

Capacity and blow-up for the $3 + 1$
dimensional wave operator

by

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This paper is dedicated to the memory of
L.I. Hedberg (1935-2005)

1. Introduction.

The wave operator in Euclidean \mathbf{R}_+^{3+1} space is

$$\square = \frac{\partial^2}{\partial t^2} - \Delta$$

where Δ is the Laplacian on \mathbf{R}^3 ; $\mathbf{R}_+^{3+1} = \mathbf{R}^3 \times (0, \infty)$. In this note we begin a potential theory development for \square that is analogous to the familiar potential theories for $-\Delta$; see [AH]. Here a potential theory development means that we want to study the pointwise behavior of weak solutions to

$$\begin{cases} \square w = F(x, t), & (x, t) \in \mathbf{R}_+^{3+1}, \\ w(x, 0) = \frac{\partial}{\partial t} w(x, 0) = 0, & x \in \mathbf{R}^3 \end{cases} \quad (1)$$

where F belongs to a certain function space. And by the pointwise behavior, we mean a study of the sets where w is discontinuous and/or w becomes infinite.

Developing a potential theory based on (1) is a reasonable possibility here, because classically there is a very simple formula for the solution to (1), namely the so called “retarded potential”:

$$w(x, t) = RF(x, t) = \int_{|y| < t} F(x - y, t - |y|) |y|^{-1} dy.$$

In particular, the fundamental solution for \square is non-negative, consequently $F \geq 0$ implies $w \geq 0$. This is no longer the case for higher space dimensions, which brings in added difficulties for a corresponding development there; see [S]. We treat only the case of three space dimensions in this note.

The pointwise property we concentrate on in this note is the nature of the blow-up set; the set

$$\{(x, t) : RF(x, t) = +\infty\} \quad (2)$$

for $F \geq 0$ and belonging the mixed norm function space $L_x^{p_1} L_t^{p_2}(\mathbf{R}_+^{3+1})$, i.e. those F for which

$$\|f\|_{L_x^{p_1} L_t^{p_2}} = \left[\int_0^\infty \left(\int_{\mathbf{R}^3} |F(x, t)|^{p_1} dx \right)^{p_2/p_1} dt \right]^{1/p_2}$$

is finite. Here $1 \leq p_1, p_2 < \infty$ with the usual modification when either p_1 or p_2 is infinite. The standard method of measuring a set like (2) is with some sort of capacity based on the differential operator and the given function space. In this paper, we propose such a capacity, a wave capacity, and study its properties and its relationships to sets like (2), the “blow-up set” for the solution $w = RF$.

The reader may already be familiar with the various $L^p - L^q$ estimates for solutions to problems (1), estimates of the form

$$(*) \quad \|w\|_{L_{x,t}^q} \leq c \|F\|_{L_{x,t}^p},$$

and similar estimates for the solution to the companion problem

$$\begin{cases} \square u = 0, & \mathbf{R}_+^{3+1} \\ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x), & x \in \mathbf{R}_+^{3+1} \end{cases} \quad (3)$$

in terms of norms of the initial data f and g . Such estimates are generally used to prove existence, uniqueness, and function space regularity results for various nonlinear versions of the wave equation, e.g.

$$\begin{cases} \square u = |u|^k, & \mathbf{R}^3 \times (0, T) \\ \text{plus initial data.} \end{cases} \quad (4)$$

Here $k > 1$ and $T \leq \infty$; see [S].

However, estimates like (*) play only a minor role in this note – they always give a lower bound on our wave capacity (defined below), but they

are not in general sharp enough nor numerous enough to do much good in estimating the wave capacity. Also, they never give upper bounds on the wave capacity. But because wave capacity is translation invariant in the space variable, we can obtain a sharp a priori estimate of (*) type for radial (radial in the space variable) retarded potentials that helps give precise results for the wave capacity of a ball in 4-space. Also, certain trace type estimates are needed to give a priori estimates for the wave capacity on lower dimensional sets.

We organize our discussion as

- A. defining a wave capacity associated with operator (1) and the mixed norm Lebesgue spaces,
- B. establishing basic properties of our wave capacity
- C. computing upper and lower matching growth estimates for the wave capacity of a 4-ball, and then a 3-ball,
- D. comparing wave capacity to Hausdorff measures, especially in the case of the blow-up set (2).

A. Wave capacity.

1. Definition.

Let K be a compact subset of \mathbb{R}_+^{3+1} and set

$$D_{p_1 p_2}(K) = \inf \{ \|f\|_{L_x^{p_1} L_t^{p_2}}^{p_1 \wedge p_2} : f \geq 0 \text{ and } Rf \geq 1 \}$$

where $p_1 \wedge p_2 = \min \{p_1, p_2\}$ and $1 \leq p_1, p_2 \leq \infty$. $D_{p_1 p_2}(\cdot)$ can be extended to all subsets of \mathbb{R}_+^{3+1} in the standard way; see [AH]. It is clear now that

various a priori estimates of the type

$$\|Rf\|_{L_x^{q_1} L_t^{q_2}} \leq c \|f\|_{L_x^{p_1} L_t^{p_2}}$$

will give lower bounds for $D_{p_1 p_2}(K)$. In particular, the classical Strichartz estimate

$$\|Rf\|_{L_{x,t}^4} \leq c \|Rf\|_{L_{x,t}^{4/3}}$$

yields

$$\mathcal{L}_4(K)^{1/3} \leq c \cdot D_{4/3,4/3}(K). \quad (5)$$

(Here and throughout, the letter c will denote a constant independent of the relevant quantities – in (5), c is independent of K , in Strichartz’s estimate it is independent of F , etc.). Here \mathcal{L}_4 denotes Lebesgue measure on \mathbb{R}_+^{3+1} .

If we apply (5) to $K = Q_r(x_0, t_0) = \{(y, s) : |y - x_0| < r, |s - t_0| < r\}$, $t_0 > 0$, a “4-ball” in \mathbb{R}_+^{3+1} , then we have

$$cr^{4/3} \leq D_{4/3,4/3}(Q_r(x_0, t_0)), \quad (6)$$

for all $r > 0$. To test the sharpness of (6), consider

$$\int_{|y|<t} \chi_{Q_{2r}}(x - y, t - |y|) |y|^{-1} dy \geq \int_{|y|<r} |y|^{-1} dy = c \cdot r^2$$

since if $|x - x_0| < r$ and $|t - t_0| < r$ and $|y| < r < t_0/2$, then $|x - y - x_0| < 2r$, $|t - |y| - t_0| < 2r$ and $r < t$, i.e.

$$R\chi_{Q_{2r}} \geq cr^2 \text{ on } Q_r(x_0, t_0). \quad (7)$$

Here Q_{2r} denotes $Q_{2r}(x_0, t_0)$. Hence by the definition of wave capacity

$$D_{4/3,4/3}(Q_r) \leq \left\| \frac{\chi_{Q_{2r}}}{cr^2} \right\|_{L_x^{4/3} L_t^{4/3}}^{4/3} = cr^{4/3}, \quad (8)$$

for all $r < t_0/2$, matching (6).

This calculation sets the tone for this note. However, there are difficulties. There are not enough embeddings of the Strichartz type or otherwise to get good lower bounds, as in (6), and the simple estimates of type (7) and (8) are not sharp for some p_1, p_2 (even among those for which $p_1 = p_2$). Hence it will be necessary to resort to other devices. One such is to formulate the wave capacity in its dual mode; cf. [AH].

2. The adjoint retarded potential.

In order to formulate the wave capacity in its dual mode, we need the adjoint retarded potential of a Borel measure, denoted as $R^*\mu$. We get this by first noting that the adjoint retarded potential applied to a non-negative function is just a solution of the backwards wave equation, i.e. with Cauchy data at time $t = T > 0$. So here we denote $S_T = \{(x, t) : x \in \mathbf{R}^3, 0 < t < T\}$. Now note that if $F, G \in C_0^\infty(\mathbf{R}_+^{3+1})$, then

$$\iint_{S_T} RF \cdot G \, dx \, dt = \iint_{S_T} F \cdot R^*G \, dx \, dt$$

where

$$R^*G(x, t) = \int_{|y| < T-t} G(x - y, t + |y|) |y|^{-1} \, dy,$$

the adjoint retarded potential of G . To extend R^* to the class of Borel measures μ with compact support in S_T , we proceed as follows: consider

$$\left| \iint_{S_T} RF \, d\mu \right| \leq cT^2 \|F\|_{L_{xt}(S_T)}^\infty \cdot \|\mu\|_1$$

for all $F \in C^0(S_T)$, $\|\mu\|_1$ denoting the total variation of μ ; in this case just $\mu(K)$. Thus by the Riesz representation theorem, there is a measure ν on

S_T such that

$$\iint_{S_T} RF d\mu = \iint_{S_T} F d\nu, \quad F \geq 0.$$

Consequently, we say $R^*\mu = \nu$ on S_T .

3. The dual wave capacity.

If K is a compact set in S_T , let

$$d_{p_1, p_2}(K) = \sup \{ \|\mu\|_1 : \mu \in \mathcal{M}^+(K) \text{ and } \|R^*\mu\|_{L_x^{p'_1} L_t^{p'_2}(S_T)} \leq 1 \},$$

i.e. we are considering all those Borel measures μ supported by K and for which $R^*\mu = \nu \ll \mathcal{L}_4$ and the Radon-Nikodym derivative $d\nu/d\mathcal{L}_4$ belongs to $L_x^{p'_1} L_t^{p'_2}(S_T)$ with mixed norm ≤ 1 . Here $p' = p/(p-1)$. It is easy to see that

$$d_{p_1, p_2}(K)^{p_1 \wedge p_2} \leq D_{p_1 p_2}(K) \tag{9}$$

for all compact sets $K \subset S_T$, because

$$\mu(K) \leq \iint_{S_T} RF d\mu = \iint_{S_T} FR^*\mu dx dt \leq \|F\|_{L_x^{p_1} L_t^{p_2}} \|R^*\mu\|_{L_x^{p'_1} L_t^{p'_2}}.$$

And if we resort to the min-max principle as in [AH], it follows that equality actually holds in (9), and then, that the equality can be extended to all Borel sets in S_T ; again see [AH].

B. Some basic properties of wave capacity.

4. Wave capacity is countably subadditive.

Here we see the reason for using the power $p_1 \wedge p_2$ – it ensures that the wave capacity is countably subadditive and it produces the “correct”

Hausdorff dimension for sets K . So, let $\{K_j\}$ be a sequence of subsets of \mathbb{R}_+^{3+1} , then we claim

$$D_{p_1, p_2}(\cup_j K_j) \leq \sum_j D_{p_1 p_2}(K_j).$$

Case 1. $p_2 \leq p_1$. Choose F_j so that $RF_j \geq 1$ on K_j , then $\sup_j F_j \equiv F$ has the property that $RF \geq 1$ on $\cup_j K_j$ and

$$\|F\|_{L_x^{p_1} L_t^{p_2}}^{p_2} \leq \int \left(\sum_j \int F_j^{p_1} dx \right)^{p_2/p_1} dt \leq \sum_j \int \left(\int F_j^{p_1} dx \right)^{p_2/p_1} dt.$$

Case 2. $p_1 < p_2$. Now

$$\|F\|_{L_x^{p_1} L_t^{p_2}}^{p_1} \leq \left[\int \left(\sum_j \int F_j^{p_1} dx \right)^{p_2/p_1} dt \right]^{p_1/p_2} \leq \sum_j \left[\left(\int F_j^{p_1} dx \right)^{p_2/p_1} dt \right]^{p_1/p_2},$$

by Minkowski's inequality.

5. Wave capacity is translation invariant in \mathbb{R}^3 .

Here we are claiming

$$D_{p_1 p_2}(K + (x_0, 0)) = D_{p_1 p_2}(K) \tag{10}$$

for all $x_0 \in \mathbb{R}^3$. Indeed, (10) follows from the fact that $\|F_{x_0}\| = \|F\|$, where the norms are the mixed $L_x^{p_1} L_t^{p_2}$ Lebesgue norms and $F_{x_0}(x, t) = F(x + x_0, t)$.

Below, we note that this no longer works for translations in the time t variable.

6. Wave capacity of an initial ball.

Here we calculate the wave capacity of the initial ball $\{\hat{Q}_r(0, 0) = \{(x, t) : |x| < r, 0 < t < r\}$ using dialation. We prove

$$D_{p_1 p_2}(\hat{Q}_r(0, 0)) = c(r^{3/p_1 + 1/p_2 - 2})^{p_1 \wedge p_2}, \quad (11)$$

for all $r > 0$. Indeed, if $RF(x, t) \geq 1$ for $(x, t) \in \hat{Q}_r(0, 0)$, then the change of variables $\xi = x/r$, $s = t/r$ produces a function $G = r^2 F_r$ for which $RG \geq 1$ on $\hat{Q}_1(0, 0)$; $F_r(\xi, s) = F(r\xi, rs)$. Thus

$$D_{p_1 p_2}(\hat{Q}_1(0, 0)) \leq \|r^2 F_r\|_{L_x^{p_1} L_t^{p_2}}^{p_1 \wedge p_2} = r^{(2 - 3/p_1 - 1/p_2)p_1 \wedge p_2} \|F\|_{L_x^{p_1} L_t^{p_2}}.$$

Now interchange the roles of \hat{Q}_r and \hat{Q}_1 .

7. A relation between wave capacities.

Here we note that there is a constant c independent of the set K such that

$$D_{p_1 p_2}(K)^{1/p_1 \wedge p_2} \leq c D_{\bar{p}_1 \bar{p}_2}(K)^{\bar{p}_1 \wedge \bar{p}_2} \quad (12)$$

for all compact sets $K \subset \hat{Q}_R(0, 0)$ for some fixed R sufficiently large and with

$$1/p_1 \geq 1/\bar{p}_1, \quad 1/p_2 \geq 1/\bar{p}_2.$$

In fact, to see (12) we first notice that for $(x, t) \in K$ and R sufficiently large

$$R(F\chi_{\hat{Q}_R(0,0)})(x, t) = RF(x, t)$$

for all $(x, t) \in K$. The result now follows by applying Holder's inequality to the mixed norm of $F \cdot \chi_{\hat{Q}_r}$ and recalling that these norms tend to the norm of F as $R \rightarrow \infty$.

8. An explicit form for $R^*\mu$.

It would aid the calculations of wave capacity if we had an explicit representation for the expression $R^*\mu$ when μ is a Borel measure on \mathbb{R}_+^{3+1} . We don't have such a representation in the general case, but if $d\mu(x, t) = d\nu(x)g(t) dt$, we can write

$$R^*\mu(x, t) = c \int_{\mathbb{R}^3} |x - y|^{-1} g(|x - y| + t) d\nu(y) \quad (13)$$

where ν is a Borel measure on \mathbb{R}^3 with compact support, g is a locally integrable function on \mathbb{R}_+^1 , and c is some constant.

To see (13), we compute the averages of $R^*\mu$ over balls in \mathbb{R}_+^{3+1} and pass to the limit in a standard way. Thus first consider

$$\begin{aligned} \iint_{Q_r(x_0, t_0)} R^*\mu(x, t) dx dt &= \iint_{S_T} R\chi_{Q_r} d\mu \\ &= \iint_{S_T} \int_M |x - z|^{-1} dz d\mu(x, t) \end{aligned}$$

where $M = \{z \in \mathbb{R}^3 : |x - z| < t, |z - x_0| < r, ||x - z| - (t - t_0)| < r\}$.

And so the above integral is

$$= \int_{\mathbb{R}^3} |x - z|^{-1} \int_{|z - x_0| < r} \int_{|x - z| + t_0 - r}^{|x - z| + t_0 + r} g(t) dt dz d\nu(x)$$

provided $r < t_0$. Now divide by $\mathcal{L}_4(Q_r(x_0, t_0))$ and pass to the limit as $r \rightarrow 0$.

This gives the desired result.

C. Wave capacity of balls and rectangles

9. $D_{p_1 p_2}(Q_R(x_0, t_0))$, $t_0 \geq 0$

The motivation for this section is: if we compute the wave capacity of the ball $Q_r(x_0, t_0)$ as $r \rightarrow 0$ for all p_1, p_2 such that $(\frac{1}{p_1}, \frac{1}{p_2}) \in (0, 1) \times (0, 1)$, then

we can get a Hausdorff measure estimate on $D_{p_1 p_2}(K)$ and consequently, a Hausdorff measure estimate for the nonhomogeneous wave equation blow up set, which clearly satisfies

$$D_{p_1 p_2}([RF = +\infty]) = 0$$

for $F \in L_x^{p_1} L_t^{p_2}$, $F \geq 0$.

So, for our result below, we divide the square $(0, 1) \times (0, 1)$ into the regions Δ_j , $j = 1, 2, 3, 4$, where

$$\Delta_1 : 1/p_2 \geq 1/p_1, 1/p_2 < 2 - 2/p_1;$$

$$\Delta_2 : 1/p_2 \leq 1/p_1, 1/p_2 < 2 - 2/p_1;$$

$$\Delta_3 : 1/p_2 \geq 1/p_1, 1/p_2 > 2 - 2/p_1;$$

$$\Delta_4 : 1/p_2 \leq 1/p_1, 1/p_2 > 2 - 2/p_1.$$

We now prove

Theorem 1. If

$$\sigma(p_1, p_2) = \begin{cases} p_2/p_1, & \text{for } (\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_1 \\ 1, & \text{for } (\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_2 \\ (\frac{3}{p_1} + \frac{1}{p_2} - 2)p_2, & \text{for } (\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_3 \\ (\frac{3}{p_1} + \frac{1}{p_2} - 2)p_1, & \text{for } (\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_4 \end{cases}$$

then for any $t_0 > 0$

$$D_{p_1 p_2}(Q_R(x_0, t_0)) \sim R^{\sigma(p_1/p_2)}, \text{ as } R \rightarrow 0.$$

Also, when $2 - 2/p_1 = 1/p_2$

$$D_{p_1 p_2}(Q_R(x_0, t_0)) \sim \begin{cases} R^{p_2/p_1} (\log 1/R)^{1-p_2}, & \text{for } 1/p_2 \geq 1/p_1 \\ R (\log (1/R))^{p_1(1-p_2)/p_2}, & \text{for } 1/p_2 \leq 1/p_1. \end{cases}$$

Here \sim means that the ratio is bounded above and below by some positive constants for all $R < r_0$, for some fixed $r_0 > 0$; r_0 is independent of R , but depends on t_0, p_1 and p_2 .

Remark 1. According to Theorem 1, when $2 - 2/p_1 = 1/p_2$ holds, embeddings of the form

$$\|RF\|_{L_x^{q_1} L_t^{q_2}} \leq c \|F\|_{L_x^{p_1} L_t^{p_2}}$$

are impossible for $\frac{3}{q_1} + \frac{1}{q_2} = \frac{1}{p_1}$. In particular, there is no embedding of the form

$$\|RF\|_{L_{x,t}^6} \leq c \|F\|_{L_{x,t}^{3/2}}.$$

The interested reader should compare this with Theorem 2.5 of reference [RS].

Proof of Theorem 1. We consider the upper and lower bounds for $D_{p_1 p_2}(Q_R(x_0, t_0))$ as $R \rightarrow 0$ separately. Clearly, we need only consider $x_0 = 0$. The lower

bounds can then be achieved by looking at only radial F , i.e. $F(x, t) = F(|x|, t)$. For the upper bounds, it seems to be necessary to look at what we shall call “mixed nonlinear” retarded potentials.

Lower bounds. We first recall that the radial case can be represented (see [S]) as: $|x| = r$,

$$\begin{aligned} 2rRF(x, t) &= \int_0^t \int_{|r-(t-s)|}^{r+(t-s)} F(\rho, s) \rho \, d\rho \, ds \\ &\leq \int_0^t \|F(\cdot, s)\|_{L_x^{p_1}} \left(\frac{[r+(t-s)]^\alpha - |r-(t-s)|^\alpha}{\alpha} \right)^{1/p_1'} ds \end{aligned}$$

where $\alpha = (1 - 2/p_1)p_1' + 1$, $p_1' = p_1/(p_1 - 1)$. Next, Holder’s inequality and a change of variables, gives the upper bound

$$c \cdot r^{\alpha/p_1' + 1/p_2'} \left(\int_0^{T/r} |(u+1)^\alpha - |u-1|^{\alpha}|^{p_2'/p_1'} du \right)^{1/p_2'}$$

The integrand in the above integral is clearly locally integrable when $3 - 3/p_1 - 1/p_2 > 0$. Thus we need only consider

$$\int_2^{T/r} |(u+1)^\alpha - |u-1|^{\alpha}|^{p_2'/p_1'} du \leq c \begin{cases} r^{-(\alpha-1)1/p_1' - 1/p_2'}, & 2 - 2/p_1 > 1/p_2 \\ 1, & 2 - 2/p_1 < 1/p_2 \\ (\log 1/r)^{1/p_2'}, & 2 - 2/p_1 = 1/p_2. \end{cases}$$

Consequently, we have

$$rRF(x, t) \leq c \|F\|_{L_x^{p_1} L_t^{p_2}} \begin{cases} r^{\alpha/p_1' + 1/p_2'}, & 2 - 2/p_1 < 1/p_2 \\ r^{1/p_1'}, & 2 - 2/p_1 > 1/p_2 \\ r^{1/p_1'} (\log 1/r)^{1/p_2'}, & 2 - 2/p_1 = 1/p_2. \end{cases}$$

which gives the desired lower bound for $D_{p_1 p_2}(Q_R(0, t_0))$ provided $3 - 3/p_1 - 1/p_2 > 0$.

When $3 - 3/p_1 \leq 1/p_2$, we recognize the bound

$$rRF(x, t) \leq c \int_0^t \|F(\cdot, t - u)\|_{L_x^{p_1}} |r - u|^{3/p_1' - 1} du.$$

Hence we have the Riesz potential estimate

$$rRF(x, t) \leq c \cdot I_{3/p_1'} * g(r) \tag{14}$$

where $g(u) = \|F(\cdot, t - u)\|_{L_x^{p_1}}$ and we are setting $F(p, s) = 0$ for $s < 0$;

$$I_\beta * g(r) = \int_{-\infty}^{\infty} |r - u|^{\beta-1} g(u) du.$$

We now use the Hardy-Littlewood convolution inequality on (14) when $3 - 3/p_1 < 1/p_2$ i.e. $3/p_1' \cdot p_2 < 1$, to get

$$\left(\int_0^\infty [rRF(x, t)]^{q_1} dr \right)^{1/q_1} \leq c \|g\|_{L^{p_2}} = c \|F\|_{L_x^{p_1} L_t^{p_2}}$$

for $\frac{1}{q_1} = \frac{1}{p_2} - \frac{3}{p_1}$. And this gives the desired lower bound in this case.

And finally, the case $3 - 3/p_1 = 1/p_2$ is resolved using the well known exponential Sobolev estimate (see Theorem 3.14 of [AH])

$$\frac{1}{R} \int_0^R \exp b \left[\frac{rRF(x, t)}{\|F\|} \right]^{p_2'} dr \leq c$$

which gives $D_{p_1 p_2}(Q_R) \geq c \cdot R^{p_1 \wedge p_2}$ in this case, which again is our desired result.

Upper bounds. Before we bite into the general situation, we should note that there are some special cases where the desired upper bound can be achieved with very little effort. However, these considerations do not seem to work in general, especially in the troublesome case that occurs with Δ_2 .

First, notice that estimate (7) can be used (as in (8)) to get the correct upper bounds for Δ_3 and Δ_4 . Secondly, the diagonal case $1/p_1 = 1/p_2$ is rather easily resolved by noticing that

$$\|R^* \chi_Q\|_{L'_{x,t}}^{p'} = \iint_Q R[R^* \chi_Q]^{1/(p-1)} dx dt. \quad (15)$$

One simply needs to show that on $Q = Q_R(0, t_0)$,

$$R[R^* \chi_Q]^{1/(p-1)}(x, t) \sim c \begin{cases} R^{3/(p-1)}, & 1/p < 2/3 \\ R^6(\log 1/R), & 1/p = 2/3 \\ R^{2p'}, & 2/3 < 1/p. \end{cases}$$

for all $R < R_0$. We will not discuss this here since our considerations below will subsume this.

Thirdly, one can notice that via the relations inequality (12), one can achieve the correct upper bounds for Δ_1 from the diagonal case, i.e.

$$D_{p_1 p_2}(Q_R)^{1/p_2} \leq c D_{p_1 p_1}(Q_R)^{1/p_1} \sim c R^{1/p_1}.$$

This does not work for Δ_2 nor for points $(1/p_1, 1/p_2)$ on the line $2 - 2/p_1 = 1/p_2$.

To treat the general case upper bound on $D_{p_1 p_2}(Q_R(0, t_0))$, we find it necessary to introduce the following ‘‘mixed nonlinear’’ retarded potentials:

$$J_Q(x, t) = R[(R^* \chi_Q)^{1/(p_1-1)} \|R^* \chi_Q\|_{L_x^{p_1}}^\lambda]_{L_t^{p_2}}(x, t)$$

where $\lambda = \frac{1}{p_2-1} - \frac{1}{p_1-1}$. Notice that

$$\iint_Q J_Q(x, t) dx dt = \|R^* \chi_Q\|_{L_x^{p_1} L_t^{p_2}}^{p_2'},$$

the $p_1 \neq p_2$ analogue of (15). Notice further, that if we can show

$$J_Q(x, t) \sim R^B, \quad R < R_0,$$

for $(x, t) \in Q = Q_R(0, t_0)$, then

$$D_{p_1 p_2}(Q_R)^{1/p_1 \wedge p_2} \sim R^{4/p_2 - B/p_2'}. \quad (16)$$

And then we only need prove that $B = \left(\frac{4}{p_2} - \frac{1}{p_1}\right) p_2'$ on Δ_1 and Δ_2 and $B = \left(\frac{3}{p_2} - \frac{3}{p_1} + 2\right) p_2'$ on Δ_3 and Δ_4 (and with an appropriate modification with $\log 1/R$ on the line $2 - 2/p_1 = 1/p_2$) to achieve our goals.

So we begin with the case $\lambda > 0$ and we look first at

$$\begin{aligned} \|R^* \chi_Q(\cdot, t - |y|)\|_{L_x^{p_1'}}^{p_1'} &= \int_{\mathbf{R}^3} R^* \chi_Q(z, t - |y|) (R^* \chi_Q(z, t - |y|))^{1/(p_1-1)} dz \quad (17) \\ &= \int_{|z-\xi| < R} \int_{\substack{|\xi| < T-t+|y| \\ |t-|y|+|\xi|-t_0| < R}} |\xi|^{-1} d\xi (R^* \chi_Q(z, t - |y|))^{1/p_1-1} dz. \end{aligned}$$

Now notice that if $|t - t_0| < R/2$ and $\||y| - |\xi|\| < R/2$ then we get

$|t - |y| + |\xi| - t_0| < R$, and $|t - t_0| < R$ with $|t - |y| + |\xi| - t_0| < R$ implies $\||y| - |\xi|\| < 2R$. Hence the above integral is equivalent to

$$\int_{|z-\xi| < R} \int_{\||y| - |\xi|\| < \alpha R} |\xi|^{-1} d\xi (R^* \chi_Q(z, t - |y|))^{1/(p_1-1)} dz$$

for some choices of $\alpha > 0$. Here we have omitted the condition $|\xi| < T - t + |y|$ by choosing T sufficiently large (relative to t_0 and R). Thus (17) is equivalent to

$$\int_{\||y| - |\xi|\| < \alpha R} |\xi|^{-1} \int_{|z-\xi| < R} \left(\int_{\substack{\||y| - |\eta|\| < \alpha \\ |z - \eta| < 1}} |\eta|^{-1} d\eta \right)^{1/(p_1-1)} dz d\xi \quad (18)$$

Now we change variables: $\xi = R\xi'$, $\eta = R\eta'$, $z = Rz'$ and $y = Ry'$, which make (18) into

$$R^{5+2/p_1-1} \int_{\|y'\|-\|\xi'\|<\alpha} |\xi'|^{-1} \int_{|z'-\xi'|<1} \left(\int_{\substack{\|y'\|-\|\eta'\|<\alpha \\ |z'-\eta'|<1}} |\eta'|^{-1} d\eta' \right)^{1/(p_1-1)} dz' d\xi'$$

$$= R^{5+2/(p_1-1)} W(|y'|).$$

Also, it is easy to see that $W(|y'|) \sim |y'|^{1-1/(p_1-1)}$ as $|y'| \rightarrow \infty$, and W is bounded for $|y'| \leq 2$.

So now put this all into J_Q and then upon performing the same tricks as before there, we finally get that $J_Q(x, t)$ is bounded above on Q and below on $\frac{1}{2}Q$ by a constant multiple of

$$R^{2/(p-1)+2} (R^{5+1/(p_1-1)})^{\lambda/p'_1} \int_{2<|y'|<\beta t_0/R} |y'|^{-1/(p_1-1)} (|y'|^{1-1/(p_1-1)})^{\lambda/p'_1} |y'|^{-1} dy' \quad (19)$$

for some choices of $\beta > 0$. The integral in (19) has

$$-p'_1 + (1 - 1/(p_1 - 1)) \frac{\lambda}{p'_1} + 3 = (p_1 - 1)p_1 p_2 [2 - 2/p_1 - 1/p_2]$$

hence when $2 - 2/p_1 - 1/p_2 < 0$, (19) behaves like

$$R^{p'_2(2+3/p_2-3/p_1)}, \quad \text{as } R \rightarrow 0,$$

exactly as needed! When $2 - 2/p_1 - 1/p_2 > 0$, we get

$$R^{(4/p_2-1/p_1)p'_2}, \quad \text{as } R \rightarrow 0,$$

And when $2 - 2/p_1 - 1/p_2 = 0$,

$$R^{p_2'(2+3/p_2-3/p_1)}(\log 1/R), \quad \text{as } R \rightarrow 0.$$

The desired estimates for wave capacity now follow.

However, when $\lambda < 0$, we need to be more careful. To get an upper bound on $J_Q(x, t)$, $(x, t) \in Q$, we clearly need a lower bound on (17), but now we are not allowed to reduce from Q to $\frac{1}{2}Q$ since ultimately we need an upper bound on

$$\iint_Q J_Q \, dx \, dt.$$

To get around this problem, we first get estimates for $J_Q(x, t)$ with $(x, t) \in Q^+$ = ‘‘upper half’’ of Q : $|x| < R$ and $t_0 < t < t_0 + R$. This allows for the set where $|t - |y| + |\xi| - t_0| < R$ in (17) to be replaced by the smaller set where $|y| - R < |\xi| < |y|$ and this accomplishes the same goal. The lower half of Q is treated similarly (for we are only interested in an estimate as a function of R , as $R \rightarrow 0$). This concludes our proof of Theorem 1.

Remark 2. Reexamining the proof of Theorem 1 reveals that a similar argument also implies that the capacity $D_{p_1 p_2}(B_r(x_0) \times \{t_0\})$ has the same growth in R as does $D_{p_1 p_2}(Q_R(x_0, t_0))$, for each $t_0 > 0$.

D. Estimates for wave capacity in terms of Hausdorff measures.

10. Hausdorff measure, Hausdorff capacity, and Hausdorff dimension

If $\phi(r)$ is a continuous non-decreasing function of $r \geq 0$, with $\phi(0) = 0$,

then we set

$$H_\epsilon^\phi(K) = \inf \sum_j \phi(r_j) \quad (20)$$

where K is a compact subset of \mathbb{R}_+^{3+1} and the infimum in (20) is taken over all covers of K by balls $\{Q_{r_j}(x_j, t_j)\}$ with $r_j \leq \epsilon$. For each fixed ϵ , we will refer to H_ϵ^ϕ as a Hausdorff capacity associated with the measure function ϕ . The Hausdorff measure associated with ϕ is

$$\lim_{\epsilon \rightarrow 0} H_\epsilon^\phi(K) \equiv H^\phi(K).$$

When $\phi(r) = r^d$, for some $d > 0$, we shall write more simply H_ϵ^d and H^d .

The Hausdorff dimension of the set K , written as $H\text{-dim}(K)$, is

$$\inf \{d : H^d(K) = 0\},$$

$K \subset \mathbb{R}_+^{3+1}$.

Now using the countably subadditivity of wave capacity and Theorem 1, we easily have

Corollary (to Theorem 1). Setting $\phi(r) = r^{\sigma(p_1, p_2)}$ for $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_i$, $i = 1, 2, 3, 4$, and

$$\phi(r) = \begin{cases} r^{p_2/p_1} (\log 1/r)^{1-p_2}, & 1/p_2 \geq 1/p_1 \\ r (\log 1/r)^{p_1(1-p_2)/p_2}, & 1/p_2 \leq 1/p_1 \end{cases}$$

when $(\frac{1}{p_1}, \frac{1}{p_2})$ lies on the line $2 - 2/p_1 = 1/p_1$, then there is a constant c independent of the Borel set $K \subset \mathbb{R}_+^{3+1}$ such that

$$D_{p_1 p_2}(K) \leq c H^\phi(K). \quad (21)$$

Also note that from Theorem 1 we could bound $D_{p_1 p_2}(K)$ by the Hausdorff capacity $H_\epsilon^\phi(K)$, which in a sense is better due to the fact that Hausdorff

capacity is finite on bounded subsets of \mathbf{R}_+^{3+1} and still H^ϕ and H_ϵ^ϕ have the same null sets; see [AH].

The Corollary gives a Hausdorff measure upper bound for the wave capacity of the blow-up set $[Rf = +\infty]$, $f \in L_x^{p_1} L_t^{p_2}$, $f \geq 0$, and we can then immediately make the following conclusions:

- $H - \dim ([Rf = +\infty]) < 2$, $(\frac{1}{p_1}, \frac{1}{p_2}) \in (0, 1) \times (0, 1)$,
- $H - \dim ([Rf = +\infty]) < 1$, $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_1$, $p_1 \neq p_2$,
- $H - \dim ([Rf = +\infty]) \leq 1$, $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_2$.

From the first of these conclusions, it is clear that the blow-up set cannot contain a set of the type $E \times \{0, t_0\}$, where $E \subset \mathbf{R}^2$, $\mathcal{L}_2(E) > 0$. This can also be seen simply from the estimate

$$\|Rf(\cdot, \cdot, 0, t_0)\|_{L_{x_1, x_2}^{p_1}} \leq c \|f\|_{L_x^{p_1} L_t^{p_2}} \quad (22)$$

where the left hand integral is taken over \mathbf{R}^2 .

To see (22), just use Minkowski's inequality getting

$$\int_{\|y\| < t_0} \|f(\cdot, \cdot, -y_3, t_0 - |y|)\|_{L_x^{p_1}} |y|^{-1} dy$$

which does not exceed

$$2\pi \int_0^{t_0} \int_{-(t_0-s)}^{t_0-s} \|f(\cdot, \cdot, u, s)\| du ds \leq c \|f\|_{L_x^{p_1} L_t^{p_2}}$$

upon changing to spherical coordinates. Thus

$$\mathcal{L}_2(E)^{1/p_1} \leq c D_{p_1 p_2} (E \times \{0, t_0\})^{1/p_1 \wedge p_2}. \quad (23)$$

11. Wave capacity of a space section of the blow-up set

We next prove

Theorem 2. For $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Delta_2$, $1/p_1 < 2/3$, there is a constant c such that

$$\frac{1}{c} \mathcal{L}_1(E) \leq D_{p_1 p_2}(\{x_0\} \times E) \leq c \mathcal{L}_1(E), \quad (24)$$

i.e. the wave capacity behaves like Lebesgue 1-dimensional measure on time lines. In particular,

$$\mathcal{L}_1([Rf = +\infty]_{x_0}) = 0$$

where K_{x_0} is the x_0 -section; here $f \in L_x^{p_1} L_t^{p_2}$ as given above.

Proof. Without loss of generality, we can assume $x_0 = 0$ and that f is radial. Then from the radial formula (see lower bounds estimates of Theorem 1)

$$Rf(0, t) = \int_0^t f(t-s, s)(t-s) ds.$$

Hence

$$\left| \int_0^T Rf(0, t) \psi(t) dt \right| \leq \|f\|_{L_x^{p_1} L_t^{p_2}} \left[\int_0^T \left(\int_S^T |\psi(t)|^{p'_1} (t-s)^{(1-2/p_1)p'_1} dt \right)^{p'_2/p'_1} ds \right]^{1/p'_2}.$$

But since $p'_2 \leq p'_1$ on Δ_2 , the above right hand integral does not exceed $c \|\psi\|_{L^{p'_1}}$ when $1/p_1 < 2/3$. Duality then gives

$$\left(\int_0^T |Rf(0, t)|^{p_1} dt \right)^{1/p_1} \leq c \|f\|_{L_x^{p_1} L_t^{p_2}}.$$

This gives the left inequality in (24); the right side follows from Theorem 1 in that $H^1 \sim \mathcal{L}_1$ on a time line.

Now consider the following example, which is meant to test the sharpness of Theorem 2, i.e. how big can the x_0 -section of the blow-up set be? For this we set

$$f(x, t) = \int ||x| + t - s|^{\alpha-1} d\mu(s)$$

with $|x| < 1$ and $0 < t < 1$ and $0 < t < T$. Note that

$$\begin{aligned} & \int_0^T \int_0^1 \left(\int |\rho + t - s|^{\alpha-1} d\mu(s) \right)^2 \rho^2 d\rho dt \\ &= \int_0^T \int_0^1 \left(\int |\rho + t - s|^{\alpha-1} d\mu(s) \right) \left(\int |\rho + t - s'|^{\alpha-1} d\mu(s') \right) \rho^2 d\rho dt \\ &= \int_0^1 \iint \int_0^T |\rho + t - s|^{\alpha-1} |\rho + t - s'|^{\alpha-1} dt d\mu(s) d\mu(s') \rho^2 d\rho \\ &\leq c \iint |s - s'|^{2\alpha-1} d\mu(s) d\mu(s') = c \|I_\alpha^* \mu\|_{L^2}^2, \end{aligned}$$

by the Riesz convolution theorem. Here I_α is the Riesz convolution operator; see [AH]. Also, we need $0 < \alpha < 1/2$. Thus f is square integrable over \mathbb{R}_+^{3+1} -locally when $I_\alpha * \mu$ is.

Next, note

$$\begin{aligned} Rf(x, t) &= \int_{|y|<t} \int ||x - y| + (t - |y|) - s|^{\alpha-1} d\mu(s) |y|^{-1} dy \\ &\geq \int_{|y|<t} \int ||x| + |t - s|^{\alpha-1} d\mu(s) |y|^{-1} dy. \end{aligned}$$

So $Rf(0, t) \geq ct^2 \cdot I_\alpha * \mu(t)$.

We now choose $E \subset \mathbb{R}^1$ such that the Riesz capacity $C_{\alpha,2}(E) = 0$, $0 < \alpha < 1/2$, but $C_{\alpha',2}(E) > 0$, $\alpha < \alpha' < 1/2$. Then there exists a measure μ such that $\|I_\alpha * \mu\|_{L^2} < \infty$ and $I_{2\alpha} * \mu = +\infty$ on E ; see [AH]. But then $I_\alpha * \mu = +\infty$ on E .

Finally, taking α and α' arbitrarily close to zero, we can deduce that the x_0 section of the blow-up set, as in Theorem 2, can have $1 - \epsilon < H\text{-dim} \leq 1$ for any $\epsilon > 0$.

12. Wave capacity and product sets

Here we look for further estimates on sets like $E \times F$, $E \subset \mathbb{R}_+^{3+1}$, $F \subset \mathbb{R}^1$, and confine our attention to the diagonal case $p_1 = p_2 = p$ for simplicity. Our first observation is that [F], Theorem 2.10.45 implies that $H^{4-2p}(E \times F) \leq cH^{3-2p}(E) \cdot \mathcal{L}_1(F)$, whenever $p < 3/2$. Consequently, if $H^{3-2p}(E) < \infty$ and $\mathcal{L}_1(F) = 0$, we have $D_{pp}(E \times F) = 0$. In particular, if $F = \{t_0\}$, we get

Theorem 3. If $p < 3/2$ and $H^{3-2p}(E) < \infty$, then

$$D_{pp}(E \times \{t_0\}) = 0.$$

Finally, we use our explicit form for $R^*\mu$ given in (13) to deduce

Theorem 4. Let E and F be bounded Borel sets of \mathbb{R}_+^{3+1} and \mathbb{R}^1 respectively, then there is a constant c such that

$$\frac{1}{c}C_{2,p}(E) \cdot \mathcal{L}_1(F) \leq D_{pp}(E \times F) \leq cH_\epsilon^{4-2p}(E \times F) \quad (25)$$

for some $\epsilon > 0$; c independent of E, F and ϵ .

Here $C_{2,p}$ is the standard Riesz capacity, now on subsets of \mathbb{R}^3 . Note that $C_{2,p}(E) \geq cH_\infty^{3-2p+\delta}(E)$ for any $\delta > 0$, $p \leq 3/2$; see [AH], Theorem 5.1.13.

The upper bound in (25) is clear from Theorem 1. For the lower bound we estimate (13) as follows:

$$\|R^*\mu\|_{L_{x,t}^{p'}} \leq \left\| \int |\cdot - y|^{-1} \|\chi_F\|_{L^{p'}} d\nu(y) \right\|_{L^{p'}}$$

where we have set $d\mu(y, t) = d\nu(y)\chi_F ds$. Thus

$$\begin{aligned} \|\nu\|_1 \cdot \mathcal{L}_1(F) = \|\mu\|_1 &\leq \iint f R^* \mu \, dx \, dt \leq \|f\|_{L^p_{x,t}} \|R^* \mu\|_{L^p_{x,t}} \\ &\leq \|f\|_{L^p_{x,t}} \cdot \mathcal{L}_1(F)^{1/p'} \|I_2 * \nu\|_{L^{p'}} \end{aligned}$$

Thus if $\|I_2 * \nu\|_{L^{p'}} \leq 1$, then

$$\|\nu\|_1 \cdot \mathcal{L}_1(F)^{1/p} \leq D_{pp}(E \times F)^{1/p}.$$

And since $\|\nu\|_1 \geq C_{2,p}(E)^{1/p}$ we get the desired result.

Remark 3. If the Riesz kernel $I_2(x) = |x|^{2-3}$ is properly restricted to $|x| < 1$, say, then estimate (25) can be considered a special case of (24) since $C_{2,p}(\{x_0\}) \neq 0$, $p \geq 3/2$. But (25) is our attempt to get a lower bound on the wave capacity that closely matches the upper bound:

$$\frac{1}{c} H_\infty^{3-2p+\delta}(E) \cdot \mathcal{L}_1(F) \leq D_{pp}(E \times F) \leq c H_\epsilon^{4-2p}(E \times F),$$

$1/p > 2/3$ and $\delta > 0$.

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