{n+1} is given by

$$N(F)(x) = \sup_{(y,t) \in \Gamma(x)} |F(y, t)|.$$

If $N(F)$ is lower semicontinuous then the set $O_\lambda = \{x \in \mathbb{R}^n : N(F)(x) > \lambda\}$ is open. Since $\{(x, t) \in \mathbb{R}_+^{n+1} : |F(x, t)| > \lambda\} \subset T(O_\lambda)$, if μ is a non-negative Borel measure on \mathbb{R}_+^{n+1} then

$$\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : |F(x, t)| > \lambda\}) \leq \mu(T(O_\lambda))$$

holds for every measurable function F . If $f \in \dot{\Lambda}_\alpha^{p,q}$, and $F(x, t) = P_t * f(x)$, then there exists a constant $C > 0$ (independent of f) such that $N(F) \leq CM(f)$, a well-known fact; see also [St2, p.57]. Thus if $\mu(T(O)) \leq C(\text{cap}(O; \dot{\Lambda}_\alpha^{p,q}))^{q/p}$ holds for every open set $O \subset \mathbb{R}^n$, then this inequality, together with the second result of the Main Theorem, implies

$$\begin{aligned} & \int_0^\infty (\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : |F(x, t)| > \lambda\}))^{p/q} dt^p \\ & \leq \int_0^\infty (\mu(\{x \in \mathbb{R}^n : |N(F)(x)| > \lambda\}))^{p/q} dt^p \\ & \leq \int_0^\infty (\mu(\{x \in \mathbb{R}^n : |M(f)(x)| > \lambda/C\}))^{p/q} dt^p \\ & \leq C \int_0^\infty (\text{cap}(\{x \in \mathbb{R}^n : |M(f)(x)| > \lambda/C\}; \dot{\Lambda}_\alpha^{p,q}))^{p/q} dt^p \\ & \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p, \end{aligned}$$

by Corollary 1.

On the other hand, if μ satisfies

$$\int_0^\infty (\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : |P_t * f(x)| > \lambda\}))^{p/q} dt^p \leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p,$$

for all $f \in \dot{\Lambda}_\alpha^{p,q}$, then for any open set $O \subset \mathbb{R}^n$, there is, by the definition of $\dot{\Lambda}_\alpha^{p,q}$ -capacity, a function $f \in \dot{\Lambda}_\alpha^{p,q}$ ensuring $f \geq 1$ on O and $\|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p \leq 2\text{cap}(O; \dot{\Lambda}_\alpha^{p,q})$. Since the $\dot{\Lambda}_\alpha^{p,q}$ -norm of $|f|$ is not greater than that of f , we may assume that this f is nonnegative. So, when $(x, t) \in T(O)$ we obtain:

$$P_t * f(x) \geq \int_O P_t(x - y) dy \geq \int_{B(x,t)} P_t(x - y) dy := c.$$

This yields

$$\begin{aligned} (\mu(T(O)))^{p/q} &\leq \int_0^c (\mu(\{(x, t) \in \mathbb{R}_+^{n+1} : P_t * f(x) > c\}))^{p/q} dt^p \\ &\leq C \|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p \\ &\leq C \text{cap}(O; \dot{\Lambda}_\alpha^{p,q}). \end{aligned}$$

We are done.

Remarks. (i) We may call μ in Theorem 4 to be Carleson-like measure for the homogeneous Besov space. Using a different definition (from ours) of the Besov capacity, Wu [Wu] obtains a characterization of Carleson-like measure similar to the second conclusion of Theorem 4.

(ii) Instead of the Poisson kernel $P_t(x)$, we can consider more general function: $\Phi_t(x) = t^{-n} \Phi(x/t)$, where $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. Theorem 4 is still valid for Φ_t . There are many applications of Carleson-like measures in PDE's; see also [Jo].

3.4. Finally, we investigate preduality of the class of Carleson-like measures for the homogeneous Besov space.

It turns out from the arguments of Theorem 4 that for μ , locally finite regular signed Borel measure, on \mathbb{R}_+^{n+1} , $|\mu|(T(O)) \leq C \text{cap}(O; \dot{\Lambda}_\alpha^{p,q})$ holds for all open set $O \subset \mathbb{R}^n$

$$\int_{\mathbb{R}_+^{n+1}} |F(x, t)| d|\mu|(x, t) \leq C \int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : N(F)(x) > \lambda\}; \dot{\Lambda}_\alpha^{p,q}) d\lambda < \infty,$$

where $F(x, t) = P_t * f(x)$. This observation leads to the following consideration. Let $T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ consist of all continuous functions F on \mathbb{R}_+^{n+1} , for which

$$\|F\|_{T_{p,q,\alpha}^1} = \int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : N(F)(x) > \lambda\}; \dot{\Lambda}_\alpha^{p,q}) d\lambda < \infty,$$

and for which $\|F_\epsilon - F\|_{T_{p,q,\alpha}^1} \rightarrow 0$ as $\epsilon \rightarrow 0$, where $F_\epsilon(x, t) = f(x, t + \epsilon)$. In the meantime, denote by $T^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ the class of all signed Borel measures μ on \mathbb{R}^{n+1} obeying

$$\|\mu\|_{T_{p,q,\alpha}^\infty} := \sup_{O \subset \mathbb{R}_+^{n+1}} \frac{|\mu|(T(O))}{\text{cap}(O; \dot{\Lambda}_\alpha^{p,q})},$$

where the supremum is taken over all open sets $O \subset \mathbb{R}_+^{n+1}$.

Theorem 5. *The pairing*

$$\langle F, \mu \rangle_{\mathbb{R}_+^{n+1}} = \int_{\mathbb{R}_+^{n+1}} F(x, t) d\mu(x, t)$$

realizes the dual of $T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ as equivalent to the space $T^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$.

Proof. It is not hard to see that each $\mu \in T^\infty(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ induces a bounded linear functional on $T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$. In fact, for such a μ and for any $f \in T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$, we have

$$\begin{aligned} |\langle F, \mu \rangle_{\mathbb{R}_+^{n+1}}| &\leq \int_0^\infty |\mu|(\{(x, t) \in \mathbb{R}_+^{n+1} : |F(x, t)| > \lambda\}) d\lambda \\ &\leq \int_0^\infty |\mu|(T(\{x \in \mathbb{R}^n : N(F)(x) > \lambda\})) d\lambda \\ &\leq C \|\mu\|_{T_{p,q,\alpha}^\infty} \int_0^\infty \text{cap}(\{x \in \mathbb{R}^n : N(F)(x) > \lambda\}; \dot{\Lambda}_\alpha^{p,q}) d\lambda \\ &\leq C \|\mu\|_{T_{p,q,\alpha}^\infty} \|F\|_{T_{p,q,\alpha}^1}. \end{aligned}$$

For the converse, since $C_0(\mathbb{R}_+^{n+1})$ (the class of continuous functions with compact support on \mathbb{R}_+^{n+1}) is contained in $T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$, every bounded linear functional L on $T^1(\text{cap}(\cdot; \dot{\Lambda}_\alpha^{p,q}))$ with finite operator norm $\|L\|$ is given by

$$\int_{\mathbb{R}_+^{n+1}} F(y, t) d\mu(y, t), \quad F \in C_0(\mathbb{R}_+^{n+1}),$$

for some signed Borel measure μ on \mathbb{R}_+^{n+1} . However for any $G \in C_0(\mathbb{R}_+^{n+1})$, one has:

$$\begin{aligned} \left| \int_{\mathbb{R}_+^{n+1}} G d\mu \right| &\leq \int_{\mathbb{R}_+^{n+1}} |G| d|\mu| \\ &= \sup \left\{ \int_{\mathbb{R}_+^{n+1}} \Phi d\mu : \Phi \in C_0(\mathbb{R}_+^{n+1}), |\Phi| \leq |G| \right\} \\ &\leq \|L\| \sup \left\{ \|\Phi\|_{T_{p,q,\alpha}^1} : \Phi \in C_0(\mathbb{R}_+^{n+1}), |\Phi| \leq |G| \right\} \\ &\leq \|L\| \cdot \|G\|_{T_{p,q,\alpha}^1}. \end{aligned}$$

Thus, by the limit argument, we may take, for every open set $O \subset \mathbb{R}^n$, $G = 1$ on $T(O)$ and $G = 0$ on $\mathbb{R}_+^{n+1} \setminus T(O)$, and obtain:

$$|\mu|(T(O)) \leq C \|L\| \text{cap}(O; \dot{\Lambda}_\alpha^{p,q}).$$

Therefore we complete the proof.

Remarks. Theorem 5 has a root in [CMS].

REFERENCES

- [Ad1] D.R. Adams, Lectures on L^p -Potential Theory, Univ. of Umea, Dept. of Math. **2**, 1981.
- [Ad2] D.R. Adams, A note on Choquet integral with respect to Hausdorff capacity, in "Function Spaces and Applications," Lund 1986, Lecture Notes in Math. **1302**, Springer-Verlag, 1988, pp. 115-124.
- [Ad3] D.R. Adams, The classification problem for capacities associated with the Besov and Triebel-Lizorkin spaces, *PWN Polish Sci. Publ.* **22**, Banach Center Oubl., Warsaw, Poland (1989),9-24.
- [Ad4] D.R. Adams, Choquet integrals in potential theory, *Publicacions Matemàtiques* **42** (1998),3-66.
- [AdBa] D. R. Adams and R. J. Bagby, Translation-Dilation invariant estimates for Riesz potentials, *Indiana Univ. Math. J.* **23** (1974), 1051-1067.
- [BeCaPa] A. Benedek, A.P. Calderón and R. Panzone, Convolution operators on Banach spaces valued functions, *Proc. Nat. Acad. Sci.* **48** (1962),356-365.
- [CMS] R. R. Coifman, Y. Meyer and E. M. Stein, Some new function spaces and their applications to harmonic analysis, *J. Funct. Anal.* **62** (1985),304-335.
- [Da] B. Dahlberg, Regularity properties of Riesz potentials, *Indiana Univ. Math. J.* **28** (1979), 257-268.
- [FeSt] C. Fefferman and E.M. Stein, On some maximal inequalities, *Amer. J. Math.*, **93** (1971), 107-115.
- [He] C. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms, *J. Math. Mech.* **18** (1968),283-323.
- [Jo] R. Johnson, Application of Carleson measures to partial differential equations and Fourier multiplier problems, *Lecture Notes in Math.* 992, Springer-Verlag 1983, 16-72.
- [Ma] V.G. Maz'ya, On capacity strong type estimates for fractional norms, *Zup. Nauch. Sem. Leningrad otel. Math. Inst. Steklov (LOMI)* **70** (1977), 161-168 (Russian).
- [Me] N.G. Meyers, A theory of capacities for potentials of functions in Lebesgue spaces, *Math. Scand.* **26**(1970), 255-292.
- [St1] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
- [St2] E.M. Stein, *Harmonic analysis: real-variable method, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, New Jersey, 1993.
- [Steg] D. Stegenga, Multipliers of the Dirichlet spaces, *Illinois J. Math.* **24**(1980), 113-139.
- [W] Z. Wu, Strong type estimate and Carleson measures for Lipschitz spaces, *Proc. Amer. Math. Soc.* **127**(1999),3243-3249.

D.R. Adams
Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506
USA
Email:dave@ms.uky.edu

J. Xiao
Department of Mathematics and Statistics
Concordia University
1455 de Maisonneuve Blvd. West
H3G 1M8 Montreal Qc, Canada
Email: jxiao@mathstat.concordia.ca