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Lower Bounds For the Condition Number of a Real Confluent Vandermonde Matrix[1]

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ABSTRACT

Lower bounds on the condition number $\kappa_p(V_{\rm c})$ of a real confluent Vandermonde matrix $V_{\rm c}$ are established in terms of the dimension n or n and the largest absolute value among all nodes that define the confluent Vandermonde matrix or n and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\max}$ (the largest number of equal nodes), $\kappa_p(V_{\rm c})$ behaves no smaller than $\mathcal{O}_n((1+\sqrt{2}\,)^n)$, and than $\mathcal{O}_n((1+\sqrt{2}\,)^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\max}$.

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[1] This report is available on the web at `http://www.ms.uky.edu/~math/MAreport/PDF/2004-10.pdf`.

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Lower Bounds For the Condition Number of a Real Confluent Vandermonde Matrix

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Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix V_c are established in terms of the dimension n or n and the largest absolute value among all nodes that define the confluent Vandermonde matrix or n and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\max}$ (the largest number of equal nodes), $\kappa_p(V_c)$ behaves no smaller than $\mathcal{O}_n((1+\sqrt{2})^n)$, and than $\mathcal{O}_n((1+\sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\max}$.

1 Introduction

Given n numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ called *nodes*, the associated *Vandermonde Matrix* is defined as

$$V \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}. \tag{1.1}$$

It, for example, arises from polynomial interpolation and others [4]. V is invertible if all nodes α_j are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of V for nodes not all of which are distinct is the so-called *Confluent Vandermonde Matrices*, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by "differentiating" the previous column. Confluent Vandermonde matrices arises in Hermite

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interpolation [3], for example. Adopting the formulation in [8], we define *confluent Vandermonde matrix* $V_{\rm c}$ as follows. First

$$\boxed{\begin{array}{l} \{\alpha_j\}_{j=1}^n \text{ are ordered so that equal nodes are contiguous, i.e.,} \\[1em] \qquad\qquad \alpha_i = \alpha_j \quad (i<j) \quad \Rightarrow \quad \alpha_i = \alpha_{i+1} = \cdots = \alpha_j. \end{array}} \tag{1.2}$$

Define

$$V_{\rm c} = (f_1(\alpha_1)\ f_2(\alpha_2)\ \cdots\ f_n(\alpha_n)), \tag{1.3}$$

where vector function $f_j(t)$ is defined recursively by

$$f_j(t) = \begin{cases} (1\ t\ \cdots\ t^{n-1})^T, & \text{if } j=1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\ \frac{d}{dx} f_{j-1}(t), & \text{otherwise,} \end{cases} \tag{1.4}$$

where "$\cdot^T$" is the transpose of a vector or matrix. As far as defining $V_{\rm c}$ is concerned, α_j can be real or complex. But in this paper, we shall focus on real α_j. In what follows, α_j and $V_{\rm c}$, as well as $\alpha \overset{\rm def}{=} \max_j |\alpha_j|$, are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his co-author (see [7] and references therein), and more recently by Tyrtyshnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish several lower bounds on the ℓ_p-condition number $\kappa_p(V_{\rm c}) \equiv \|V_{\rm c}\|_p \|V_{\rm c}^{-1}\|_p$ in terms of n or n and α or n and the interval $[a,b]$ that contains all nodes. In particular, we will show that for fixed $k_{\max}$ (the largest number of equal nodes), $\kappa_p(V_{\rm c})$ behaves no smaller than $\mathcal{O}_n((1+\sqrt{2}\,)^n)$, where notation $a_n = \mathcal{O}_n(b_n)$ means $c_1 n^{d_1} \leq a_n/b_n \leq c_2 n^{d_2}$ for some constants c_1, c_2, d_1, and d_2. We also obtain a qualitative plot in Figure 1.1 to show how our lower bounds on $\min_{\alpha_j} \kappa_p(V_{\rm c})$ and $\min_{\alpha_j \geq 0} \kappa_p(V_{\rm c})$ behave qualitatively as functions of α. What Figure 1.1 says that initially as α increases, our lower bound for $\min_{\alpha_j} \kappa_p(V_{\rm c})$ and that for $\min_{\alpha_j \geq 0} \kappa_p(V_{\rm c})$ decrease until at $\alpha = \alpha_{\rm opt}$ when global minimums of the bounds are reached, and then they start climbing again. Notice $\alpha_{\rm opt}$ may be different for the two cases, but $\alpha_{\rm opt} = \mathcal{O}(1)$ in both cases. What that is not clear, however, is how sharp our lower bounds are, in contrast to many of those in [10] for Vandermonde matrices that were proved to be asymptotically optimal.

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which $V_{\rm c}$ is lower triangular and thus

$$\kappa_p(V_{\rm c}) \geq (n-1)! \sim \sqrt{2\pi}\, n^{n-1/2}\, e^{-n}$$

by Stirling's asymptotic formula [1, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to ∞ as fast as $(1+\sqrt{2}\,)^n$ modulo a factor n^d for $d \leq 1$ [2, 10].

The rest of this paper is organized as follows. Section 2 reviews some preliminary results from [10] in connection to the coefficients of translated Chebyshev polynomials. Two general lower bounds on $\kappa_p(V_{\rm c})$ are established in Section 3, but they are not uniform. Uniform bounds for $p = \infty$ are obtained in Section 4 for all real $V_{\rm c}$ and in Section 5 for $V_{\rm c}$ with nonnegative nodes.

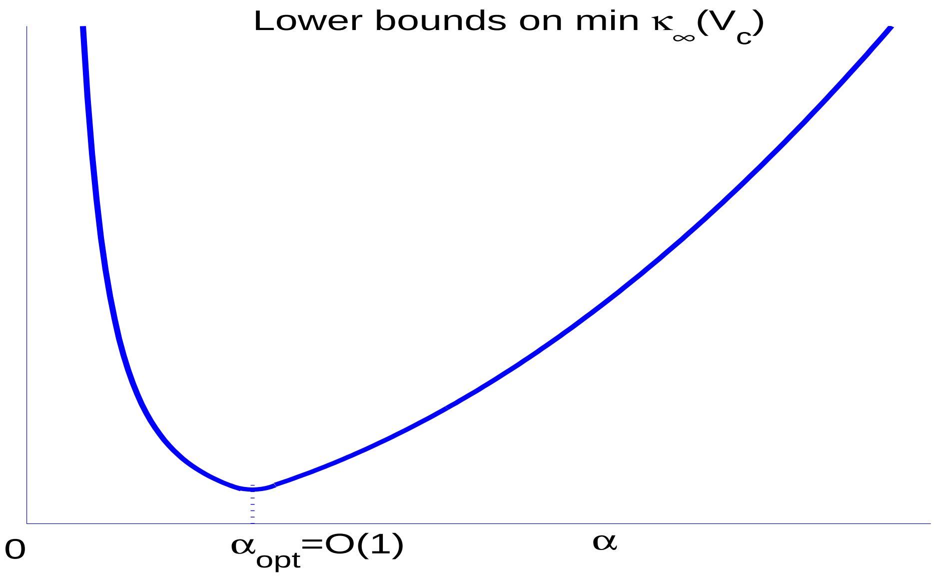


Figure 1.1: Qualitative behaviors of our lower bounds on $\min_{\alpha_j} \kappa_p(V_c)$ and $\min_{\alpha_j \geq 0} \kappa_p(V_c)$ (subject to fixed α) as α varies for any given $k_{\max}$.

2 Preliminaries

Let us start by briefly reviewing relevant notation and results from [10]. Let $[a, b]$ be the interval in which lie all α_j. $T_n(t) = \cos(n \arccos t)$ is the nth Chebyshev polynomial of the first kind. Define the nth translated Chebyshev polynomial $T_n(x; \omega, \tau) \overset{\text{def}}{=} T_n(x/\omega + \tau)$, where

$$\omega = \frac{b-a}{2}, \quad \tau = -\frac{b+a}{b-a}.$$

Let $a_{jn} \equiv a_{jn}(\omega, \tau)$ be the coefficient of x^j in $T_n(x; \omega, \tau)$, i.e.,

$$T_n(x; \omega, \tau) = a_{nn} x^n + a_{n-1\,n} x^{n-1} + \cdots + a_{1n} x + a_{0n}, \tag{2.1}$$

Let $1 \leq p \leq \infty$. In [10], $S_{n,p}(\omega, \tau)$ is defined as

$$S_{n,p}(\omega, \tau) = \left(\sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}.$$

Also explicit formulas were found for $p = 1$ and $-a = b$ (for which $\omega = 0$):

$$\begin{aligned} S_{n,1}(\omega, 0) &= \frac{1}{2} \left[\left(\frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n + (-1)^n \left(\frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^{-n} \right], & (2.2) \\ &\sim \frac{1}{2} \left(\frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n. & (2.3) \end{aligned}$$

and for $a \geq 0$ (for which $\tau \leq -1$):

$$\begin{aligned} S_{n,1}(\omega,\tau) &= \frac{1}{2}\left[\left(\frac{1}{|\omega|}+|\tau|\right)+\sqrt{\left(\frac{1}{|\omega|}+|\tau|\right)^2-1}\right]^n \\ &\quad +\frac{1}{2}\left[\left(\frac{1}{|\omega|}+|\tau|\right)+\sqrt{\left(\frac{1}{|\omega|}+|\tau|\right)^2-1}\right]^{-n}, \qquad (2.4) \\ &\sim \frac{1}{2}\left[\left(\frac{1}{|\omega|}+|\tau|\right)+\sqrt{\left(\frac{1}{|\omega|}+|\tau|\right)^2-1}\right]^n. \qquad (2.5) \end{aligned}$$

For any other p, $S_{n,p}(\omega,\tau)$ relates to $S_{n,1}(\omega,\tau)$ by

$$(n+1)^{-1/p'} S_{n,1}(\omega,\tau) \leq \ S_{n,p}(\omega,\tau) \ \leq S_{n,1}(\omega,\tau), \qquad (2.6)$$

$$\lceil (n+1)/2 \rceil^{-1/p'} S_{n,1}(\omega,0) \leq \ S_{n,p}(\omega,0) \ \leq S_{n,1}(\omega,0), \qquad (2.7)$$

where $1/p+1/p'=1$, and $\lceil \xi \rceil$ is the smallest integer that is larger than ξ.

The ℓ_p-norm of vector $u=(\mu_1 \ \mu_2 \ \cdots \mu_n)^T$ is defined as

$$\|u\|_p = \left(\sum_{j=1}^n |\mu_j|^p\right)^{1/p},$$

and $\|u\|_\infty = \lim_{p\to\infty}\|u\|_p = \max_j |\mu_j|$. The associated ℓ_p-operator norm of $m \times n$ matrix A is defined as

$$\|A\|_p = \max_{u\neq 0} \frac{\|Au\|_p}{\|u\|_p}. \qquad (2.8)$$

It can be proved that [9] $\|A\|_p = \|A^T\|_{p'}$, and [10]

$$n^{-2/p}\kappa_p(V_{\rm c}) \leq \kappa_\infty(V_{\rm c}) \leq n^{2/p}\kappa_p(V_{\rm c}) \qquad (2.9)$$

which is useful in deriving bounds on $\kappa_p(V_{\rm c})$ from these for $\kappa_\infty(V_{\rm c})$ as in Sections 4 and 5.

3 Lower bounds on $\kappa_p(V_{\rm c})$

For the sake of presentation, we assume, in addition to (1.2),

> There are ℓ distinct nodes α_j, having multiplicities $k_1, k_2, \ldots, k_\ell$, respectively, where $k_1+k_2+\cdots+k_\ell = n$. This implies that the first k_1 α_j's are equal, the next k_2 α_j's are equal, and so on. Define $k_{\max} = \max_j k_j$. (3.1)

Lemma 3.1 *Assume* (1.2) *and* (3.1). *Then*

$$\|V_{\rm c}\|_p \ \geq \ \max\left\{\ell^{1/p'}, \alpha^{n-1}\right\}, \qquad (3.2)$$

$$\|V_{\rm c}\|_p \ \geq \ \left(\sum_{j=1}^n \alpha^{(j-1)p}\right)^{1/p}. \qquad (3.3)$$

Proof: Let e_j be the jth column of the $n \times n$ identity matrix I_n (or simply I if n is clear from the context). Use $\|V_{\rm c}\|_p \geq \|V_{\rm c}^T e_1\|_{p'}$ and $\|V_{\rm c}\|_p \geq \|V_{\rm c}^T e_n\|_{p'}$ to get (3.2), and use $\|V_{\rm c}\|_p \geq \max_j \|V_{\rm c}^T e_j\|_p$ to get (3.3). ∎

Lemma 3.2 *For* $0 \leq k \leq n$,

$$\left|\frac{d}{dx^k} T_n(x;\omega,\tau)\right| \leq \frac{[n(n-1)\cdots(n-k+1)]^2}{\omega^k} \quad \textit{for } x \in [a,b]. \tag{3.4}$$

Proof: It follows from $T_n(x;\omega,\tau) = T_n(x/\omega+\tau) \equiv T_n(t)$ that

$$\frac{d^k}{dx^k} T_n(x;\omega,\tau) = \frac{1}{\omega^k} T_n^{(k)}(t),$$

where $t \equiv t(x) = x/\omega + \tau$. It suffices to show that $|T_n^{(k)}(t)| \leq [n(n-1)\cdots(n-k+1)]^2$ for $t \in [-1,1]$ since $t(x)$ maps $x \in [a,b]$ to $t \in [-1,1]$. By Markov's Inequality [5, Page 233],

$$\begin{aligned}
\max_{t\in[-1,1]} |T_n^{(k)}(t)| &\leq (n-k+1)^2 \max_{t\in[-1,1]} |T_n^{(k-1)}(t)| \\
&\leq \cdots \\
&\leq [n(n-1)\cdots(n-k+1)]^2 \max_{t\in[-1,1]} |T_n(t)| \\
&= [n(n-1)\cdots(n-k+1)]^2,
\end{aligned}$$

as expected. ∎

Lemma 3.3 *Assume* (1.2) *and* (3.1). *Then*

$$\|V_{\rm c}^{-1}\|_p \geq \min_{1\leq k\leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}. \tag{3.5}$$

Proof: Let v be the vector of the coefficients of the translated Chebyshev polynomial $T_{n-1}(x;\omega,\tau)$, i.e., $v = (a_{0\,n-1} \ a_{1\,n-1} \ \cdots \ a_{n-1\,n-1})^T$. Then

$$V_{\rm c}^T v = (T_{n-1}(\alpha_1;\omega,\tau) \ T'_{n-1}(\alpha_1;\omega) \ \cdots \ T_{n-1}^{(k_1-1)}(\alpha_1;\omega) \ \cdots \ \cdots)^T$$

which yields, by Lemma 3.2, for $1 \leq p' < \infty$

$$\|V_{\rm c}^T v\|_{p'}^{p'} \leq \sum_{j=1}^{\ell} \left(1^{p'} + \left[\frac{(n-1)^2}{\omega}\right]^{p'} + \cdots + \left[\frac{[(n-1)(n-2)\cdots(n-k_j+1)]^2}{\omega^{k_j-1}}\right]^{p'}\right) \tag{3.6}$$

$$\leq \sum_{j=1}^{\ell} k_j \times \left(\max_{1\leq k\leq k_j} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}}\right)^{p'}$$

$$\leq n \times \left(\max_{1\leq k\leq k_{\max}} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}}\right)^{p'} \tag{3.7}$$

which gives

$$\|V_{\rm c}^T v\|_{p'} \le n^{1/p'} \times \max_{1\le k\le k_{\max}} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}}. \tag{3.8}$$

This is proved so far for $1 \le p' < \infty$, but it can be verified that (3.8) holds for $p' = \infty$, too. We therefore have

$$\begin{aligned} \|V_{\rm c}^{-T}\|_{p'} &= \max_{u\in\mathbb{R}^n} \frac{\|u\|_{p'}}{\|V_{\rm c}^T u\|_{p'}} \ge \frac{\|v\|_{p'}}{\|V_{\rm c}^T v\|_{p'}} \\ &\ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}, \end{aligned}$$

as was to be shown. ∎

Theorem 3.1 *Assume* (1.2) *and* (3.1)*. Then*

$$\kappa_p(V_{\rm c}) \ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \max\{\ell^{1/p'}, \alpha^{n-1}\} \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}. \tag{3.9}$$

Let $\omega = \eta\alpha$*. Then*

$$\kappa_p(V_{\rm c}) \ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \frac{S_{n-1,1}(\eta,\tau)}{n^{1/p'}}. \tag{3.10}$$

Proof: Inequality (3.9) is a consequence of Lemmas 3.1 and 3.3. We now prove (3.10). Since $a_{jn}(\omega,\tau) = \omega^{-j}\, a_{jn}(1,\tau)$ [10], $S_{n-1,p'} = \left(\sum_{j=0}^{n-1} |\omega|^{-jp'} |a_{j\,n-1}(1,\tau)|^{p'}\right)^{1/p'}$. Therefore it follows from (3.3) and (3.5) that

$$\begin{aligned} & n^{1/p'} \max_{1\le k\le k_{\max}} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}} \times \kappa_p(V_{\rm c}) \\ \ge\ & \left(\sum_{j=0}^{n-1} \alpha^{jp}\right)^{1/p} \left(\sum_{j=0}^{n-1} |\omega|^{-jp'} |a_{j\,n-1}(1,\tau)|^{p'}\right)^{1/p'} \\ \ge\ & \sum_{j=0}^{n-1} |\eta|^{-j} |a_{j\,n-1}(1,\tau)| \\ =\ & S_{n-1,1}(\eta,\tau), \end{aligned}$$

by Hölder inequality. This yields (3.10). ∎

For $k_{\max} = 1$, i.e., $\ell = n$ and $k_1 = s = k_n = 1$ (and thus $V_{\rm c} = V$), (3.9) becomes one of the lower bounds for $\kappa_p(V)$ in [10]. In general, we may also use (3.6), instead of (3.7), in estimating $\|V_{\rm c}^{-1}\|_p$. Doing so will lead to a more complicated lower bound on $\kappa_p(V_{\rm c})$.

Corollary 3.1 *Assume* (1.2) *and* (3.1)*.*

$$\kappa_p(V_{\rm c}) \ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \alpha^{k-1} \times \frac{S_{n-1,1}(1,0)}{n^{1/p'}}. \tag{3.11}$$

If, in addition, all $\alpha_j \ge 0$*, then*

$$\kappa_p(V_{\rm c}) \ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \left[\frac{\alpha}{2}\right]^{k-1} \times \frac{S_{n-1,1}(1/2,1)}{n^{1/p'}}. \tag{3.12}$$

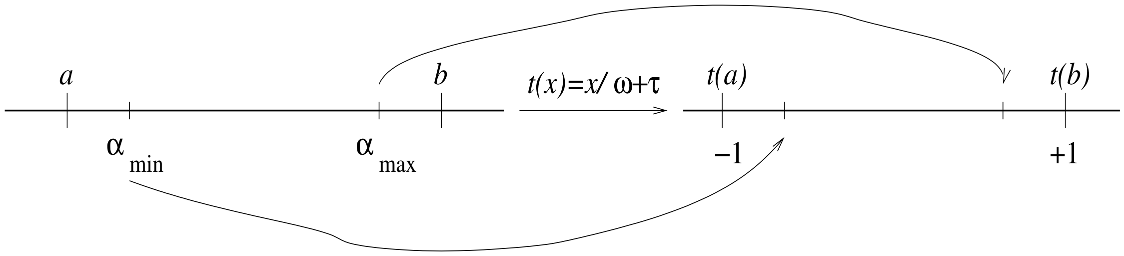
Proof: Apply (3.10) to $[a,b]=[-\alpha,\alpha]$ (and thus $\eta=1$, $\tau=0$, and $\omega=\alpha$) to get (3.11). Apply (3.10) to $[a,b]=[0,\alpha]$ (and thus $\eta=1/2$, $\tau=-1$, and $\omega=\alpha/2$) to get (3.12). ∎

Neither bounds in (3.11) and (3.12) are uniform, because both depend on α. They do not yield useful lower bounds on $\min_{\alpha_j}\kappa_p(V_c)$ or $\min_{\alpha_j\ge 0}\kappa_p(V_c)$. In fact, the minimums of both right-hand sides of (3.11) and (3.12) over either all $\alpha_j\in\mathbb{R}$ or all $\alpha_j\ge 0$ are zero! In the next two sections, we shall establish two uniform bounds using (3.9) for all real V_c and for those with nonnegative nodes.

Remark **3.1** Lemma 3.3 is made possible by Lemma 3.2 which was proved with the help of Markov's Inequality. Another classical inequality for the same purpose is Bernstein's Inequality [5, Page 233], using which we can obtain the following: *For* $0\le k\le n$*, if* $a<\alpha_{\min}<\alpha_{\max}<b$*, then*

$$\left|\frac{d}{dx^k}T_n(x;\omega,\tau)\right| \le \frac{n(n-1)s(n-k+1)}{\left[\omega\sqrt{1-\left(\frac{\max\{b-\alpha_{\max},\,\alpha_{\min}-a\}}{\omega}\right)^2}\right]^k} \quad \textit{for } x\in[a,b], \tag{3.13}$$

where $\alpha_{\max}\equiv\max_j\alpha_j$ *and* $\alpha_{\min}\equiv\min_j\alpha_j$*.* This inequality improves (3.4) in the numerator part, but has complications in the denominator, and also it requires the interval $[a,b]$ be (slightly) larger than the smallest interval containing all nodes, as follows.



This can be bad because larger $[a,b]$ will weaken the effectiveness of $S_{n,p'}(\omega,\tau)$ in the later bounds on $\kappa_p(V_c)$, for example $S_{n,p'}(\omega,\tau)$ is decreasing in ω [10].

4 Uniform bounds for all real V_c

We'll restrict ourselves to $p=\infty$ because the availability of formulas for $S_{n,1}$ in Section 2 that allow us to do analysis below. Equivalent relation (2.9) makes it possible to derive uniform lower bounds on $\kappa_p(V_c)$ for $p\ne\infty$.

Let Φ be the right-hand side of (3.9) with $-a=b=\alpha$ for $p=\infty$ (and thus $p'=1$):

$$\Phi=\min_{1\le k\le k_{\max}}\left[\frac{(n-k)!}{(n-1)!}\right]^2\alpha^{k-1}\times\max\{\ell,\alpha^{n-1}\}\frac{S_{n-1,1}(\alpha,0)}{n}\equiv\max\{\Phi_1,\Phi_2\},$$

where

$$\Phi_1=\min_{1\le k\le k_{\max}}\left[\frac{(n-k)!}{(n-1)!}\right]^2\alpha^{k-1}\times\frac{\ell\,S_{n-1,1}(\alpha,0)}{n}, \tag{4.1}$$

$$\Phi_2=\min_{1\le k\le k_{\max}}\left[\frac{(n-k)!}{(n-1)!}\right]^2\alpha^{k-1}\times\frac{\alpha^{n-1}\,S_{n-1,1}(\alpha,0)}{n}. \tag{4.2}$$

Φ is Φ_1 for $\alpha \leq \ell^{1/(n-1)}$ and Φ_2 for $\alpha \geq \ell^{1/(n-1)}$. Φ_2 is increasing in α for $\alpha > 0$ because $\alpha^{n-1} S_{n-1,1}(\alpha, 0)$ is increasing in α for $\alpha > 0$ [10]. Lemma 4.1 shows that Φ_1 is decreasing in α for $\alpha \leq 1$ for $k_{\max}$ approximately no bigger than $1 + (n-1)/\sqrt{2}$. Between $1 \leq \alpha \leq \ell^{1/(n-1)}$, $\Phi = \Phi_1 \geq \Phi_2$ and $\Phi_1 = \mathcal{O}_n(\Phi_2)$ because for $1 \leq \alpha \leq \ell^{1/(n-1)}$,

$$\left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{S_{n-1,1}(n^{1/(n-1)}, 0)}{n} \leq \Phi_1 \leq \frac{n\, S_{n-1,1}(1,0)}{n},$$

$$\left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{S_{n-1,1}(1,0)}{n} \leq \Phi_2 \leq \frac{n\, S_{n-1,1}(n^{1/(n-1)}, 0)}{n},$$

noting that Φ_2 is increasing in α and that $S_{n-1,1}(\alpha, 0)$ is decreasing in α. These inequalities, together with,

$$\begin{aligned} n^{1/(n-1)} &= 1 + \frac{\ln(n-1)}{n-1} + \frac{2 + \ln^2(n-1)}{2(n-1)^2} + s, \\ S_{n-1,1}(n^{1/(n-1)}, 0) &\sim \frac{(1+\sqrt{2})^{n-1}}{2\,(n-1)^{1/\sqrt{2}}} \end{aligned}$$

imply $\Phi_1 = \mathcal{O}_n(\Phi_2)$ for $1 \leq \alpha \leq \ell^{1/(n-1)}$. Therefore we have the qualitative plot in Figure 1.1 for Φ.

Lemma 4.1 *Let $k \geq 0$. $\alpha^k S_{n-1,1}(\alpha, 0)$ is decreasing in α for $\alpha \leq 1$ if*

$$k \leq \frac{n-1}{\sqrt{2}} \left[1 - (1+\sqrt{2}\,)^{-2n+2}\right] \sim \frac{n-1}{\sqrt{2}}. \tag{4.3}$$

Proof: We claim that under (4.3) $\frac{d}{d\alpha} \alpha^k S_{n-1,1}(\alpha, 0) \leq 0$ for $\alpha \leq 1$. To this end, we notice

$$\frac{d}{d\alpha} \alpha^k S_{n-1,1}(\alpha, 0) = k \alpha^{k-1} S_{n-1,1}(\alpha, 0) + \alpha^k \frac{d}{d\alpha} S_{n-1,1}(\alpha, 0).$$

Now for $\alpha \leq 1$, by (2.2), we have

$$\begin{aligned} S_{n-1,1}(\alpha, 0) &\leq \frac{1}{2}\left[\frac{1}{\alpha} + \sqrt{1 + \frac{1}{\alpha^2}}\right]^{n-1} \left[1 + \epsilon^{-2n+2}\right], \\ -\frac{d}{d\alpha} S_{n-1,1}(\alpha, 0) &\geq \frac{n-1}{2}\left[\frac{1}{\alpha} + \sqrt{1 + \frac{1}{\alpha^2}}\right]^{n-2} \left[1 - \delta^{-2n+2}\right] \\ &\quad \times \left[\frac{1}{\alpha^2} + \frac{1}{\alpha^2\sqrt{1+\alpha^2}}\right], \end{aligned}$$

where $\epsilon = 1 + \sqrt{2}$ and $\delta = 0$ for even $n-1$, and $\epsilon = 0$ and $\delta = 1 + \sqrt{2}$ for odd $n-1$. Therefore for $\alpha \leq 1$

$$\begin{aligned} \frac{\frac{d}{d\alpha} \alpha^k S_{n-1,1}(\alpha, 0)}{(n-1)\alpha^{k-1} S_{n-1,1}(\alpha, 0)} &= \frac{k}{n-1} + \frac{\alpha \frac{d}{d\alpha} S_{n-1,1}(\alpha, 0)}{(n-1) S_{n-1,1}(\alpha, 0)} \\ &\leq \frac{k}{n-1} - \frac{\alpha\left[\frac{1}{\alpha^2} + \frac{1}{\alpha^2\sqrt{1+\alpha^2}}\right]}{\frac{1}{\alpha} + \sqrt{1 + \frac{1}{\alpha^2}}} \frac{1 - \delta^{-2n+2}}{1 + \epsilon^{-2n+2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{n-1} - \frac{1}{\sqrt{1+\alpha^2}}\,\frac{1-\delta^{-2n+2}}{1+\epsilon^{-2n+2}} \\
&\le \frac{k}{n-1} - \frac{1}{\sqrt{2}}\,\frac{1-\delta^{-2n+2}}{1+\epsilon^{-2n+2}} \\
&\le \frac{k}{n-1} - \frac{1}{\sqrt{2}}\left[1-(1+\sqrt{2}\,)^{-2n+2}\right] \\
&\le 0
\end{aligned}
$$

upon using (4.3). ■

Theorem 4.1 *If* (4.3) *with* $k = k_{\max} - 1$ *holds, then*

$$
\begin{aligned}
\kappa_\infty(V_{\rm c}) &\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{S_{n-1,1}(1,0)}{n} \\
&\sim \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{\left[1+\sqrt{2}\right]^{n-1}}{n}.
\end{aligned}
$$

Proof: It can be verified that $\Phi \ge \Phi_2|_{\alpha=1}$. ■

5 Uniform bounds for $V_{\rm c}$ with $\alpha_i \in [a,b]$ and $0 = a < b$

Let Ψ be the right-hand side of (3.9) with $0 = a < b = \alpha$ for $p = \infty$ (and thus $p' = 1$):

$$
\Psi = \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \left[\frac{\alpha}{2}\right]^{k-1} \times \max\{\ell, \alpha^{n-1}\} \frac{S_{n-1,1}(\alpha/2,1)}{n} = \max\{\Psi_1, \Psi_2\},
$$

where

$$
\begin{aligned}
\Psi_1 &= \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \left[\frac{\alpha}{2}\right]^{k-1} \times \frac{\ell\, S_{n-1,1}(\alpha/2,1)}{n}, && (5.1)\\
\Psi_2 &= \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \left[\frac{\alpha}{2}\right]^{k-1} \times \frac{\alpha^{n-1}\, S_{n-1,1}(\alpha/2,1)}{n}. && (5.2)
\end{aligned}
$$

Ψ is Ψ_1 for $\alpha \le \ell^{1/(n-1)}$ and Ψ_2 for $\alpha \ge \ell^{1/(n-1)}$. Ψ_2 is increasing in α for $\alpha > 0$ because $\alpha^{n-1} S_{n-1,1}(\alpha/2,1)$ is increasing in α for $\alpha > 0$ [10]. Lemma 5.1 shows that Ψ_1 is decreasing in α for $\alpha \le 1$ for $k_{\max}$ approximately no bigger than $1 + (n-1)/\sqrt{2}$. Between $1 \le \alpha \le \ell^{1/(n-1)}$, $\Psi = \Psi_1 \ge \Psi_2$ and $\Psi_1 = \mathcal{O}_n(\Psi_2)$ because for $1 \le \alpha \le \ell^{1/(n-1)}$,

$$
\begin{aligned}
\left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{2^{k_{\max}-1}} \frac{S_{n-1,1}(n^{1/(n-1)}/2,1)}{n} &\le \Psi_1 \le \frac{n\, S_{n-1,1}(1/2,1)}{n}, \\
\left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{2^{k_{\max}-1}} \frac{S_{n-1,1}(1/2,1)}{n} &\le \Psi_2 \le \frac{n\, S_{n-1,1}(n^{1/(n-1)}/2,1)}{n},
\end{aligned}
$$

noting that Ψ_2 is increasing in α and that $S_{n-1,1}(\alpha/2,1)$ is decreasing in α. These inequalities, together with

$$
\begin{aligned}
n^{1/(n-1)} &= 1 + \frac{\ln(n-1)}{n-1} + \frac{2+\ln^2(n-1)}{2(n-1)^2} + s, \\
S_{n-1,1}(n^{1/(n-1)}/2,1) &\sim \frac{(3+2\sqrt{2})^{n-1}}{2(n-1)^{1/\sqrt{2}}},
\end{aligned}
$$

imply $\Psi_1 = \mathcal{O}_n(\Psi_2)$ for $1 \le \alpha \le \ell^{1/(n-1)}$. Therefore we have the qualitative plot in Figure 1.1 for Ψ.

Lemma 5.1 *Let $k \ge 0$. $\alpha^k S_{n-1,1}(\alpha/2,1)$ is decreasing in α for $\alpha \le 1$ if*

$$k \le \frac{n-1}{\sqrt{2}}\left[1-(1+\sqrt{2}\,)^{-4(n-1)}\right]^{-1} \sim \frac{n-1}{\sqrt{2}}. \tag{5.3}$$

Proof: We claim that under (5.3) $\frac{d}{d\alpha}\alpha^k S_{n-1,1}(\alpha/2,1) \le 0$ for $\alpha \le 1$. To this end, we notice

$$\frac{d}{d\alpha}\alpha^k S_{n-1,1}(\alpha/2,1) = k\alpha^{k-1}S_{n-1,1}(\alpha/2,1) + \alpha^k\frac{d}{d\alpha}S_{n-1,1}(\alpha/2,1).$$

Now for $\alpha \le 1$, by (2.4), we have

$$\begin{aligned}
S_{n-1,1}(\alpha/2,1) &\le \frac{1}{2}\left[\frac{2}{\alpha}+1+\sqrt{\left(\frac{2}{\alpha}+1\right)^2-1}\,\right]^{n-1}\left[1+(3+2\sqrt{2}\,)^{-2n+2}\right],\\
-\frac{d}{d\alpha}S_{n-1,1}(\alpha/2,1) &\ge \frac{n-1}{2}\left[\frac{2}{\alpha}+1+\sqrt{\left(\frac{2}{\alpha}+1\right)^2-1}\,\right]^{n-2}\\
&\quad\times\frac{2}{\alpha^2}\left[1+\frac{2+\alpha}{2\sqrt{1+\alpha}}\right].
\end{aligned}$$

Therefore for $\alpha \le 1$

$$\begin{aligned}
\frac{\frac{d}{d\alpha}\alpha^k S_{n-1,1}(\alpha/2,1)}{(n-1)\alpha^{k-1}S_{n-1,1}(\alpha/2,1)} &= \frac{k}{n-1}+\frac{\alpha\frac{d}{d\alpha}S_{n-1,1}(\alpha/2,1)}{(n-1)S_{n-1,1}(\alpha/2,1)}\\
&\le \frac{k}{n-1}-\frac{\alpha\frac{2}{\alpha^2}\left[1+\frac{2+\alpha}{2\sqrt{1+\alpha}}\right]}{\frac{2}{\alpha}+1+\sqrt{\left(\frac{2}{\alpha}+1\right)^2-1}}\,\frac{1}{1+(3+2\sqrt{2}\,)^{-2n+2}}\\
&= \frac{k}{n-1}-\frac{1}{\sqrt{1+\alpha}}\left[1+(3+2\sqrt{2}\,)^{-2n+2}\right]^{-1}\\
&\le \frac{k}{n-1}-\frac{1}{\sqrt{2}}\left[1+(3+2\sqrt{2}\,)^{-2n+2}\right]^{-1}\\
&\le 0
\end{aligned}$$

upon using (5.3). ∎

Theorem 5.1 *If* (5.3) *with $k = k_{\max}-1$ holds and all $\alpha_i \ge 0$, then*

$$\begin{aligned}
\kappa_\infty(V_{\rm c}) &\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2\frac{1}{2^{k_{\max}-1}}\frac{S_{n-1,1}(1/2,1)}{n}\\
&\sim \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2\frac{1}{2^{k_{\max}-1}}\frac{\left[1+\sqrt{2}\right]^{2(n-1)}}{n}.
\end{aligned}$$

Proof: It can be verified that $\Psi \ge \Psi_2|_{\alpha=1}$. ∎

6 Conclusions

We have obtained several lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix V_c. Two of them are uniform in the sense that they depend on n, the dimension of V_c only, while the others are either functions of n and α or n and the interval $[a, b]$ that contains all α_j. These bounds grow exponentially for any fixed $k_{\max}$, much as expected. Qualitative behaviors of our general lower bound (3.9) for $-a = b = \alpha$ and for $0 = a < b = \alpha$ are plotted in Figure 1.1. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if $k_{\max}$ also grows, e.g., linearly in n. This is illustrated by the extreme example $k_{\max} = n$, as we commented in Section 1.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an tuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [6, 11]. We plan to investigate this issue in future work.

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