An Improved Alternating-Direction Method for a Viscous Wave Equation

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Abstract

We introduce an accurate and efficient alternating-direction method for solving a viscous wave equation which is based on a three-level, second-order correct implicit algorithm and which has a splitting error not significantly larger than the truncation error of the base method.

Key words. Viscous wave, microscale heat transfer.

AMS subject classifications. 65M06, 65M12, 65M15.

1 Introduction

Let $\Omega = [0,1]^m$, $2 \le m \le 3$, and let $\Gamma = \partial\Omega$ be its boundary; also, set $J = (0, T]$, the time interval. Consider the wave equation in the form

$$
\begin{aligned}
&\text{(a)} \quad \gamma_1 \frac{\partial u}{\partial t} + \gamma_2 \frac{\partial^2 u}{\partial t^2} - \nabla\cdot(A\nabla u) - \frac{\partial}{\partial t}\nabla\cdot(Q\nabla u) = S, \quad (\mathbf{x}, t) \in \Omega \times J,\\
&\text{(b)} \quad u = 0, \quad (\mathbf{x}, t) \in \Gamma \times J,\\
&\text{(c)} \quad u(\mathbf{x}, 0) = g_0(\mathbf{x}), \quad \frac{\partial u}{\partial t}(\mathbf{x}, 0) = g_1(\mathbf{x}), \quad \mathbf{x} \in \Omega,\ t = 0,
\end{aligned}
\tag{1.1}
$$

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where $\gamma_k = \gamma_k(\mathbf{x})$, $k = 1, 2$, are positive coefficients, S denotes a source, and $A = A(\mathbf{x})$ and $Q = Q(\mathbf{x})$ are diagonal, nonnegative diffusion tensors:

$$\begin{aligned} A = A^T, \quad \xi \cdot A\xi \geq 0, \\ Q = Q^T, \quad \xi \cdot Q\xi \geq 0, \end{aligned} \qquad \forall\, \mathbf{x} \in \Omega, \ \ \forall\, \xi \in \mathbb{R}^m. \tag{1.2}$$

Here, we have selected a homogeneous Dirichlet boundary condition for simplicity. The method to be treated below can be adjusted to accommodate other boundary conditions.

The equation (1.1) governs various physical phenomena; examples include damped acoustic waves and microscale heat transfer [1, 2, 3, 8, 9, 10, 11, 12, 13, 14]. The main goal of the paper is to introduce and analyze a three-level, improved alternating-direction method for solving (1.1) in m-dimensional space, $2 \leq m \leq 3$, which is second-order correct in time.

An outline of the paper is as follows. In §2, we present a three-level, second-order correct, implicit time-stepping algorithm, along with a second-order initialization scheme for u^1, the solution at the first time level. In §3, a standard alternating-direction procedure is introduced and then modified to obtain an efficient perturbation of the implicit algorithm. In §4, stability for both the three-level implicit algorithm and the final alternating-direction procedure is discussed.

2 Discretization Schemes

In this section, we introduce a three-level numerical algorithm for (1.1) which is a natural generalization of a standard time-stepping procedure [4, 7] for wave equations not including the time-differentiated diffusive (Sobolev) term. Then, we evaluate the truncation/consistency error and present an effective scheme for approximating the solution u^1 at the first time level.

2.1 Notation

Let $\mathbf{x_i} = (i_1 h, \ldots, i_m h)$, $0 \leq i_j \leq N = h^{-1}$ and $t^n = n\Delta t$, $n \geq 0$. Denote $f(\mathbf{x_i}, t^n)$ by $f_{\mathbf{i}}^n$; these indices will be omitted where unneeded for clarity. The difference operators

$$\overline{\partial}_t u^n = \frac{u^{n+1} - u^{n-1}}{2\Delta t} \quad \text{and} \quad \overline{\partial}_{tt} u^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} \tag{2.1}$$

approximate the first and second time derivatives, respectively, to second order in Δt. Also, for $1 \leq i_\ell \leq N-1$, let

$$\begin{aligned} \mathcal{A}_\ell u^n_{\mathbf{i}} \;=\; & \frac{1}{h^2}\Big(a_\ell(\mathbf{x}_{\mathbf{i}+\frac{1}{2}\mathbf{e}_\ell})\,[u^n(\mathbf{x}_{\mathbf{i}+\mathbf{e}_\ell}) - u^n(\mathbf{x}_{\mathbf{i}})] \\ & - a_\ell(\mathbf{x}_{\mathbf{i}-\frac{1}{2}\mathbf{e}_\ell})\,[u^n(\mathbf{x}_{\mathbf{i}}) - u^n(\mathbf{x}_{\mathbf{i}-\mathbf{e}_\ell}]\Big), \end{aligned} \tag{2.2}$$

where $\mathbf{e}_\ell$ is the unit vector in the ℓ^{th} direction, and define $\mathcal{Q}_\ell$ analogously. Then, set

$$\mathcal{A} = \sum_{\ell=1}^{m} \mathcal{A}_\ell, \qquad \mathcal{Q} = \sum_{\ell=1}^{m} \mathcal{Q}_\ell. \tag{2.3}$$

These centered second difference operators are second order correct in h approximations to $\nabla \cdot (a \nabla u)$ and $\nabla \cdot (d \nabla u)$ for smooth u. Finally, let

$$u^{n,\theta} = \theta u^{n+1} + (1-2\theta) u^n + \theta u^{n-1}. \tag{2.4}$$

2.2 The implicit time-stepping algorithm

An implicit, second-order time-stepping algorithm for (1.1) can be defined as follows. Given $v^0, \cdots, v^n$, $n \geq 1$, find v^{n+1} as the solution of

$$\gamma_1 \overline{\partial}_t v^n + \gamma_2 \overline{\partial}_{tt} v^n + \mathcal{A} v^{n,\theta} + \overline{\partial}_t (\mathcal{Q} v)^n = S^n, \quad 1 \leq i_\ell \leq N-1, \tag{2.5}$$

where $\theta \in [.25, 0.5]$, subject to the boundary condition

$$v^n = 0, \qquad \mathbf{x} \in \Gamma, \ \forall n, \tag{2.6}$$

and the initial conditions

$$v^0(\mathbf{x_i}) = g_0(\mathbf{x_i}), \qquad \mathbf{x_i} \in \Omega, \tag{2.7}$$

and

$$v^1(\mathbf{x_i}) = g_0 + \Delta t g_1 + \frac{(\Delta t)^2}{2\gamma_2}\Big[S^0 - \gamma_1 g_1 - \mathcal{A} g_0 - \mathcal{Q} g_1\Big], \tag{2.8}$$

which can be seen to be correct to order $\mathcal{O}((\Delta t)^3 + h^2(\Delta t)^2)$.

The symmetric, positive-definite algebraic system for (2.5) has the form

$$\Big(\frac{\Delta t}{2}\gamma_1 + \gamma_2 + (\Delta t)^2\theta\mathcal{A} + \frac{\Delta t}{2}\mathcal{Q}\Big)v^{n+1} = \mathbf{b}^n(v), \tag{2.9}$$

where

$$\begin{aligned}\mathbf{b}^n(v) &= (\Delta t)^2 S^n + \frac{\gamma_1 \Delta t}{2}v^{n-1} + \gamma_2(2v^n - v^{n-1}) \\ &\quad - (\Delta t)^2\mathcal{A}\left((1-2\theta)v^n + \theta v^{n-1}\right) + \frac{\Delta t}{2}\mathcal{Q}v^{n-1}.\end{aligned} \tag{2.10}$$

It is easy to see that the truncation error in (2.5) is $\mathcal{O}((\Delta t)^2 + h^2)$. A modest modification of the argument given in [7] shows that, for $.25 \le \theta \le .5$, the solution of (2.5) converges to that of (1.1) with a global rate of convergence given by $\mathcal{O}(h^2 + (\Delta t)^2)$.

3 The Alternating-Direction Procedure

In this section we first introduce a standard [5] alternating-direction perturbation of (2.5) and then define a modification of this procedure that reduces significantly the so-called splitting error, as was done in [6] for parabolic and simpler hyperbolic problems. Then, we note that the improved algorithm can be obtained in a simpler fashion.

3.1 The standard alternating-direction algorithm

The standard alternating-direction method [5] is defined as follows. Given $w^k,\ k \le n$, determine $w^{n+1,1}, \ldots, w^{n+1,m} = w^{n+1}$ recursively as the solutions of

$$\left(\frac{1}{2}\gamma_1\Delta t + \gamma_2\right) w^{n+1,j} + \sum_{\ell=1}^{j}\mathcal{B}_\ell w^{n+1,\ell} + \sum_{\ell=j+1}^{m}\mathcal{B}_\ell w^n = \mathbf{b}^n(w) \tag{3.1}$$

for $j = 1, \ldots, m$, where

$$\mathcal{B}_\ell = (\Delta t)^2\theta\mathcal{A}_\ell + \frac{1}{2}\Delta t\mathcal{Q}_\ell. \tag{3.2}$$

Equivalently, find $w^{n+1,1}$ by (3.1) and then find $w^{n+1,j},\ j \ge 2$, as the solutions of

$$\left(\frac{1}{2}\gamma_1\Delta t + \gamma_2\right)(w^{n+1,j} - w^{n+1,j-1}) + \mathcal{B}_j(w^{n+1,j} - w^n) = 0. \tag{3.3}$$

Note that in each substep of the algorithm specified by (3.1) or (3.1)+(3.3), the algebraic equations to be solved are tridiagonal; thus, the number of floating point operations required to complete a time step is proportional to h^{-m}, the minimal order with respect to h.

For the purposes of analysis, it is desirable to eliminate the intermediate solutions $w^{n+1,j},\ j = 1, \dots, m-1$ [5, 6]. Set

$$\zeta = \left(\frac{1}{2}\gamma_1 \Delta t + \gamma_2\right)^{-1},$$

and rewrite (3.3) as

$$w^{n+1,j-1} = w^{n+1,j} + \zeta \mathcal{B}_j (w^{n+1,j} - w^n). \tag{3.4}$$

Then,

$$w^{n+1,m-1} = w^{n+1} + \zeta \mathcal{B}_m (w^{n+1} - w^n), \tag{3.5}$$

and, if $m = 3$,

$$\begin{aligned} w^{n+1,1} &= w^{n+1} + \zeta(\mathcal{B}_2 + \mathcal{B}_3)(w^{n+1} - w^n) \\ &\quad + \zeta^2 \mathcal{B}_2 \mathcal{B}_3 (w^{n+1} - w^n). \end{aligned} \tag{3.6}$$

A bit of manipulation leads to the recursion relation

$$\gamma_1 \overline{\partial}_t w^n + \gamma_2 \overline{\partial}_{tt} w^n + \mathcal{A} w^{n,\theta} + \overline{\partial}_t (\mathcal{Q} w)^n + \mathcal{P}(w^{n+1} - w^n) = S^n, \tag{3.7}$$

where

$$\mathcal{P} = \zeta (\Delta t)^{-2} \mathcal{B}_1 \mathcal{B}_2 = \frac{1}{4} \zeta (2 \Delta t \mathcal{A}_1 + \mathcal{Q}_1)(\Delta t \mathcal{A}_2 + \mathcal{Q}_2), \quad m = 2, \tag{3.8}$$

or

$$\mathcal{P} = \zeta (\Delta t)^{-2} \left(\sum_{1 \le i < j \le 3} \mathcal{B}_i \mathcal{B}_j + \zeta \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_3 \right), \quad m = 3. \tag{3.9}$$

Thus, if $m = 2$ or 3, when $\mathcal{Q}$ is nontrivial (*i.e.*, when $d_\ell \ge d^* > 0,\ \ell = 1, \dots, m$), then the splitting error term

$$\mathcal{P}(w^{n+1} - w^n) = \mathcal{O}(\Delta t) \tag{3.10}$$

and can be expected to dominate the underlying second order truncation error associated with (2.5). In the case of acoustic waves ($\mathcal{Q} = 0$), the splitting error is $\mathcal{O}((\Delta t)^3)$ and should produce a solution only slightly differing from that of the standard alternating-direction method; however, the improved algorithm (3.13)-(3.14) should be a bit better than (3.1) and costs very little more in computational effort.

3.2 Improvement by modification of the right-hand side

In [6], Douglas and Kim modified the right-hand sides of standard alternating-direction procedures in order to replace the first difference in (3.10) with a higher order time difference. This choice is available to us here; let the equation for $w^{n+1,1}$ be changed to

$$\left(\frac{1}{2}\gamma_1 \Delta t + \gamma_2\right) w^{n+1,1} + \mathcal{B}_1 w^{n+1,1} + \sum_{\ell=2}^{m} \mathcal{B}_\ell w^n = \mathbf{b}^n(w) + \mathcal{P}(w^n - w^{n-1}), \tag{3.11}$$

while followed by (3.3) for evaluating $w^{n+1,2}, \ldots, w^m$. Eliminating the intermediate solutions now gives the recursion relation

$$\gamma_1 \overline{\partial}_t w^n + \gamma_2 \overline{\partial}_{tt} w^n + \mathcal{A} w^{n,\theta} + \overline{\partial}_t (\mathcal{Q} w)^n + (\Delta t)^2 \mathcal{P} \overline{\partial}_{tt} w^n = S^n. \tag{3.12}$$

An alternate, and slightly cheaper way, to obtain (3.12) is described immediately below.

3.3 The recommended improved procedure

In (3.1), w^{n+1} was approximated by w^n in the discretization of the spatial derivatives for $\ell > j$ in the j^{th} fractional step. If, instead, it is approximated by $2w^n - w^{n-1}$, then the splitting error term should be of second order in Δt. Indeed, this is the case. Let

$$\left(\frac{1}{2}\gamma_1 \Delta t + \gamma_2\right) w^{n+1,1} + \mathcal{B}_1 w^{n+1,1} + \sum_{\ell=2}^{m} \mathcal{B}_\ell (2w^n - w^{n-1}) = \mathbf{b}^n(w), \tag{3.13}$$

followed for $j = 2, \ldots, m$ by

$$\left(\frac{1}{2}\gamma_1 \Delta t + \gamma_2\right) (w^{n+1,j} - w^{n+1,j-1}) + \mathcal{B}_j (w^{n+1,j} - 2w^n + w^{n-1}) = 0. \tag{3.14}$$

Then, it follows easily that (3.7) is again replaced by

$$\gamma_1 \overline{\partial}_t w^n + \gamma_2 \overline{\partial}_{tt} w^n + \mathcal{A} w^{n,\theta} + \overline{\partial}_t (\mathcal{Q} w)^n + (\Delta t)^2 \mathcal{P} \overline{\partial}_{tt} w^n = S^n, \tag{3.15}$$

so that the splitting error becomes $\mathcal{O}((\Delta t)^2)$, which is consistent with the truncation error of (2.5). In the acoustic wave case, the splitting error is $\mathcal{O}((\Delta t)^4)$.

4 Stability and Convergence

The stability of the three-level implicit algorithm (2.5) can be established by a fairly simple extension of the stability proof by Dupont [7] when $\mathcal{Q} = 0$. Our alternating-direction procedure corresponds to a difference approximation to a differential equation of the form

$$\frac{\partial}{\partial t}\left(\gamma_1 u - \nabla \cdot (d\nabla u)\right) + \frac{\partial^2}{\partial t^2}\left(\gamma_2 u - \mathcal{D}u\right) - \nabla \cdot (a\nabla u) = S. \tag{4.1}$$

Extending Dupont's proof to (4.1) and (3.15) directly is blocked by the $\mathcal{D}$-term. However, if we assume that the entries in

$$\{\mathcal{A}_1, \mathcal{Q}_1, \ldots, \mathcal{Q}_m\}$$

are pairwise commutative, then both Dupont's argument and one based on spectral analysis apply, since then $\mathcal{D}$ is positive definite. This condition is essentially equivalent to requiring the coefficients a_ℓ and d_ℓ for $\ell = 1, \ldots, m$ to be constants. In that case, the stability of (3.12) follows, so that the solution of (3.12) converges to that of (1.1) with an error bounded by $\mathcal{O}(h^2 + (\Delta t)^2)$ without the necessity of imposing a CFL constraint.

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