NONLINEAR POTENTIAL ANALYSIS ON MORREY SPACES AND THEIR CAPACITIES

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Dedicated to the memory of Thomas H. Wolff

Abstract. To study the existence and regularity, even just partial regularity (smoothness except for a closed exceptional set of some measure or capacity zero) for higher order (nonlinear) elliptic equations (systems), we intend to develop a nonlinear potential analysis on Morrey spaces, fractional Riesz potentials, fractional maximal functions, and Morrey capacities of two different types.

1. Introduction

As early as 1938, C.B. Morrey had introduced an L^p-growth condition on the gradient of solutions of certain partial differential equations (PDE's) in two space dimension that insured their continuity in their domain of existence. This condition, which now bares his name, has become a well known and highly useful tool for studying existence and regularity, even just partial regularity (smoothness except for a closed exceptional set of some measure or capacity zero) for single higher order elliptic equations (e.g. biharmonic equation) or second order nonlinear elliptic systems.

Since such solutions can usually be represented as a potential – perhaps as a Riesz or modified Riesz potential of their derivatives – it is clear that a systematic study of potentials of functions that satisfy Morrey's condition would be of interest in PDE research, the purpose of this note (motivated by the success of the study of potentials of functions that belong to L^p, the so called nonlinear or L^p potential theory) is to begin such an investigation. Some of the elements are already in the literature, especially beginning with the early systematic work of G. Stampacchia [S] and S. Campanato [C] in the mid 1960's; see also [P]. A lemma that has also been of some use (Lemma 4.3 below, taken from [A1])

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is a version of the classical Sobolev Embedding Theorem for Morrey class potentials rather than the usual Lebesgue class potentials.

Here, we intend to develop some of the analogues of the L^p-potential theory including: (i) a study of the pre-dual space of a Morrey space; (ii) a study of the equivalence of the Riesz potential of a Borel measure versus the fractional maximal function of the same measure, now in the Morrey norm; and (iii) to begin to develop a theory of capacities that is naturally associated with the potentials of functions from a Morrey space or from its pre-dual space.

There is prior work on pre-dual spaces for a fixed Morrey space $L^{p,\lambda}$, due to C.T. Zorko [Z], the $Z^{q,\lambda}$ space, and E.A. Kalita [K], the space $K^{q,\lambda}$. In this note, we produce a third space, $H^{q,\lambda}$, pre-dual to $L^{p,\lambda}$ and then we demonstrate that all three predual spaces coincide (with equivalent norms); Theorem 3.3. Furthermore, it is this versatility of pre-dual spaces that really facilitates all the calculations involving the capacities of Sections 5, 6 and 7.

The equivalence of the Morrey norms of the Riesz potential of a Borel measure and the fractional maximal function of the same measure is an extension of an earlier result of Muckenhoupt-Wheeden [MuW]; see Theorem 4.2 below. The 1974 M-W result corresponds to $\lambda = n$ in our result. It is the equivalence for $\lambda < n$ that allows us to estimate the capacity $C_\alpha(\cdot; H^{q,\lambda})$ on a ball $B(x, r)$ from below as the radius r tends to zero; Theorem 6.4. Here we find an analogue of the Wolff potential from the standard L^p-theory; see also [AH, p. 110].

The theory of capacities for potentials of functions of either the Morrey space $L^{p,\lambda}$ or its pre-dual $H^{q,\lambda}$, $q = p/(p-1)$, $p > 1$, is developed in Sections 5 and 6 below; see definitions for $C_\alpha(\cdot; L^{p,\lambda})$ in Section 5 and $C_\alpha(\cdot; H^{q,\lambda})$ in Section 6. We note here that one is based on a dual space, $L^{p,\lambda}$, and the other on a space, $H^{q,\lambda}$, not necessarily a dual space. If we represent the first by a general symbol $C(\cdot; X^*)$ and the second by $C(\cdot; X)$, we now propose to classify capacities like $C(\cdot; X)$ as type I capacities, and $C(\cdot; X^*)$ as type II capacities. Note that if the space X is reflexive, then $C(\cdot; X)$ will be both of type I and type II. Moreover, a standard capacity in classical potential theory can be considered as type I, with $X = \mathbf{L}^1 \subset \mathbf{M}$; where $\mathbf{L}^1$=the set of all Radon measures on $\mathbb{R}^n$ with finite total variation, and $\mathbf{M}$ = the set of all Radon measures (locally finite regular signed Borel measures) on $\mathbb{R}^n$. The corresponding type II capacity is rather uninteresting (based loosely on $X^* = L^\infty$). Further, we show below (Theorem 5.1 and Corollary 6.2) that $C(\cdot; X^*) \approx \text{Cap}(\cdot; X)$, and $C(\cdot; X) \approx \text{Cap}(\cdot; X^*)$, $\approx$ denoting "having the same null sets". Here Cap represents the usual "dual capacity" – it is usually based on the potentials of measures in $\mathbf{M}$.

Finally, we investigate capacity strong-type inequalities (CSI) for our capacities $C_\alpha(\cdot; H^{q,\lambda})$ and $C_\alpha(\cdot; L^{p,\lambda})$. We get a very strong version for the type I case (Theorem 7.4) but *no* CSI for the type II case (Example 5.4). Note that the CSI for $C_\alpha(\cdot; L^p)$, the standard L^p-Riesz capacity (see [AH]), is intermediate between these two extreme situations. Putting everything into Choquet-Lorentz notation (Remark 5.6), the first case is $I_\alpha H^{q,\lambda} \subseteq L^{(q,1)}(C_\alpha(\cdot; H^{q,\lambda}))$, whenever $0 < \alpha < \lambda < n$ and $\lambda/\alpha < q < \infty$; the second is $I_\alpha L^{p,\lambda} \nsubseteq L^{(p,\gamma)}(C_\alpha(\cdot; L^{p,\lambda}))$ for $\lambda, \alpha \in (0, n)$, $1 < p < \lambda/\alpha$, and any $0 < \gamma < \infty$.

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2. Morrey Spaces

2.1. Definition. *Let $1 < p < \infty$, $0 \le \lambda \le n$, and $f \in L^p_{loc}(\mathbb{R}^n)$. Then we shall say that f belongs to the Morrey space $L^{p,\lambda}$ provided*

$$\|f\|_{L^{p,\lambda}} = \sup_{r>0, x\in\mathbb{R}^n} \Big(r^{\lambda-n} \int_{B(x,r)} |f|^p \Big)^{1/p} < \infty,$$

where $B(x,r) \subset \mathbb{R}^n$ stands for the ball centered at x and with radius r.

Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals. Clearly, $\|\cdot\|_{L^{p,\lambda}}$ is a norm. When $\lambda = n$, then $L^{p,\lambda} = L^p$, the classical L^p-space on $\mathbb{R}^n$.

In the case of $p = 1$, f is allowed to be a measure on $\mathbb{R}^n$ and the Morrey space is linked with Hausdorff capacity and its Choquet integral. More precisely, $L^{1,\lambda}$ consists of all Radon measures (locally finite regular signed Borel measures) μ on $\mathbb{R}^n$ satisfying

$$\|\mu\|_{L^{1,\lambda}} = \sup_{r>0, x\in\mathbb{R}^n} r^{\lambda-n} |\mu|(B(x,r)) < \infty,$$

where $|\mu|$ is the total variation measure of μ.

2.2 Proposition [A3, Proposition 1]. *The predual space of $L^{1,\lambda}$ is given by $L^1(\Lambda^{(\infty)}_{n-\lambda})$ which consists of $\Lambda^{(\infty)}_{n-\lambda}$-quasi continuous functions u on $\mathbb{R}^n$ for that*

$$\|u\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} = \int_{\mathbb{R}^n} |u| d\Lambda^{(\infty)}_{n-\lambda} = \int_0^\infty \Lambda^{(\infty)}_{n-\lambda}(\{x \in \mathbb{R}^n : |u(x)| \ge t\}) dt < \infty.$$

In particular, the duality implies

$$\Big| \int_{\mathbb{R}^n} \omega d\mu \Big| \le \|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\alpha})} \|\mu\|_{L^{1,\lambda}}.$$

Here and henceforth, $\Lambda^{(\infty)}_d$, $0 < d \le n$, denotes the d-dimensional Hausdorff capacity, that is,

$$\Lambda^{(\infty)}_d(E) = \inf \sum r_j^d,$$

where the infimum is taken over all countable coverings of $E \subset \mathbb{R}^n$ by open balls of radius r_j.

The following is a new characterization of the Morrey space:

2.3. Theorem. *Let $1 < p < \infty$ and $0 < \lambda < n$. Then*

$$\|f\|_{L^{p,\lambda}} = \sup_{\omega} \Big(\int_{\mathbb{R}^n} |f|^p \omega \Big)^{1/p}, \tag{2.1}$$

where the supremum is taken over all nonnegative functions ω on $\mathbb{R}^n$ with

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \le 1. \tag{2.2}$$

Proof. From Proposition 2.2, it follows that

$$\int_{\mathbb{R}^n} |f|^p \omega \le \|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \|f\|^p_{L^{p,\lambda}} \le \|f\|^p_{L^{p,\lambda}}$$

for $\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \le 1$. Consequently, the right-hand side of (2.1) is not greater than the left-hand side of (2.1).

On the other hand, let $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ and

$$\omega_0 = 1_{B(x_0,r_0)} r_0^{\lambda-n}.$$

Then

$$\|\omega_0\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} = \int_{B(x_0,r_0)} r_0^{\lambda-n} d\Lambda^{(\infty)}_{n-\lambda} = 1.$$

Therefore ω_0 satisfies (2.2). In the meantime,

$$\begin{aligned} \|f\|_{L^{p,\lambda}} &= \sup_{r_0>0, x_0\in\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f|^p r_0^{\lambda-n} 1_{B(x_0,r_0)} \Big)^{1/p} \\ &= \sup_{r_0>0, x_0\in\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f|^p \omega_0 \Big)^{1/p} \\ &\le \sup_{\omega} \Big(\int_{\mathbb{R}^n} |f|^p \omega \Big)^{1/p} \end{aligned}$$

where the supremum is taken over any nonnegative ω on $\mathbb{R}^n$ with (2.2). The proof is complete.

More importantly, (2.1) leads to a consideration of new space describing the predual space of a Morrey space. For $q = p/(p-1)$, $p \in (1,\infty)$ and $0 < \lambda < n$, we say that g is in $H^{q,\lambda}$ if

$$\|g\|_{H^{q,\lambda}} = \inf_{\omega} \Big(\int_{\mathbb{R}^n} |g|^q \omega^{1-q} \Big)^{1/q} < \infty, \tag{2.3}$$

where the infimum is over all nonnegative functions ω on $\mathbb{R}^n$ satisfying (2.2).

2.4 Theorem. *Let $1 < p < \infty$, $q = p/(p-1)$ and $0 < \lambda < n$. Then the pre-dual space of $L^{p,\lambda}$ is $H^{q,\lambda}$ under the following pairing:*

$$\langle f, g \rangle = \int_{\mathbb{R}^n} fg.$$

Moreover

$$\|f\|_{L^{p,\lambda}} = \sup_g \Big| \int_{\mathbb{R}^n} fg \Big|, \quad f \in L^{p,\lambda}, \tag{2.4}$$

where the supremum is taken over all functions $g \in H^{q,\lambda}$ with $\|g\|_{H^{q,\lambda}} \le 1$.

Proof. If $f \in L^{p,\lambda}$ and $g \in H^{q,\lambda}$, then an application of Hölder's inequality yields

$$\big|\langle f, g\rangle\big| \le \int_{\mathbb{R}^n} |f||g| \omega^{1/p} \omega^{-1/p} \le \Big(\int_{\mathbb{R}^n} |f|^p \omega \Big)^{1/p} \Big(\int_{\mathbb{R}^n} |g|^q \omega^{1-q} \Big)^{1/q}$$

since if $g \in H^{q,\lambda}$, then take ω such that $\int_{\mathbb{R}^n} |g|^q \omega^{1-q} < \infty$. Then it follows that $|g|\omega^{-1/p}$ is finite almost everywhere. This implies that g must be zero a.e. ω is zero. Thus,

$$\left|\langle f, g\rangle\right| \le \|f\|_{L^{p,\lambda}} \|g\|_{H^{q,\lambda}}.$$

In particular, every function $f \in L^{p,\lambda}$ induces a bounded linear functional on $H^{q,\lambda}$.

Conversely, suppose L is a bounded linear functional on $H^{q,\lambda}$ with the norm $\|L\| < \infty$. Fix a ball $B(x_0, r_0) \subset \mathbb{R}^n$. If g is supported in $B(x_0, r_0)$ and $f \in L^q(B(x_0, r_0))$ (the q-Lebesgue integrable space on $B(x_0, r_0)$), then

$$\|g\|_{H^{q,\lambda}} = \inf_{\omega} \left(\int_{B(x_0,r_0)} |g|^q \omega^{1-q} \right)^{1/q}.$$

Now, let $\omega_0 = 1_{B(x_0,r_0)}(x) r_0^{\lambda-n}$. Then

$$\|g\|_{H^{q,\lambda}} \le \left(r_0^{(\lambda-n)(1-q)} \int_{B(x_0,r_0)} |g|^q \right)^{1/q}.$$

Hence L induces a bounded linear functional on $L^q(B(x_0, r_0))$, and acts with some function $f^B \in L^p(B(x_0, r_0))$. By taking $B_j = B(0, j)$, $j = 1, 2, 3, ...$, we have $f^{B_j} = f^{B_{j+1}}$ on B_j, so we get a single function f on $\mathbb{R}^n$ that is locally in L^p, and such that $L(g) = \int_{\mathbb{R}^n} fg$ when $g \in H^{q,\lambda}$ with support in some ball of $\mathbb{R}^n$. If $g = 1_{B(x_0,r_0)} |f|^p f^{-1}$ then

$$|L(g)| = \int_{B(x_0,r_0)} |f|^p \le \|L\| \left(r_0^{(\lambda-n)(1-q)} \int_{B(x_0,r_0)} |f|^{(p-1)q} \right)^{1/q}.$$

This implies that $f \in L^{p,\lambda}$ with

$$\|f\|_{L^{p,\lambda}} \le \|L\|.$$

Clearly, (2.4) is a direct consequence of the previous argument.

3. Equivalent Predual Spaces

For $p > 1$ there are already two characterizations of the predual space of a Morrey space in the literature. First, in 1986, C.T. Zorko essentially proved the following theorem.

3.1 Theorem [Z, Proposition 5]. *Let $p \in (1, \infty)$, $1/p + 1/q = 1$ and $\lambda \in (0, n)$. Then a predual space of $L^{p,\lambda}$ is $Z^{q,\lambda}$ in the following sense: if $g \in L^{p,\lambda}$ and $f \in Z^{q,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(Z^{q,\lambda})^*$. Moreover, for any $L \in (Z^{q,\lambda})^*$, there exists $g \in L^{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \quad f \in Z^{q,\lambda}.$$

The space $Z^{q,\lambda}$ is defined by the set of all functions f on $\mathbb{R}^n$ with the norm

$$\|f\|_{Z^{q,\lambda}} = \inf\Big\{\|\{c_k\}\|_{l^1} : f = \sum_k c_k a_k\Big\} < \infty,$$

where a_k is a $(q, n-\lambda)$-atom and $\|\{c_k\}\| = \sum_k |c_k| < \infty$, and the infimum is taken over all possible atomic decompositions of f. Additionally, we say that a function a on $\mathbb{R}^n$ is an $(q, n-\lambda)$-atom provided that a is supported on a ball $B \subset \mathbb{R}^n$ and satisfies

$$\|a\|_q \le \frac{1}{|B|^{(n-\lambda)/(np)}}; \quad 1/p + 1/q = 1.$$

Second, in 1998, E.A. Kalita obtained another description of the predual space of a Morrey space as follows.

3.2 Theorem [K, Theorem 1]. *Let $p \in (1, \infty)$, $1/p + 1/q = 1$ and $\lambda \in (0, n)$. Then a predual space of $L^{p,\lambda}$ is $K^{q,\lambda}$ in the following sense: if $g \in L^{p,\lambda}$ and $f \in K^{q,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(K^{q,\lambda})^*$. Moreover, for any $L \in (K^{q,\lambda})^*$, there exists $g \in L^{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \quad f \in K^{q,\lambda}.$$

The $K^{q,\lambda}$ consists of all functions f on $\mathbb{R}^n$ with the quasi-norm

$$\|f\|_{K^{q,\lambda}} = \inf_\sigma \Big(\int_{\mathbb{R}^n} |f|^q \omega_\sigma^{1-q}\Big)^{1/q},$$

where

$$\omega_\sigma(x) = \int_{\mathbb{R}^{n+1}_+} r^{-(n-\lambda)} 1_{\mathbb{R}^1_+}(r - |x-y|)\, d\sigma(y, r),$$

and where the infimum is taken over all $\sigma \in \mathbf{M}^+(\mathbb{R}^{n+1}_+)$ (the class of all nonnegative Radon measures on the upper half space $\mathbb{R}^{n+1}_+ = \{(x, r) : x \in \mathbb{R}^n, r > 0\}$) with normalization $\sigma(\mathbb{R}^{n+1}_+) = 1$. Here and afterwards, 1_E is the characteristic function of the set E.

Note that throughout we will often use the notation $A \sim B$ to denote comparability of the quantities, i.e., there are two finite positive constants c_1 and c_2 (independent of A and B) satisfying $c_1 B \le A \le c_2 B$. Similarly, we say that $A \gtrsim B$ resp. $A \lesssim B$ if only the first inequality resp. the second inequality holds.

Therefore, a very natural question arises: What is the relationship between those three predual spaces? Below is the answer.

3.3 Theorem. *Let $q = p/(p-1)$, $p \in (1, \infty)$ and $\lambda \in (0, n)$. Then $Z^{q,\lambda} = K^{q,\lambda} = H^{q,\lambda}$ with*

$$\|\cdot\|_{Z^{q,\lambda}} \sim \|\cdot\|_{K^{q,\lambda}} \sim \|\cdot\|_{H^{q,\lambda}}. \tag{3.1}$$

Proof. In what follows, we always make the following convention: $1/p+1/q=1$, $p\in(1,\infty)$ and $\lambda\in(0,n)$.

We prove Theorem 3.3 by verifying

$$K^{q,\lambda}\subseteq H^{q,\lambda}\subseteq Z^{q,\lambda}\subseteq K^{q,\lambda}.$$

Step 1. $K^{q,\lambda}\subseteq H^{q,\lambda}$.

Note that if $\omega(x)=\int_{|y-x|<r} r^{-(n-\lambda)}d\sigma(y,r)$ where σ is as above, then by the Corollary to Proposition 1 in [A3], it follows that

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})}\lesssim\int_{\mathbb{R}^{n+1}_+} r^{-(n-\lambda)}\int_{\mathbb{R}^n} 1_{\mathbb{R}^1_+}(|x-y|<r)d\Lambda^{(\infty)}_{n-\lambda}(x)d\sigma(y,r)\lesssim 1.$$

In other words, $K^{q,\lambda}$ is contained in $H^{q,\lambda}$ with

$$\|f\|_{H^{q,\lambda}}\lesssim\|f\|_{K^{q,\lambda}},\quad f\in K^{q,\lambda}. \tag{3.2}$$

Step 2. $H^{q,\lambda}\subseteq Z^{q,\lambda}$.

Let $f\in H^{q,\lambda}$. Consider $E_k=\{x\in\mathbb{R}^n:\omega(x)>2^k\}$, $k\in\mathbb{Z}$. Then

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})}\sim\sum_k 2^k\Lambda^{(\infty)}_{n-\lambda}(E_k).$$

Since $f\in H^{q,\lambda}$, there is an ω such that

$$\Big(\int_{\mathbb{R}^n}|f|^q\omega^{1-q}\Big)^{1/q}\le 2\|f\|_{H^{q,\lambda}},$$

and

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})}\le 1.$$

Thanks to the definition of $\Lambda^{(\infty)}_{n-\lambda}$ and its dyadic equivalence (cf. [Ad3]), we can always select a sequence of dyadic cubes $\{J_{j,k}\}$ (at the level $k\in\mathbb{Z}$) such that

$$E_k\subset\bigcup_j J_{j,k}\quad\text{and}\quad\sum_j\ell(J_{j,k})^{n-\lambda}\lesssim\Lambda^{(\infty)}_{n-\lambda}(E_k).$$

Here and hereforth, $\ell(J)$ denotes the edge length of a cube J of sides parallel to the axes of $\mathbb{R}^n$.

Clearly,

$$\left(\sum_j\ell(J_{j,k})^n\right)^{(n-\lambda)/n}\le\sum_j\ell(J_{j,k})^{n-\lambda}\lesssim 2^{-k}\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})}<\infty.$$

So $\cup_j J_{j,k}$ is bounded. Now, let $\mathcal{I}$ be the family of all dyadic cubes $I \subset \mathbb{R}^n$ with

$$I = \bigcup_{J_{j,k} \subseteq I} J_{j,k}.$$

Then there exists a sequence of (maximal) dyadic cubes $\{I_{m,k}\}_{m=1}^{\infty} \subset \mathcal{I}$ (cf. [DX]) such that

$$\mathrm{Int}(I_{m,k}) \cap \mathrm{Int}(I_{m',k}) = \emptyset, \quad \text{if} \quad m \neq m',$$

and

$$\bigcup_j J_{j,k} = \bigcup_m I_{m,k}.$$

Here $\mathrm{Int}(S)$ stands for the interior of a set $S \subset \mathbb{R}^n$.

Noticing

$$\ell(I_{m,k})^n \leq \sum_{J_{j,k} \subseteq I_{m,k}} \ell(J_{j,k})^n,$$

we find that if $0 < \lambda < n$ then

$$\ell(I_{m,k})^{n-\lambda} \leq \sum_{J_{j,k} \subseteq I_{m,k}} \ell(J_{j,k})^{n-\lambda}$$

and hence

$$\sum_m \ell(I_{m,k})^{n-\lambda} \leq \sum_m \sum_{J_{j,k} \subseteq I_{m,k}} \ell(J_{j,k})^{n-\lambda} = \sum_m \ell(I_{m,k})^{n-\lambda} \lesssim \Lambda_{n-\lambda}^{(\infty)}(E_k).$$

Since $\mathbb{R}^n = \bigcup_k E_k$, it follows that $\mathbb{R}^n = \bigcup_{k,j} I_{j,k}$. Upon defining

$$\Delta_{j,k} = I_{j,k} \setminus \bigcup_l I_{l,k+1},$$

we see $\mathbb{R}^n = \cup_{j,k} \Delta_{j,k}$, and with this, we obtain $f = \sum_{j,k} c_{j,k} a_{j,k}$, where

$$c_{j,k} = \ell(I_{j,k})^{(n-\lambda)/p} \Big(\int_{\Delta_{j,k}} |f|^q \Big)^{1/q}$$

and

$$a_{j,k} = f 1_{\Delta_{j,k}} \ell(I_{j,k})^{-(n-\lambda)/p} \Big(\int_{\Delta_{j,k}} |f|^q \Big)^{-1/q}.$$

It is easy to check that each $a_{j,k}$ is a $(q, n-\lambda)$-atom. To prove that $f \in Z^{q,\lambda}$, it remains to verify that $\{c_{j,k}\}$ is l^1-summable. For this, noting the following two facts: $\|\omega\|_{L^1(\Lambda_{n-\lambda}^{(\infty)})} \leq 1$ and $\omega(x) \leq 2^{k+1}$ as $x \in \Delta_{j,k}$ (in fact, $x \in \Delta_{j,k}$ implies $x \notin \bigcup_l I_{j,k+1}$, then $x \notin E_{k+1}$ and hence $\omega(x) \leq 2^{k+1}$), we apply Hölder's inequality to get

$$
\begin{aligned}
\|\{c_{j,k}\}\|_{l^1} &\lesssim \sum_{j,k} \ell(I_{j,k})^{(n-\lambda)/p} 2^{k/p} \Big(\int_{\Delta_{j,k}} |f|^q \omega^{1-q} \Big)^{1/q} \\
&\lesssim \Big(\sum_{j,k} 2^k \ell(I_{j,k})^{n-\lambda} \Big)^{1/p} \Big(\sum_{j,k} \int_{\Delta_{j,k}} |f| \omega^{1-q} \Big)^{1/q} \\
&\lesssim \Big(\sum_{k} 2^k \big(\sum_{j} \ell(I_{j,k})^{n-\lambda} \big) \Big)^{1/p} \Big(\sum_{j,k} \int_{\Delta_{j,k}} |f| \omega^{1-q} \Big)^{1/q} \\
&\lesssim \Big(\sum_{k} 2^k \Lambda^{(\infty)}_{n-\lambda}(E_k) \Big)^{1/p} \Big(\sum_{j,k} \int_{\Delta_{j,k}} |f| \omega^{1-q} \Big)^{1/q} \\
&\lesssim \|\omega\|^{1/p}_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \Big(\int_{\mathbb{R}^n} |f| \omega^{1-q} \Big)^{1/q} \\
&\lesssim \|f\|_{H^{q,\lambda}}.
\end{aligned}
$$

Consequently,

$$
\|f\|_{Z^{q,\lambda}} \lesssim \|f\|_{H^{q,\lambda}}, \quad f \in Z^{q,\lambda}. \tag{3.3}
$$

Step 3. $Z^{q,\lambda} \subseteq K^{q,\lambda}$.

Suppose that $f \in Z^{q,\lambda}$. So $f = \sum_i c_j a_j$ with $\{c_j\} \in l^1$ and each a_j is a $(q, n-\lambda)$-atom. Assume that x_j and r_j are the center and radius of the support ball B_j of a_j, respectively. Define the following two functions:

$$
A_j(x) = \max\{r_j, |x - x_j|\},
$$

and

$$
\omega(x) = \|\{c_j\}\|_{l^1}^{-1} \sum_j \frac{|c_j| r_j^{\epsilon}}{A_j(x)^{n-\lambda+\epsilon}}, \quad \epsilon > 0.
$$

It is clear that ω can be written as that integral form required for the definition of $K^{q,\lambda}$.

A further application of Hölder's inequality implies

$$
|f|^q \le \Big(\sum_j |c_j| r_j^{\epsilon} A_j^{-(n-\lambda)-\epsilon} \Big)^{q-1} \Big(\sum_j |c_j| \big(r_j^{-\epsilon} A_j^{n-\lambda+\epsilon} \big)^{q-1} |a_j|^q \Big).
$$

Thus

$$
\begin{aligned}
\int_{\mathbb{R}^n} |f|^q \omega^{1-q} &\le \|\{c_j\}\|_{l^1}^{q-1} \sum_j |c_j| r_j^{-\epsilon(q-1)} \int_{\mathbb{R}^n} A_j^{(n-\lambda+\epsilon)(q-1)} |a_j|^q \\
&= \|\{c_j\}\|_{l^1}^{q-1} \sum_j |c_j| r_j^{-\epsilon(q-1)} \int_{B_j} A_j^{(n-\lambda+\epsilon)(q-1)} |a_j|^q \\
&\le \|\{c_j\}\|_{l^1}^{q-1} \sum_j |c_j| r_j^{(n-\lambda)(q-1)} \int_{\mathbb{R}^n} |a_j|^q \\
&\lesssim \|\{c_j\}\|_{l^1}^{q-1} \sum_j |c_j| r_j^{(n-\lambda)(q-1)} r_j^{-(n-\lambda)(q-1)} \\
&\lesssim \|\{c_j\}\|_{l^1}^{q}.
\end{aligned}
$$

That is to say,

$$\|f\|_{K^{q,\lambda}} \lesssim \|f\|_{Z^{q,\lambda}}, \quad f \in Z^{q,\lambda}. \tag{3.4}$$

Clearly, (3.1) is a direct consequence of (3.2), (3.3) and (3.4).

3.4 Remark. Note that $\|f\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} < \infty$ if and only if $f = \sum_j c_j a_j$ where $\{c_j\} \in l^1$ and a_j is a $(\infty, n-\lambda)$-atom. Here a function a on $\mathbb{R}^n$ is called $(\infty, n-\lambda)$-atom if the support of a is contained in a ball $B \subset \mathbb{R}^n$ and $\|a\|_\infty \le |B|^{(n-\lambda)/n}$.

On the one hand, if f has such a decomposition, then the sublinearity of the Choquet integral with respect to dyadic Hausdorff capacity (cf. [A3]) yields

$$\|f\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \le \int_{\mathbb{R}^n} \sum_j |c_j||a_j| d\Lambda^{(\infty)}_{n-\lambda} \lesssim \sum_j |c_j| \int_{\mathbb{R}^n} |a_j| d\Lambda^{(\infty)}_{n-\lambda} \lesssim \|\{c_j\}\|_{l^1}.$$

But, one should observe that f is not necessarily in the space $L^1(\Lambda^{(\infty)}_{n-\lambda})$ since those $(\infty, n-\lambda)$-atoms may not be $\Lambda^{(\infty)}_{n-\lambda}$-quasi-continuous, see also [A3, p. 123]. This lack of quasi-continuity, however, does not hinder the applications of the atomic decomposition. If we want to restore the quasi-continuity of atoms, we can modify the construction above, and obtain the majorization $|f| \le \sum_j |c_j||a_j|$, which for most purposes is a satisfactory substitute.

Conversely, suppose $\|f\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} < \infty$. Using the argument for Theorem 3.3, we consider

$$c_{j,k} = \ell(I_{j,k})^{n-\lambda} 2^{k+1},$$

and

$$a_{j,k} = f 1_{\Delta_{j,k}} \ell(I_{j,k})^{-(n-\lambda)} 2^{-(k+1)}.$$

Then $f = \sum_{j,k} c_{j,k} a_{j,k}$. It is obvious that $a_{j,k}$ is a $(\infty, n-\lambda)$-atom, since $|f(x)| \le 2^{k+1}$ whenever $x \in \Delta_{j,k}$, and $\{c_{j,k}\}$ belongs to l^1 since

$$\|\{c_{j,k}\}\|_{l^1} \lesssim \sum_{j,k} \ell(I_{j,k})^{n-\lambda} 2^{k+1} \lesssim \|f\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})}.$$

4. Riesz Potentials and Maximal Functions with Fractional Orders

Recall that $I_\alpha * \mu$ and $M_\alpha \mu$ denote the fractional Riesz potential and the fractional maximal function associated with a nonnegative measure on $\mathbb{R}^n$, respectively. That is,

$$I_\alpha * \mu(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} d\mu(y), \quad 0 \le \alpha \le n;$$

and

$$M_\alpha \mu(x) = \sup_{r>0} r^{\alpha-n} \mu\big(B(x,r)\big), \quad 0 \le \alpha \le n.$$

It is evident that $I_\alpha * \mu \geq M_\alpha \mu$ due to the estimate

$$r^{\alpha-n}\mu\big(B(x,r)\big) \leq I_\alpha * \mu(x),$$

for $x \in \mathbb{R}^n$ and $r > 0$. However, if $d\mu(y) = |y|^{-\alpha}dy$ and $x = 0$ then the reverse inequality is false. In view of this, the Morrey norm equivalence of $I_\alpha * \mu$ and $M_\alpha \mu$ is quite surprising. To do so, let us use the Fefferman-Stein sharp function $F^{\#}$ of a function $F \in L^1_{loc}(\mathbb{R}^n)$:

$$F^{\#}(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |F(y) - F_Q| dy \tag{4.1}$$

where the supremum is taken over all "coordinate" cubes Q containing x, and F_Q denotes the integral average of F over Q.

4.1 Lemma. *Let $\alpha \in [0, n]$ and let μ be a nonnegative measure with compact support on $\mathbb{R}^n$ and for which $I_\alpha * \mu \in L^1_{loc}(\mathbb{R}^n)$. Then:*

(i) $(I_\alpha * \mu)^{\#}(x) \sim M_\alpha \mu(x)$ for $x \in \mathbb{R}^n$;

(ii) Given a cube $Q \subset \mathbb{R}^n$ and numbers $t, \epsilon > 0$,

$$\begin{aligned}|\{x \in Q : I_\alpha * \mu(x) > t\}| \leq |\{x \in Q : (I_\alpha * \mu)^{\#}(x) > 2^{-1}\epsilon t\}| \\ + \epsilon|\{x \in Q : I_\alpha * \mu(x) > 2^{-n-1}t\}|.\end{aligned}$$

Proof. (i) For this comparability, see also [A2, Theorem 2.2] or [A1, Propositions 3.3 and 3.4].

(ii) For $t > 0$, let $s = 2^{-n-1}t$ and $M(t) = |\{x \in Q : I_\alpha * \mu(x) > t\}|$. Then an application of the C-Z (Calderón-Zygmund) decomposition theorem to Q, t and $I_\alpha * \mu$ gives $Q = P^t \cup Q^t$ and $P^t \cap Q^t = \emptyset$ with the following three properties:

1) $Q^t = \bigcup_{k=1}^\infty Q^t_k$, where $Q^t_1, Q^t_2, ...$ are cubes with $\mathrm{Int}(Q^t_k) \cap \mathrm{Int}(Q^t_{k'}) = \emptyset$ for $k \neq k'$;

2) $I_\alpha * \mu(x) \leq t$ a.e. $x \in P^t$;

3) $t < |Q^t_k|^{-1} \int_{Q^t_k} I_\alpha * \mu < 2^n t$ for any $Q^t_k \in \{Q^t_k\}$.

It is worth remarking that for t and s we may choose two C-Z decompositions such that every cube of the C-Z decomposition associated with t is contained in a cube of the C-Z decomposition associated with s.

Let $\mathcal{F}_1$ be the family of cubes Q^s_j of the C-Z decomposition associated with s such that

$$Q^s_j \subseteq \{x \in Q : (I_\alpha * \mu)^{\#}(x) > 2^{-1}\epsilon t\} \tag{4.2}$$

and $\mathcal{F}_2$ the family of the remaining cubes of the C-Z decomposition associated with s.

Now, if $Q' \in \mathcal{F}_2$ then there is $x \in Q'$ such that $(I_\alpha * \mu)^{\#}(x) \leq 2^{-1}\epsilon t$ and by the definition of $(I_\alpha * \mu)^{\#}$, one has

$$|Q'|^{-1} \int_{Q'} |I_\alpha * \mu - (I_\alpha * \mu)_{Q'}| \leq 2^{-1}\epsilon t \tag{4.3}$$

and

$$(I_\alpha * \mu)_{Q'} = |Q'|^{-1}\int_{Q'} I_\alpha * \mu \le 2^n s = 2^{-1}t. \tag{4.4}$$

For the cubes $Q_k^t \subseteq Q'$ from the C-Z decomposition associated with t, we use the third property of the C-Z decomposition, (4.3) and (4.4) to imply

$$\begin{aligned} t\sum_{Q_k^t\subseteq Q'}|Q_k^t| &< \sum_{Q_k^t\subseteq Q'}\int_{Q_k^t} I_\alpha * \mu \\ &\le \sum_{Q_k^t\subseteq Q'}\int_{Q_k^t}|I_\alpha*\mu-(I_\alpha*\mu)_{Q'}| + (I_\alpha*\mu)_{Q'}\sum_{Q_k^t\subseteq Q'}|Q_k^t| \\ &\le \int_{Q'}|I_\alpha*\mu-(I_\alpha*\mu)_{Q'}| + (I_\alpha*\mu)_{Q'}\sum_{Q_k^t\subseteq Q'}|Q_k^t| \\ &\le 2^{-1}\epsilon t|Q'| + 2^{-1}t\sum_{Q_k^t\subseteq Q'}|Q_k^t|. \end{aligned}$$

Consequently,

$$\sum_{Q_k^t\subseteq Q'}|Q_k^t| < \epsilon|Q'|.$$

Thus,

$$\sum_{Q'\in\mathcal{F}_2}\Big(\sum_{Q_k^t\subseteq Q'}|Q_k^t|\Big) < \epsilon\sum_{Q'\in\mathcal{F}_2}|Q'| \le \epsilon M(s) = \epsilon M(2^{-n-1}t).$$

Meanwhile, it follows from (4.2) that

$$\sum_{Q'\in\mathcal{F}_1}\Big(\sum_{Q_k^t\subseteq Q'}|Q_k^t|\Big) = \sum_{Q'\in\mathcal{F}_1}|Q'| \le |\{x\in Q : (I_\alpha*\mu)^\#(x) > 2^{-1}\epsilon t\}|.$$

So

$$\begin{aligned} M(t) &= \Big(\sum_{Q'\in\mathcal{F}_1}+\sum_{Q'\in\mathcal{F}_2}\Big)\Big(\sum_{Q_k^t\subseteq Q'}|Q_k^t|\Big) \\ &\le \epsilon M(2^{-n-1}t) + |\{x\in Q : (I_\alpha*\mu)^\#(x) > 2^{-1}\epsilon t\}|. \end{aligned} \tag{4.5}$$

This (4.5) concludes the proof.

4.2 Theorem. *Let $1 < p < \infty$ and $0 \le \alpha, \lambda \le n$. If μ is a nonnegative measure on $\mathbb{R}^n$ then*

$$\|I_\alpha * \mu\|_{L^{p,\lambda}} \sim \|M_\alpha\mu\|_{L^{p,\lambda}}. \tag{4.6}$$

Proof. From Lemma 4.1 (i) as well as the estimate $I_\alpha * \mu \ge M_\alpha\mu$, we need only to prove

$$\|I_\alpha * \mu\|_{L^{p,\lambda}} \lesssim \|(I_\alpha*\mu)^\#\|_{L^{p,\lambda}}. \tag{4.7}$$

We first suppose that μ has compact support. Then (4.7) is a consequence of Lemma 4.1 (ii). In fact, by Lemma 4.1 (ii) we have that for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}\int_Q (I_\alpha*\mu)^p &= \int_0^\infty |\{x\in Q: I_\alpha*\mu(x)>t\}|dt^p\\ &\le \epsilon\int_0^\infty M(2^{-n-1}t)dt^p+\int_0^\infty |\{x\in Q:(I_\alpha*\mu)^\#(x)>2^{-1}\epsilon t\}|dt^p\\ &=\epsilon 2^{p(n+1)}\int_0^\infty M(t)dt^p+(2\epsilon^{-1})^p\int_0^\infty|\{x\in Q:(I_\alpha*\mu)^\#(x)>t\}|dt^p\\ &=\epsilon 2^{p(n+1)}\int_Q (I_\alpha*\mu)^p+(2\epsilon^{-1})^p\int_Q\big((I_\alpha*\mu)^\#\big)^p.\end{aligned}$$

If $\epsilon=2^{-1-p(n+1)}$, then

$$2^{-1}\int_Q (I_\alpha*\mu)^p\le 2^{p(2+p(n+1))}\int_Q\big((I_\alpha*\mu)^\#\big)^p. \tag{4.8}$$

Given a ball $B(x,r)$ centered at $x\in\mathbb{R}^n$ and with radius r, we may find two cubes Q_1 and Q_2 which have the center x and the side lengths $2r/\sqrt{n}$ and $2r$ respectively. This, together with (4.8) and the definition of $\|\cdot\|_{L^{p,\lambda}}$, deduces that (4.7) is true for the measure μ with compact support on $\mathbb{R}^n$.

If μ does not have compact support, we let μ_j be the restriction of μ to the ball $B(0,j)$ (with radius j about the origin 0) for $j=1,2,....$ By (4.7) for μ_j, we have

$$\|I_\alpha*\mu_j\|_{L^{p,\lambda}}\lesssim\|M_\alpha\mu\|_{L^{p,\lambda}} \tag{4.9}$$

for all j, with the constant in (4.9) that does not depend on j. The inequality (4.7) for the general μ now follows from (4.9) and monotone convergence.

Of course, Lemma 4.1 and the argument for Theorem 4.2 may infer

$$\|I_\alpha*\mu\|_{L^p}\sim\|M_\alpha\mu\|_{L^p} \tag{4.10}$$

which is the theorem of B. Muckenhoupt and R. L. Wheeden. For a proof of (4.10) using the so-called "good λ inequality", see also [AH, pp. 72-73].

4.3 Lemma [A2, Theorem 4.1 and Remark 4.1; A]. *Let $0\le\alpha\le n$ and $0<\lambda\le n$.*
(i) *If $1<p<\lambda/\alpha$ and $\tilde p=\lambda p/(\lambda-\alpha p)$, then*

$$\|I_\alpha*f\|_{L^{\tilde p,\lambda}}\lesssim\|f\|_{L^{p,\lambda}}.$$

(ii) *If $1<p=\lambda/\alpha$, then*

$$\|I_\alpha*f\|_{BMO}\lesssim\|f\|_{L^{p,\lambda}}.$$

Here and henceforth, BMO is the John-Nirenberg class of all locally integrable functions f on $\mathbb{R}^n$ with bounded mean oscillation: $\|f\|_{BMO}=\sup_{x\in\mathbb{R}^n}f^\#(x)<\infty$.

An immediate corollary of Lemma 4.3 and Theorem 4.2 is

4.4 Corollary. *Let $0 \le \alpha \le n$ and $0 < \lambda \le n$.*
(i) [A2, Theorem 3.2] *If $1 < p < \lambda/m$, $m \in \mathbb{N}$, $\tilde{p}_m = \lambda p/(\lambda - mp)$, and f is a C^m function on $\mathbb{R}^n$, then*

$$\|f\|_{L^{\tilde{p}_m,\lambda}} \lesssim \|\nabla^m f\|_{L^{p,\lambda}},$$

where $\nabla^m f$ denotes the m-th order gradient of f; i.e.

$$\nabla^m f = \begin{cases} \Delta^{m/2} f, & m = 2,4,6,... \\ \nabla \Delta^{(m-1)/2} f, & m = 1,3,5,... \end{cases}$$

for which Δ is the Laplacian.
(ii) *If $1 < p < \lambda/\alpha$ and $\tilde{p} = \lambda p/(\lambda - \alpha p)$, then*

$$\|M_\alpha f\|_{L^{\tilde{p},\lambda}} \lesssim \|f\|_{L^{p,\lambda}}.$$

(iii) [A1, Corollary (i)] *If $1 < p = \lambda/\alpha$, then*

$$\|M_\alpha f\|_{L^\infty} \lesssim \|f\|_{L^{p,\lambda}}.$$

Note that in case $\alpha = 0$ of Corollary 4.4 (ii) is well-known; see for example [CFr, Theorem 1].

5. Morrey Type II Capacities

To begin with, we define the Morrey type II capacity. For a set $E \subset \mathbb{R}^n$, $\alpha \in [0, n]$, $\lambda \in (0, n)$, and $p \in (1, \infty)$, let

$$\mathrm{C}_\alpha(E; L^{p,\lambda}) = \inf\left\{\|f\|^p_{L^{p,\lambda}} : \quad f \ge 0 \quad \& \quad I_\alpha * f \ge 1_E\right\}$$

be the Morrey capacity of E. It is not hard to see that $\mathrm{C}_\alpha(\cdot; L^{p,\lambda})$ is a monotone, countably subadditive set function on the class of all subsets of $\mathbb{R}^n$ which vanishes on the empty set. Furthermore, it is an outer capacity in the sense that

$$\mathrm{C}_\alpha(E; L^{p,\lambda}) = \inf \mathrm{C}_\alpha(G; L^{p,\lambda})$$

where the infimum is over all open sets $G \supseteq E$.

The dual to a Morrey Type II capacity is denoted $\mathrm{Cap}_\alpha(\cdot)$. Due to Theorem 3.3 we bring $K^{q,\lambda}$ (as the predual of $L^{p,\lambda}$, $q = p/(p-1)$, $p > 1$) into play to define the dual capacity.

For $E \subset \mathbb{R}^n$ let

$$\mathrm{Cap}_\alpha(E; H^{q,\lambda}) = \sup\left\{\mu(E) : \quad \mu \in \mathbf{M}^+(E) \quad \& \quad \|I_\alpha * \mu\|_{K^{q,\lambda}} \le 1\right\},$$

where $\mathbf{M}^+(E)$ denotes those Radon measures that are nonnegative and have their support in E.

Meanwhile, we consider the corresponding weighted Riesz capacity: For $E \subset \mathbb{R}^n$ and a nonnegative function ω on $\mathbb{R}^n$, let

$$R^\omega_{\alpha,p}(E) = \inf\left\{\|f\|^p_{L^p(\omega)} : \quad f \ge 0 \quad \& \quad I_\alpha * f \ge 1_E\right\},$$

where

$$\|f\|^p_{L^p(\omega)} = \int_{\mathbb{R}^n} |f|^p \omega.$$

The following theorem reveals the relationship between these three capacities.

5.1 Theorem. *Let $\alpha \in [0,n]$, $\lambda \in (0,n)$, $1<p<\infty$ and $q=p/(p-1)$. Then*

$$C_\alpha(E;L^{p,\lambda}) = \big(Cap_\alpha(E;H^{q,\lambda})\big)^p = \sup_\sigma R^{\omega_\sigma}_{\alpha,p}(E), \quad E\subset \mathbb{R}^n, \tag{5.1}$$

where the supremum is over all $\sigma \in \mathbf{M}^+(\mathbb{R}^{n+1}_+)$ with $\|\sigma\|_1 = \sigma(\mathbb{R}^{n+1}_+) = 1$.

Proof. We start with working on the first equality in (5.1). Without loss of generality, we may assume that $E\subset\mathbb{R}^n$ is compact. From [K] (or the proof of Theorem 2.3) it follows that

$$\|f\|_{L^{p,\lambda}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|, \quad f\in L^{p,\lambda}, \tag{5.2}$$

where the supremum is over all $g\in H^{q,\lambda}=K^{q,\lambda}$ with $\|g\|_{K^{q,\lambda}}\le 1$.

If $f\ge 0$ and $I_\alpha * f\ge 1$ on E, an application of Hölder's inequality implies

$$\mu(E)\le \int_E I_\alpha * f\,d\mu = \int_{\mathbb{R}^n} fI_\alpha*\mu \le \Big(\int_{\mathbb{R}^n} f^p\omega_\sigma\Big)^{1/p}\Big(\int_{\mathbb{R}^n}\big(I_\alpha*\mu\big)^q\omega_\sigma^{1-q}\Big)^{1/q}.$$

So by (5.2) and the definition of $\|\cdot\|_{K^{q,\lambda}}$,

$$\mu(E)\le \|f\|_{L^{p,\lambda}}\|I_\alpha*\mu\|_{K^{q,\lambda}}.$$

And hence it follows that

$$\mathrm{Cap}_\alpha(E;H^{q,\lambda})\le \big(\mathrm{C}_\alpha(E;L^{p,\lambda})\big)^{1/p}. \tag{5.3}$$

The inequality (5.3) is the half of what we want; the other half is not as easy. Using the Minimax Theorem in [AH, Theorem 2.4.1] we have

$$\sup_g\inf_\mu\inf_\sigma \int_{\mathbb{R}^n} I_\alpha*(g\omega_\sigma^{-1/p})d\mu = \inf_\mu\inf_\sigma\sup_g\int_{\mathbb{R}^n} I_\alpha*(g\omega_\sigma^{-1/p})d\mu. \tag{5.4}$$

Here the supremum is over all $g\in L^p(\mathbb{R}^n)$ with $\|g\|_{L^p}\le 1$, as well as the infimums are over all $\sigma\in\mathbf{M}^+(\mathbb{R}^{n+1}_+)$ with $\|\sigma\|_1=\sigma(\mathbb{R}^{n+1}_+)=1$ and over $\mu\in\mathbf{M}^+(E)$ with $\|\mu\|_1=\mu(E)=1$, respectively. Of course, ω_σ must an admissible weight or else the integral involving $\omega_\sigma^{-1/p}$ will be undefined; i.e., we take $\omega_\sigma>0$ on $\mathbb{R}^n$ for otherwise we can replace it by one that is positive on $\mathbb{R}^n$ and make the integral smaller as a function of σ.

Note that the integrand in (5.4) is jointly convex in the argument $\sigma\times\mu$ (the eigenvalues of the Hessian are nonnegative). If $(\sigma,\mu)\in\mathbf{M}^+(\mathbb{R}^{n+1}_+)\times\mathbf{M}^+(E)$ is equipped with the norm $\max\{\|\sigma\|_1,\|\mu\|_1\}$, then the set

$$\{(\sigma,\mu): \|\sigma\|_1=1, \quad \|\mu\|_1=1\}$$

is vaguely compact.

Now it turns out from the definition of $\|\cdot\|_{K^{q,\lambda}}$ that the right-hand-side of (5.4) is equal to

$$\inf_{\mu}\inf_{\sigma}\|I_\alpha*\mu\|_{L^q(\omega_\sigma^{-q/p})}=\inf_{\sigma}\|I_\alpha*\mu\|_{K^{q,\lambda}}=\mathrm{Cap}_\alpha(E;K^{q,\lambda})^{-1}. \tag{5.5}$$

Also noting that if μ is taken as the Dirac measure concentrated at $x\in E$ then

$$\int_{\mathbb{R}^n} I_\alpha*(g\omega_\sigma^{-1/p})d\mu\le I_\alpha*(g\omega_\sigma^{-1/p})(x),\quad x\in E.$$

we apply lower semi-continuity to obtain an $x_E\in E$ such that

$$I_\alpha*(g\omega_\sigma^{-1/p})(x)\ge I_\alpha*(g\omega_\sigma^{-1/p})(x_E),\quad x\in E.$$

Thus

$$I_\alpha*\left(\frac{g\omega_\sigma^{-1/p}}{I_\alpha*(g\omega_\sigma^{-1/p})(x_E)}\right)(x)\ge 1,\quad x\in E.$$

Consequently

$$\mathrm{C}_\alpha(E;L^{p,\lambda})\le\frac{\|g\omega_\sigma^{-1/p}\|_{L^{p,\lambda}}^p}{\Big(I_\alpha*(g\omega_\sigma^{-1/p})(x_E)\Big)^p}.$$

This, together with (5.5) and (5.2), infers

$$\inf_{\mu}\inf_{\sigma}\int_{\mathbb{R}^n} I_\alpha*(g\omega_\sigma^{-1/p})d\mu\cdot \mathrm{C}_\alpha(E;L^{p,\lambda})^{1/p}\le\sup_h\left|\int_{\mathbb{R}^n} g\omega_\sigma^{-1/p}h\right|,$$

where the supremum is taken over all h with $\|h\|_{K^{q,\lambda}}\le 1$. But the right-hand-side above does not exceed

$$\sup_h\|g\|_{L^p}\|h\omega_\sigma^{-1/p}\|_{L^q}.$$

Since we are considering h with the condition

$$\|h\|_{K^{q,\lambda}}=\Big(\inf_{\sigma}\int_{\mathbb{R}^n}|h|^q\omega_\sigma^{1-q}\Big)^{1/q}\le 1,$$

we can take any such σ so that $\|h\omega_\sigma^{-1/p}\|_{L^q}\le 1+\epsilon$ for any $\epsilon>0$. Hence

$$\sup_h\|g\|_{L^p}\|h\omega_\sigma^{-1/p}\|_{L^q}\le\|g\|_{L^p}(1+\epsilon).$$

Thus

$$\left(\inf_{\mu}\inf_{\sigma}\int_{\mathbb{R}^n} I_\alpha*(g\omega_\sigma^{-1/p})d\mu\right)\mathrm{C}_\alpha(E;L^{p,\lambda})^{1/p}\le\|g\|_{L^p}(1+\epsilon).$$

The inequality

$$\Big(\mathrm{Cap}_\alpha(E;H^{q,\lambda})\Big)^{-1}\mathrm{C}_\alpha(E;L^{p,\lambda})^{1/p}\le 1 \tag{5.6}$$

follows right away upon using (5.4), taking supremum over g with $\|g\|_{L^p}\le 1$ and noticing that $\epsilon>0$ is arbitrary. Also observe that the order of the two infimums over (σ,μ) is not an issue because the integrand is jointly convex in these two arguments. Clearly, (5.6) is just the other half of what we desire.

For the second equality in (5.1), we again use the Minimax Theorem (quoted before), but this time write

$$\sup_f\inf_\mu\int_{\mathbb{R}^n} I_\alpha f\,d\mu=\inf_\mu\sup_f\int_{\mathbb{R}^n} I_\alpha f\,d\mu, \tag{5.7}$$

where now the supremum ranges over all $f\in L^p(\omega_\sigma)$ with

$$\|f\|^p_{L^p(\omega_\sigma)}=\int_{\mathbb{R}^n}|f|^p\omega_\sigma\le 1,$$

while the infimum is over all $\mu\in\mathbf{M}^+(E)$ with $\|\mu\|_1=1$.

Taking the infimum over both sides of (5.7) with respect to all $\sigma\in\mathbf{M}^+(\mathbb{R}^{n+1}_+)$ satisfying $\|\sigma\|_1=1$, we again see that the right-hand-side of (5.7) then becomes

$$\inf_\mu\inf_\sigma\|I_\alpha*\mu\|_{L^q(\omega_\sigma^{-q/p})}=\inf_\mu\|I_\alpha*\mu\|_{K^{q,\lambda}}=\Big(\mathrm{Cap}_\alpha(E;H^{q,\lambda})\Big)^{-1},$$

where again we have taken advantage of the convexity of $\|I_\alpha*\mu\|_{L^q(\omega_\sigma^{-q/p})}$ with respect to (σ,μ) to interchange the two infimums. Meanwhile, the left-hand-side of (5.7) becomes

$$\inf_\sigma\Big(\mathrm{R}^{\omega_\sigma}_{\alpha,p}(E)\Big)^{-1/p}=\Big(\sup_\sigma\mathrm{R}^{\omega_\sigma}_{\alpha,p}(E)\Big)^{-1/p}.$$

This follows from the standard Meyers format; see page 274 in [Me]. Therefore, the proof of (5.1) is complete.

The above theorem enables us to estimate the Morrey capacity of a ball in $\mathbb{R}^n$. First we need:

5.2. Remark. Recall that 1_E stands for the characteristic function of a set E. Then $1_{B(x_0,r_0)}$ of the ball $B(x_0,r_0)\subset\mathbb{R}^n$ is given by

$$1_{B(x_0,r_0)}(x)=\begin{cases}1, & \text{if}\quad x\in B(x_0,r_0)\\ 0, & \text{if}\quad x\notin B(x_0,r_0).\end{cases}$$

Thus there exists a constant c independent of $B(x_0,r_0)$ such that

$$\begin{aligned}\|1_{B(x_0,r_0)}\|^p_{L^{p,\lambda}}&=\sup_{r>0,x\in\mathbb{R}^n} r^{\lambda-n}|B(x,r)\cap B(x_0,r_0)|\\ &\sim\sup_{r>0}\begin{cases}r^\lambda, & \text{if}\quad r\le 2r_0\\ r^{\lambda-n}r_0^n, & \text{if}\quad r>2r_0.\end{cases}\\ &=cr_0^\lambda\end{aligned}$$

5.3 Theorem. *Let* $\alpha \in [0, n]$, $\lambda \in (0, n)$.
(i) *If* $1 < p < \lambda/\alpha$, *then*

$$C_\alpha(B(x,r); L^{p,\lambda}) \sim r^{\lambda-\alpha p}, \quad B(x,r) \subset \mathbb{R}^n. \tag{5.8}$$

(ii) *If* $1 < p = \lambda/\alpha$, *then*

$$C_\alpha(B(x,r); L^{p,\lambda}) \sim \left(\log \frac{1}{r}\right)^{-p}, \quad B(x,r) \subset \mathbb{R}^n, \quad r \to 0. \tag{5.9}$$

Proof. (i) It is clear that for any $y \in B(x,r)$,

$$I_\alpha * 1_{B(x,r)}(y) \gtrsim r^\alpha.$$

Thus, by Remark 5.2 we have

$$\mathrm{C}_\alpha(B(x,r); L^{p,\lambda}) \lesssim \|1_{B(x,r)} r^{-\alpha}\|^p_{L^{p,\lambda}} \sim r^{\lambda-\alpha p}.$$

In order to obtain the lower bound, note that for any $y \in B(x,r)$,

$$I_\alpha * f(y) \geq 1_{B(x,r)}(y).$$

Hence, it follows from Remark 5.2 and Lemma 4.3 (i) that for $\tilde{p} = \lambda p/(\lambda - \alpha p)$,

$$\left(\mathrm{C}_\alpha(B(x,r); L^{p,\lambda})\right)^{\tilde{p}/p} \gtrsim \|1_{B(x,r)}\|^{\tilde{p}}_{L^{\tilde{p},\lambda}} \sim r^\lambda.$$

Consequently

$$\mathrm{C}_\alpha(B(x,r); L^{p,\lambda}) \gtrsim r^{\lambda-\alpha p}.$$

(ii) Fix $x_0 \in \mathbb{R}^n$ and $r \in (0,1)$, and consider the test function

$$f(x) = \begin{cases} 0, & x \in \mathbb{R}^n \setminus B(x_0, 1); \\ \left(\frac{r}{|x-x_0|}\right)^\alpha, & x \in B(x_0,1) \setminus B(x_0,r); \\ 1, & x \in B(x_0, r). \end{cases}$$

Then for $x \in B(x_0, r)$ we get

$$\begin{aligned} I_\alpha * f(x) &= \int_{B(x_0,r)} + \int_{B(x_0,1)\setminus B(x_0,r)} + \int_{\mathbb{R}^n \setminus B(x_0,1)} \left(\cdots\right) \\ &\geq r^\alpha \int_{B(x_0,1)\setminus B(x_0,r)} |y-x|^{\alpha-n} |y-x_0|^{-\alpha} dy \\ &\gtrsim r^\alpha \int_{B(x_0,1)\setminus B(x_0,r)} |y-x_0|^{-n} dy \\ &\gtrsim r^\alpha \log \frac{1}{r}. \end{aligned}$$

That is to say,

$$I_\alpha * f \gtrsim 1_{B(x_0,r)} r^\alpha \log\frac{1}{r}.$$

On the other hand, using Remark 5.2 and making some elementry estimates, we obtain

$$\begin{aligned}\|f\|_{L^{p,\lambda}}^p &\lesssim \sup_{x\in\mathbb{R}^n, t>0} t^{\lambda-n}\left(|B(x,t)\cap B(x_0,r)| + \int_{B(x,t)\cap\left(B(x_0,1)\setminus B(x_0,r)\right)} \left(\frac{r}{|y-x_0|}\right)^{\alpha p} dy\right)\\ &\lesssim r^\lambda.\end{aligned}$$

Thus

$$\left\|\frac{f}{r^\alpha \log\frac{1}{r}}\right\|^p \lesssim \left(\log\frac{1}{r}\right)^{-p},$$

namely, the desired upper bound is achieved.

To get the lower bound, we assume $f \geq 0$, and use Lemma 4.3 (ii) as well as the well-known John-Nirenberg lemma to obtain a constant β_0 depending only on n such that

$$\sup_{B\subset\mathbb{R}^n} |B|^{-1}\int_B \exp\left(\beta\left|I_\alpha * f(x) - (I_\alpha * f)_B\right|\right)dx \lesssim \|f\|_{L^{p,\lambda}}$$

for all $\beta \leq \beta_0$, where the suppremum is taken over all balls B in $\mathbb{R}^n$. Thus for any ball $B \subset \mathbb{R}^n$ we have

$$|B|^{-1}\int_B \exp\left(\beta I_\alpha * f(x)\right)dx \lesssim \|f\|_{L^{p,\lambda}} \exp\left((I_\alpha * f)_B\right).$$

Without loss of generality, we may assume supp$f \subset B_0$ for a large ball $B_0 \subset \mathbb{R}^n$. Then there is a constant $c(r_0)$ depending only on n and the radius r_0 of B_0 such that

$$\int_{B_0} I_\alpha * f(x)dx \leq c(r_0)\|f\|_{L^{p,\lambda}},$$

and hence

$$\int_{B_0} \exp\left(\beta\|f\|_{L^{p,\lambda}}^{-1} I_\alpha * f(x)\right)dx \lesssim c(r_0).$$

So if $I_\alpha * f \geq 1$ on $B(x,r) \subset B_0$ with r being small enough, then

$$\exp\left(\|f\|_{L^{p,\lambda}}^{-1}\right)|B(x,r)| \lesssim c(r_0),$$

or

$$\|f\|_{L^{p,\lambda}}^p \gtrsim \left(\log\frac{1}{r}\right)^{-p}.$$

This gives the desired lower bound.

Clearly, the countable subadditivity of $\mathrm{C}_\alpha(\cdot; L^{p,\lambda})$ yields that if $1 < p < \lambda/\alpha$ then

$$\mathrm{C}_\alpha(E; L^{p,\lambda}) \lesssim \Lambda_{\lambda-\alpha p}^{(\infty)}(E), \quad E \subset \mathbb{R}^n.$$

Moreover, using Theorem 5.3 we can obtain a local isoperimetric-type inequality attached to the Morrey capacity.

5.4 Theorem. *Let $\alpha \in [0, n]$, $\lambda \in (0, n)$, $r \in (0, \infty)$ and $\tilde{p} = \lambda p/(\lambda - \alpha p)$ with $1 < p < \lambda/\alpha$. Then*

$$|E|^{(\lambda-\alpha p)/\lambda} \lesssim r^{p(n-\lambda)/\tilde{p}} C_\alpha(E; L^{p,\lambda}), \quad E \subseteq B(x, r) \subset \mathbb{R}^n. \tag{5.10}$$

Proof. We fix the set $E \subseteq B(x, r)$ in the definition of $\mathrm{C}_\alpha(E; L^{p,\lambda})$. By Hölder's inequality we have that if $I_\alpha * f \geq 1$ on E then

$$|E| \leq \int_E |I_\alpha * f| \leq \Big(\int_E |I_\alpha * f|^{\tilde{p}} \Big)^{1/\tilde{p}} |E|^{1-1/\tilde{p}}.$$

This, together with the boundedness of I_α sending $L^{p,\lambda}$ to $L^{\tilde{p},\lambda}$ (see also [A1, Theorem 3.1]), yields

$$\begin{aligned} |E|^{(\lambda-\alpha p)/(p\lambda)} &\leq \Big(\int_E |I_\alpha * f|^{\tilde{p}} \Big)^{1/\tilde{p}} \\ &\leq \Big(\int_{B(x,r)} |I_\alpha * f|^{\tilde{p}} \Big)^{1/\tilde{p}} \\ &= r^{(n-\lambda)/\tilde{p}} \Big(r^{\lambda-n} \int_{B(x,r)} |I_\alpha * f|^{\tilde{p}} \Big)^{1/\tilde{p}} \\ &\leq r^{(n-\lambda)/\tilde{p}} \|I_\alpha * f\|_{L^{\tilde{p},\lambda}} \\ &\lesssim r^{(n-\lambda)/\tilde{p}} \|f\|_{L^{p,\lambda}}. \end{aligned}$$

We are done.

We close this section by showing that there exists no strong type inequality for the potentials of Morrey functions:

5.5 Example. *Let $\lambda, \alpha \in (0, n)$, $\gamma \in (0, \infty)$ and $1 < p < \lambda/\alpha$. There is a function $f \in L^{p,\lambda}$ such that*

$$\int_0^\infty \Big(C_\alpha\big(\{x \in \mathbb{R}^n : I_\alpha * f(x) \geq t\}; L^{p,\lambda}\big) t^p \Big)^{\gamma/p} \frac{dt}{t} = \infty. \tag{5.11}$$

Proof. The example is simple: set

$$f(y) = |y|^{-\lambda/p} 1_{B(0,1)}(y)$$

for those λ and p assumed above. Then it is easy to see that $\|f\|_{L^{p,\lambda}} < \infty$. Next notice that

$$I_\alpha * f(x) = \int_{B(0,1)} |x - y|^{\alpha-n} |y|^{-\lambda/p} dy = |x|^{\alpha-\lambda/p} \big(1 + O(1)\big), \quad |x| \to 0.$$

Thus there are constants r_0, c_1 and c_2 such that

$$c_1 |x|^{\alpha-\lambda/p} \leq I_\alpha * (x) \leq c_2 |x|^{\alpha-\lambda/p}, \quad |x| < r_0.$$

But (5.11) holds because

$$\mathrm{C}_\alpha\big(\{x\in\mathbb{R}^n: |x|<r_0 \quad \& \quad |x|^{\alpha-\lambda/p}\ge t\}; L^{p,\lambda}\big)\sim t^{-p}, \quad t\to\infty.$$

So if we use the Choquet-Lorentz notation, then we have a function $f\in L^{p,\lambda}$ for which

$$I_\alpha * f \notin L^{(p,\gamma)}\big(\mathrm{C}_\alpha(\cdot; L^{p,\lambda})\big), \quad \forall\gamma<\infty.$$

Note that we trivially have

$$I_\alpha * f \in L^{(p,\infty)}\big(\mathrm{C}_\alpha(\cdot; L^{p,\lambda})\big).$$

5.6 Remark. Here by the Choquet-Lorentz notation in relation to the capacity C, $u\in L^{(p,\gamma)}(C)$ means

$$\int_0^\infty \Big(t^p C\big(\{x\in\mathbb{R}^n: |u(x)|\ge t\}\big)\Big)^{\gamma/p}\frac{dt}{t}<\infty, \quad \gamma<\infty;$$

and

$$\sup_{t>0} t^p C\big(\{x\in\mathbb{R}^n: |u(x)|\ge t\}\big)<\infty, \quad \gamma=\infty.$$

It is worth remarking the following facts: First, when $\lambda=n$, we have

$$\|I_\alpha * f\|_{L^{(p^*,p)}}\lesssim\bigg(\int_0^\infty\Big(\mathrm{C}_\alpha\big(\{x\in\mathbb{R}^n: I_\alpha*f(x)\ge t\}; L^p\big)\Big)dt^p\bigg)^{1/p}\lesssim\|f\|_{L^{(p,p)}}$$

where $L^{(p,p)}=L^p$, and $L^{(p^*,p)}$ is a Lorentz space on $\mathbb{R}^n$ with $p^*=np/(n-\alpha p)$. Second, if $\lambda\in(0,n)$, then

$$\|I_\alpha * f\|_{L^{\tilde p,\lambda}}\lesssim(\bullet)\lesssim\|f\|_{L^{p,\lambda}},$$

where again $\tilde p=\lambda p/(\lambda-\alpha p)$. It appears that there is no Choquet space norm for $I_\alpha * f$ to insert in $(\bullet)$, i.e., no capacitary strongtype inequality (CSI) for the Riesz potentials of functions in a Morrey space with respect to the Morrey type II capacity: $\mathrm{C}_\alpha(\cdot, L^{p,\lambda})$.

6. Morrey Type I Capacities

This section is devoted to an investigation of the Morrey type I capacity associated with the predual space of a Morrey space.

We start with a capacity type functional that is naturally attached to the space of M. Riesz potentials of functions in $H^{q,\lambda}$. The functional is:

$$R_{\alpha,q,\lambda}(\phi)=\inf\{\|f\|^q_{Z^{q,\lambda}}: \quad f\ge0,\ I_\alpha*f\ge\phi \quad \text{on} \quad \mathrm{supp}\phi\}.$$

And, its dual functional is determined by

$$S_{\alpha,p,\lambda}(\phi)=\sup\Big\{\int_{\mathbb{R}^n}\phi\,d\mu: \quad \mu\in\mathbf{M}^+(\mathrm{supp}\phi) \quad \& \quad \|I_\alpha*\mu\|_{L^{p,\lambda}}\le1\Big\}.$$

Here ϕ is a nonnegative bounded function with compact support (and its support is written as suppϕ). All such functions will be denoted by $\mathbf{O}$.

6.1 Theorem. *Let* $\alpha \in [0,n]$, $\lambda \in (0,n)$, $p \in (1,\infty)$ *and* $q = p/(p-1)$. *Suppose* $\phi \in \mathbf{O}$ *and its restriction to its support is continuous there and strictly positive. Then*

$$R_{\alpha,q,\lambda}(\phi)^{1/q} \sim S_{\alpha,p,\lambda}(\phi). \tag{6.1}$$

Proof. For our purpose, let

$$\mathbf{M}_\phi = \Big\{\nu \in \mathbf{M}^+(\mathrm{supp}(\phi)) : \quad \int_{\mathbb{R}^n} \phi d\nu = 1\Big\}$$

and

$$\mathbf{F} = \big\{f \in H^{q,\lambda} : \quad \|f\|_{Z^{q,\lambda}} \le 1\big\},$$

and take an account of the following functionals:

$$R_{\alpha,q,\lambda,1}(\phi) = \left(\sup_{\mathbf{F}} \inf_{\mathbf{M}_\phi} \int_{\mathbb{R}^n} I_\alpha * f d\nu\right)^{-1}$$

and

$$S_{\alpha,p,\lambda,1}(\phi) = \left(\inf_{\mathbf{M}_\phi} \sup_{\mathbf{F}} \int_{\mathbb{R}^n} I_\alpha * f d\nu\right)^{-1}.$$

Note that since ϕ is bounded below away from zero on suppϕ, $\mathbf{M}_\phi$ is a compact subset in the vague topology of $\mathbf{M}$ and that both $\mathbf{M}_\phi$ and $\mathbf{F}$ are convex. These facts plus the linearity and continuity of $\int_{\mathbb{R}^n} I_\alpha * f d\nu$ insure that Fan's Minimax Theorem (see again [AH, Theorem 2.4.1]) applies and yields

$$R_{\alpha,q,\lambda,1}(\phi)^{1/q} = S_{\alpha,p,\lambda,1}(\phi).$$

We conclude the proof by showing

$$R_{\alpha,q,\lambda}(\phi) \sim R_{\alpha,q,\lambda,1}(\phi); \quad S_{\alpha,p,\lambda}(\phi) \sim S_{\alpha,p,\lambda,1}(\phi).$$

Nevertheless, it is enough to verify the second comparability since the argument for the first one is similar. To do this, first notice that $S_{\alpha,p,\lambda,1}(\phi) \ge 0$ since $0 \in \mathbf{F}$. Also, note that $S_{\alpha,p,\lambda,1}(\phi) < \infty$, for if not then there would be a sequence of measures $\{\nu_j\} \subset \mathbf{M}_\phi$ such that

$$\sup_{f\in\mathbf{F}} \int_{\mathbb{R}^n} I_\alpha * f d\nu_j \to 0, \quad j \to \infty$$

and hence, by the $H^{q,\lambda}$-$L^{p,\lambda}$ duality,

$$\|I_\alpha * \nu_j\|_{L^{p,\lambda}} \to 0, \quad j \to \infty.$$

Because $M_\alpha \mu \le I_\alpha * \mu$ for any nonnegative measure μ, we obtain

$$\|M_\alpha \nu_j\|_{L^{p,\lambda}} \to 0, \quad j \to \infty,$$

This implies that there is a subsequence $\{\nu_{j_k}\}$ converging vaguely to a measure $\nu \in \mathbf{M}_\phi$. This measure ν satisfies $\int_{\mathbb{R}^n} \phi d\nu = 1$. But the convergence in $L^{p,\lambda}$ yields $\nu(B) = 0$ for all balls $B \subset \mathbb{R}^n$ and then $\int_{\mathbb{R}^n} \phi d\nu = 0$, violating $\int_{\mathbb{R}^n} \phi d\nu = 1$.

Due to $S_{\alpha,p,\lambda,1}(\phi) < \infty$ and the $H^{q,\lambda}$-$L^{p,\lambda}$ duality, we see that for any $\epsilon > 0$ there is a measure $\nu \in \mathbf{M}_\phi$ satisfying

$$S_{\alpha,p,\lambda,1}(\phi) < \left(\sup_{f\in\mathbf{F}} \int_{\mathbb{R}^n} I_\alpha * f d\nu\right)^{-1} + \epsilon \lesssim \|I_\alpha * \nu\|_{L^{p,\lambda}}^{-1} + \epsilon.$$

So if $\mu = \|I_\alpha * \nu\|_{L^{p,\lambda}}^{-1}\nu$, we get

$$S_{\alpha,p,\lambda,1}(\phi) - \epsilon \lesssim \|I_\alpha * \nu\|_{L^{p,\lambda}}^{-1} = \int_{\mathbb{R}^n} \phi d\mu \lesssim S_{\alpha,p,\lambda}(\phi),$$

and thus

$$S_{\alpha,p,\lambda,1}(\phi) \lesssim S_{\alpha,p,\lambda}(\phi).$$

To establish the reversed version of the last estimate, we may assume $S_{\alpha,p,\lambda}(\phi) < \infty$, without loss of generality. Then for any $\nu \in \mathbf{M}^+(\mathrm{supp}\phi)$ with $\|I_\alpha * \nu\|_{L^{p,\lambda}} \le 1$ and $f \in \mathbf{F}$ and $\mu = \big(\int_{\mathbb{R}^n} \phi d\nu\big)^{-1}\nu$, we have

$$\int_{\mathbb{R}^n} I_\alpha * f d\mu = \int_{\mathbb{R}^n} f I_\alpha * \mu \lesssim \Big(\int_{\mathbb{R}^n} \phi d\nu\Big)^{-1}.$$

by the $H^{q,\lambda}$-$L^{p,\lambda}$ duality once again. So

$$\int_{\mathbb{R}^n} \phi d\nu \lesssim S_{\alpha,p,\lambda,1}(\phi).$$

This implies that

$$S_{\alpha,p,\lambda}(\phi) \lesssim S_{\alpha,p,\lambda,1}(\phi).$$

The above discussion produces

$$S_{\alpha,p,\lambda}(\phi) \sim S_{\alpha,p,\lambda,1}(\phi).$$

We are done.

For $K \subset \mathbb{R}^n$ compact, $p \in (1,\infty)$, $q = p/(p-1)$, $\alpha \in [0,n]$ and $\lambda \in (0,n)$, define

$$\mathrm{C}_\alpha(E; H^{q,\lambda}) = \inf\big\{\|f\|_{Z^{q,\lambda}}^q : \quad f \ge 0 \quad \& \quad I_\alpha * f \ge 1_E\big\}.$$

Also, let

$$\mathrm{Cap}_\alpha(K; L^{p,\lambda}) = \sup\big\{\mu(K) : \mu \in \mathbf{M}^+(K) \quad \& \quad \|I_\alpha * \mu\|_{L^{p,\lambda}} \le 1\big\}.$$

In an obvious manner, these two capacities are extended to the general set $E \subset \mathbb{R}^n$.

6.2 Corollary. *Let* $\alpha,\lambda\in(0,n)$, $1<p<\infty$ *and* $q=p/(p-1)$. *Then*

$$C_\alpha(E;H^{q,\lambda})\sim\Big(Cap_\alpha(E;L^{p,\lambda})\Big)^q,\quad E\subset\mathbb{R}^n. \tag{6.2}$$

Proof. It follows from taking $\phi=1_K$ (for compact $K\subset\mathbb{R}^n$) in (6.1).

6.3 Remark. $C_\alpha(E;H^{q,\lambda})$ is never zero when $p=\lambda/(n-\alpha)>1$. This follows since by duality

$$I_\alpha*f(x)\le\|f\|_{H^{q,\lambda}}\big\|\,|x-\cdot|^{\alpha-n}\big\|_{L^{p,\lambda}}$$

and the second factor on the right is finite for all x for our choice of p. Also if the Riesz kernel is modified at infinity (say set equal to zero for large arguments or have exponential decay at infinity) then the same result holds here for $1<p\le\lambda/(n-\alpha)$.

6.4 Theorem. *Let* $\alpha,\lambda\in(0,n)$. *Then:*

(i) *For all* $p>1$,

$$Cap_\alpha(B(x,r);L^{p,\lambda})\lesssim r^{n-\alpha-\lambda/p},\quad B(x,r)\subset\mathbb{R}^n; \tag{6.3}$$

(ii) *For all* $p>\max\{\lambda/(n-\alpha),(n-\lambda)/\alpha\}$,

$$Cap_\alpha\big(B(x,r);L^{p,\lambda}\big)\gtrsim r^{n-\alpha-\lambda/p},\quad B(x,r)\subset\mathbb{R}^n. \tag{6.4}$$

Proof. We begin by establishing the lower bound in (6.4). We do this by first getting an analogue of the Wolff potential for the capacity $\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda})$. To accomplish this, we use (4.6) and estimate the quantity on the right-hand-side of (4.6). Applying Fubini's theorem and the simple inequality:

$$M_\alpha\mu(x)\lesssim\left(\int_0^\infty\Big(t^{\alpha-n}\mu\big(B(x,t)\big)\Big)^p\frac{dt}{t}\right)^{1/p},\quad x\in\mathbb{R}^n,$$

we get

$$\|M_\alpha\mu\|^p_{L^{p,\lambda}}\lesssim\sup_{x_0\in\mathbb{R}^n,r>0}r^{\lambda-n}\int_0^\infty t^{p(\alpha-n)}\left(\int_{B(x_0,r)}\Big(\mu\big(B(x,t)\big)\Big)^p dx\right)\frac{dt}{t}. \tag{6.5}$$

Since $B(x,t)\subset B(y,2t)$ whenever $|x-y|<t$, we have the following estimate on the inside integral of the right-hand-side of (6.4):

$$\begin{aligned}\int_{B(x_0,r)}\Big(\mu\big(B(x,t)\big)\Big)^p dx&=\int_{B(x_0,r)}\Big(\mu\big(B(x,t)\big)\Big)^{p-1}\left(\int_{B(x,t)}d\mu(y)\right)dx\\&=\iint_{|x-x_0|<r,|x-y|<t}\Big(\mu\big(B(x,t)\big)\Big)^{p-1}dx\,d\mu(y)\\&\le\int_{\mathbb{R}^n}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}|B(x_0,r)\cap B(y,t)|d\mu(y)\\&:=\mathrm{INT}(x_0,r,t).\end{aligned}$$

We put this back into (6.5) and get

$$\|M_\alpha \mu\|^p_{L^{p,\lambda}} \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} r^{\lambda-n}\left(\int_0^r + \int_r^\infty\right) t^{p(\alpha-n)}\Big(\mathrm{INT}(x_0,r,t)\Big)\frac{dt}{t}.$$

Looking at the first integral, we find

$$\begin{aligned}
&\sup_{x_0\in\mathbb{R}^n, r>0} r^{\lambda-n}\int_0^r \Big(\mathrm{INT}(x_0,r,t)\Big)\frac{dt}{t}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} r^{\lambda-n}\int_0^r t^{p(\alpha-n)}t^n\left(\int_{\mathbb{R}^n}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}d\mu(y)\right)\frac{dt}{t}\\
&\lesssim \int_0^\infty t^{n+p(\alpha-n)}t^{\lambda-n}\left(\int_{\mathbb{R}^n}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}d\mu(y)\right)\frac{dt}{t}\\
&\lesssim \int_{\mathbb{R}^n}\left(\int_0^\infty t^{p(\alpha-n)+\lambda}\Big(\mu\big(B(y,t)\big)\Big)^{p-1}\frac{dt}{t}\right)d\mu(y).
\end{aligned}$$

Meanwhile, looking at the second integral, we have

$$\begin{aligned}
&\sup_{x_0\in\mathbb{R}^n, r>0} r^{\lambda-n}\int_r^\infty \Big(\mathrm{INT}(x_0,r,t)\Big)\frac{dt}{t}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} r^{\lambda-n}\int_r^\infty t^{p(\alpha-n)}t^n\left(\int_{\mathbb{R}^n}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}d\mu(y)\right)\frac{dt}{t}\\
&\lesssim \int_0^\infty t^{p(\alpha-n)+\lambda}\left(\int_{\mathbb{R}^n}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}d\mu(y)\right)\frac{dt}{t}\\
&\lesssim \int_{\mathbb{R}^n}\left(\int_0^\infty t^{p(\alpha-n)+\lambda}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}\frac{dt}{t}\right)d\mu(y).
\end{aligned}$$

Therefore

$$\|M_\alpha \mu\|^p_{L^{p,\lambda}} \lesssim \int_{\mathbb{R}^n} W^\mu_{\alpha,p,\lambda}(y)\,d\mu(y), \tag{6.6}$$

where the homogeneous Wolff potential for the capacity $\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda})$ is:

$$W^\mu_{\alpha,p,\lambda}(y) = \int_0^\infty t^{p(\alpha-n)+\lambda}\Big(\mu\big(B(y,2t)\big)\Big)^{p-1}\frac{dt}{t}. \tag{6.7}$$

Now to get the lower bound in (6.4), we estimate (6.7) from above (and hence generate an upper estimation of $\|I_\alpha * \mu\|_{L^{p,\lambda}}$). To do so, we break up the integral in (6.7) as $\int_0^r + \int_r^\infty$ (where we are estimating $\mathrm{Cap}_\alpha(B(x_0,r); L^{p,\lambda})$), take $d\mu(x) = 1_{B(x_0,r)}dx$, and get

$$\int_0^r \{\cdots\} \lesssim \int_0^r t^{p(\alpha-n)+\lambda}t^{n(p-1)}\frac{dt}{t} \sim r^{\alpha p+\lambda-n}, \quad p > \frac{n-\lambda}{\alpha};$$

and

$$\int_r^\infty \{\cdots\} \lesssim \int_r^\infty t^{p(\alpha-n)+\lambda} r^{n(p-1)} \frac{dt}{t} \sim r^{\alpha p+\lambda-n}, \quad p > \frac{\lambda}{n-\alpha}.$$

To get the lower bound in (6.4), we replace the measure μ above by $\nu = cr^{-(\alpha+\lambda/p)}\mu$ with an appropriate constant $c > 0$. This, together with (6.6), (6.7) and (4.6), achieves $\|I_\alpha * \mu\|_{L^{p,\lambda}} \le 1$ and then yields the lower estimate in (6.4).

Next we give the upper bound in (6.3). If $f = 1_{B(x_0,r)}$, then $I_\alpha * f(x) \ge cr^\alpha$ for a constant $c > 0$ when $x \in B(x_0, r)$. Thus we need to calculate $\|1_{B(x_0,r)}(cr^\alpha)^{-1}\|_{Z^{q,\lambda}}$. To this end we make the following decomposition:

$$1_{B(x_0,r)}(cr^\alpha)^{-1} = 1_{B(x_0,r)}|B(x_0,r)|^{(\lambda-pn)/(pn)}|B(x_0,r)|^{-(\lambda-pn)/(pn)}(cr^\alpha)^{-1}.$$

Since $1_{B(x_0,r)}|B(x_0,r)|^{(\lambda-pn)/(pn)}$ is a $(q, n-\lambda)$-atom, one has

$$\|1_{B(x_0,r)}(cr^\alpha)^{-1}\|_{Z^{q,\lambda}} \lesssim r^{-(\lambda-pn)/p}(cr^\alpha)^{-1} = r^{n-\alpha-\lambda/p}.$$

And this in term gives

$$\Big(\mathrm{C}_\alpha\big(B(x_0,r); H^{q,\lambda}\big)\Big)^{1/q} \lesssim r^{n-\lambda/p-\alpha}.$$

Therefore, the proof is complete.

7. Choquet Integrability via Morrey Type I Capacities

In this section, we study the spaces induced by the Choquet integrals with respect to Morrey type I capacities. In particular, we prove that the Hardy-Littlewood maximal operator M_0 is bounded on these spaces under some natural assumptions, but also that the functional $\mathrm{R}_{\alpha,q,\lambda}$ may be bounded (from both below and above) by the space quasi-norms.

We denote by $L^\infty(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$ the class of all Radon measures (i.e., locally finite regular signed Borel measures) μ on $\mathbb{R}^n$ obeying

$$\|\mu\|_{L^\infty_{p,\lambda,\alpha}} = \sup_{K\subset\mathbb{R}^n} \frac{|\mu|(K)}{\mathrm{Cap}_\alpha(K; L^{p,\lambda})} < \infty,$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$, and again $|\mu|$ is the total variation measure of μ. Also, let $L^1(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$ consist of all $\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda})$ quasi-continuous functions f on $\mathbb{R}^n$ (cf. [A4]), for which

$$\begin{aligned}\|f\|_{L^1_{p,\lambda,\alpha}} &= \int_{\mathbb{R}^n} |f| d\mathrm{Cap}_\alpha\big(\cdot; L^{p,\lambda}\big)\\ &= \int_0^\infty \mathrm{Cap}_\alpha\big(\{x\in\mathbb{R}^n : |f(x)| \ge t\}; L^{p,\lambda}\big)dt < \infty. \qquad (7.1)\end{aligned}$$

The intermediate term is called the Choquet integral with respect to $\mathrm{Cap}_\alpha\big(\cdot; L^{p,\lambda}\big)$.

7.1 Theorem. *Let $\alpha \in [0,n]$, $\lambda \in (0,n)$ and $p \in (1,\infty)$. The the pairing*

$$\langle f, \mu \rangle = \int_{\mathbb{R}^n} f d\mu$$

realizes the dual of $L^1(Cap_\alpha(\cdot; L^{p,\lambda}))$ as equivalent to the space $L^\infty(Cap_\alpha(\cdot; L^{p,\lambda}))$.

Proof. It is not hard to see that each $\mu \in L^\infty(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$ induces a bounded linear functional on $L^1(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$. In fact, for such a μ, whenever $f \in L^1(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$, we have

$$\begin{aligned} |\langle f, \mu \rangle| &\leq \int_0^\infty |\mu|(\{x \in \mathbb{R}^n : |f(x)| \geq t\}) dt \\ &\leq \|\mu\|_{L^\infty_{p,\lambda,\alpha}} \int_0^\infty \mathrm{Cap}_\alpha\big(\{x \in \mathbb{R}^n : |f(x)| \geq t\}; L^{p,\lambda}\big) dt \\ &= \|\mu\|_{L^\infty_{p,\lambda,\alpha}} \|f\|_{L^1_{p,\lambda,\alpha}}. \end{aligned}$$

For the converse, since $C_0(\mathbb{R}^n)$ (the class of all continuous functions with compact support on $\mathbb{R}^n$) is contained in $L^1(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$, every bounded linear functional L on $L^1(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$ with finite operator norm $\|L\|$ is given by

$$\int_{\mathbb{R}^n} f d\mu, \quad f \in C_0(\mathbb{R}^n),$$

for some Radon measure μ on $\mathbb{R}^n$. However for any $g \in C_0(\mathbb{R}^n)$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} g d|\mu| \right| &\leq \sup \left\{ \int_{\mathbb{R}^n} \psi d\mu : \psi \in C_0(\mathbb{R}^n) \quad \& \quad |\psi| \leq |g| \right\} \\ &\leq \|L\| \sup \left\{ \|\psi\|_{L^1_{p,\lambda,\alpha}} : \psi \in C_0(\mathbb{R}^n) \quad \& \quad |\psi| \leq |g| \right\} \\ &\leq \|L\| \cdot \|g\|_{L^1_{p,\lambda,\alpha}}. \end{aligned}$$

Now, given a compact set $K \subset \mathbb{R}^n$, we force g to approach 1_K, and use the previous estimate to get

$$|\mu|(K) \lesssim \|L\| \mathrm{Cap}_\alpha(K; L^{p,\lambda}).$$

Therefore we complete the proof.

For the sake of simplicity, we employ $L^\infty(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))^+$ to denote the set of all non-negative measures $\mu \in L^\infty(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))$.

7.2 Corollary. *Let $\alpha \in [0,n]$, $\lambda \in (0,n)$ and $p \in (1,\infty)$. If $f \geq 0$ is lower semi-continuous on $\mathbb{R}^n$, then*

$$\|f\|_{L^1_{p,\lambda,\alpha}} \sim \sup \left\{ \int_{\mathbb{R}^n} f d\mu : \quad \mu \in L^\infty(Cap_\alpha(\cdot; L^{p,\lambda}))^+ \quad \& \quad \|\mu\|_{L^\infty_{p,\lambda,\alpha}} \leq 1 \right\}. \tag{7.2}$$

Proof. Theorem 7.1 implies that the canonical map of $L^1(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))$ into the second dual has the quasi-norm

$$\int_{\mathbb{R}^n}|u|d\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda})\sim\sup\Big\{\Big|\int_{\mathbb{R}^n}ud\mu\Big|:\quad \mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))^+\quad\&\quad\|\mu\|_{L^\infty_{p,\lambda,\alpha}}\le 1\Big\}$$

for any $u\in L^1(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))$. For a nonnegative lower semi-continuous f, we approximate from below by a sequence $\{\phi_j\}\subset C_0(\mathbb{R}^n)^+$ (the class of all nonnegative functions in $C_0(\mathbb{R}^n)$), $\phi_j\nearrow f$ as $j\to\infty$. Then

$$\begin{aligned}\int_{\mathbb{R}^n}\phi_j d\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda})&=\sup\Big\{\int_{\mathbb{R}^n}\phi_j d\mu:\quad \mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))^+\quad\&\quad\|\mu\|_{L^\infty_{p,\lambda,\alpha}}\le 1\Big\}\\&\le\sup\Big\{\int_{\mathbb{R}^n}fd\mu:\quad \mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))^+\quad\&\quad\|\mu\|_{L^\infty_{p,\lambda,\alpha}}\le 1\Big\}.\end{aligned}$$

It is clear that

$$\mathrm{Cap}_\alpha\big(\{x\in\mathbb{R}^n:\phi_j\ge t\};L^{p,\lambda}\big)\longrightarrow\mathrm{Cap}_\alpha\big(\{x\in\mathbb{R}^n:f\ge t\};L^{p,\lambda}\big),$$

as $j\to\infty$. Consequently, it follows that

$$\begin{aligned}&\|f\|_{L^1_{p,\lambda,\alpha}}\\&\lesssim\sup\Big\{\int_{\mathbb{R}^n}fd\mu:\quad \mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))^+\quad\&\quad\|\mu\|_{L^\infty_{p,\lambda,\alpha}}\le 1\Big\}\\&\lesssim\sup\Big\{\int_0^\infty\mu\big(\{x\in\mathbb{R}^n:f(x)\ge t\}\big)dt:\quad \mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda}))^+\quad\&\quad\|\mu\|_{L^\infty_{p,\lambda,\alpha}}\le 1\Big\}\\&\lesssim\|f\|_{L^1_{p,\lambda,\alpha}}\end{aligned}$$

The result now follows.

Next, we would like to give a comparable functional of $\mathrm{R}_{\alpha,q,\lambda}$ via the Choquet integral with respect to $\mathrm{Cap}_\alpha(\cdot;L^{p,\lambda})$.

7.3 Theorem. *Let $\alpha\in[0,n]$, $\lambda\in(0,n)$, $p\in(1,\infty)$ and $q=p/(p-1)$. Then for any $\phi\in C_0(\mathbb{R}^n)^+$,*

$$R_{\alpha,q,\lambda}(\phi)\lesssim\|\phi\|^q_{L^1_{p,\lambda,\alpha}}.\tag{7.3}$$

Proof. For $\phi\in C_0(\mathbb{R}^n)^+$ and $\epsilon>0$, set

$$\phi_\epsilon(x)=\begin{cases}\phi(x)+\epsilon, & x\in\mathrm{supp}\phi,\\ 0, & \text{otherwise.}\end{cases}$$

Then Theorem 6.1 applies to ϕ_ϵ and gives

$$\begin{aligned}
&R_{\alpha,q,\lambda}(\phi)\\
&\le R_{\alpha,q,\lambda}(\phi_\epsilon)\\
&\sim \sup\Big\{\int_{\mathbb{R}^n}\phi_\epsilon d\mu: \quad \mu\in\mathbf{M}^+(\mathrm{supp}\phi_\epsilon) \quad \& \quad \|I_\alpha*\mu\|_{L^{p,\lambda}}\le 1\Big\}\\
&\sim \sup\Big\{\int_0^\infty \mu\big(\{x\in\mathbb{R}^n: \phi_\epsilon(x)\ge t\}\big)dt: \quad \mu\in\mathbf{M}^+(\mathrm{supp}\phi_\epsilon) \quad \& \quad \|I_\alpha*\mu\|_{L^{p,\lambda}}\le 1\Big\}\\
&\lesssim \int_0^\infty \mathrm{Cap}_\alpha\big(\{x\in\mathbb{R}^n: \phi_\epsilon(x)\ge t\}; L^{p,\lambda}\big)dt\\
&\sim \int_{\mathbb{R}^n}\phi_\epsilon d\mathrm{Cap}_\alpha\big(\cdot; L^{p,\lambda}\big)\\
&\lesssim \int_{\mathbb{R}^n}\phi d\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}) + \epsilon\mathrm{Cap}_\alpha(\mathrm{supp}\phi; L_{p,\lambda}).
\end{aligned}$$

So, the result follows.

However, in order to establish the reversed inequality of (7.3), we first derive an estimate for the Hardy-Littlewood maximal operator $M_0(f)$ of a function f with (7.1).

7.4 Theorem. *Let $0<\alpha<\lambda<n, 1<p<\lambda/(\lambda-\alpha)$, and $q=p/(p-1)$. Then for any $f\in H^{q,\lambda}$,*

$$\int_{\mathbb{R}^n} M_0(I_\alpha*f)dCap_\alpha(\cdot; L^{p,\lambda}) \lesssim \|f\|_{H^{q,\lambda}}. \tag{7.4}$$

In particular,

$$I_\alpha H^{q,\lambda}\subseteq L^{(q,1)}(C_\alpha(\cdot; H^{q,\lambda})). \tag{7.5}$$

Proof. Without loss of generality, we can assume $f\ge 0$. Then $M_0(I_\alpha*f)\lesssim I_\alpha*f$. So

$$\int_{\mathbb{R}^n} M_0(I_\alpha*f)d\mu \lesssim \|f\|_{H^{q,\lambda}}\|I_\alpha*\mu\|_{L^{p,\lambda}}.$$

Hence due to Corollary 7.2, it suffices to prove that

$$\int_{B(x,r)}(I_\alpha*\mu)^p \lesssim r^{n-\lambda}, \quad B(x,r)\subset\mathbb{R}^n \tag{7.6}$$

is valid for all measures $\mu\in L^\infty(\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}))^+$. But by (6.3),

$$\begin{aligned}
\mu(B(x,r)) &\le \|\mu\|_{L^\infty_{p,\lambda,\alpha}}\mathrm{Cap}_\alpha(B(x,r); L^{p,\lambda})\\
&\lesssim \|\mu\|_{L^\infty_{p,\lambda,\alpha}} r^{n-\alpha-\lambda/p}.
\end{aligned}$$

The estimation (7.6) follows from [A1, Theorem 5.1] with $d\mu = dx =$ Lebesgue n-measure. The result there is

$$\|I_\alpha * \mu\|_{L_*^{\beta,\theta}} \lesssim \|\mu\|_{L^{1,\theta}}$$

under the conditions: $0 < \alpha < \theta = \alpha + \lambda/p$ and $\beta = \theta/(\theta - \alpha)$. Recall that $\|I_\alpha * \mu\|_{L_*^{\beta,\theta}} < \infty$ if and only if $I_\alpha * \mu$ belongs to the weak-Morrey space $L_*^{\beta,\theta}$:

$$\sup_{t>0} t^\beta \left|B(x,r) \cap \{x \in \mathbb{R}^n : I_\alpha * \mu(x) \geq t\}\right| \lesssim r^{n-\lambda}.$$

Clearly, $L_*^{\beta,\theta} \subseteq L^{p,p\theta/\beta}$ when $p < \beta$. Thus $\theta = \alpha + \lambda/p$ gives

$$\|I_\alpha * \mu\|_{L^{p,\lambda}} \lesssim \|\mu\|_{L^{1,\alpha+\lambda/p}} \lesssim \|\mu\|_{L^\infty_{p,\lambda,\alpha}}$$

since $\frac{p}{\beta}\theta = \lambda$ and $p < \beta$ becomes $p < \lambda/(\lambda - \alpha)$. We are done.

7.5 Theorem. *Let* $0 < \alpha < \lambda < n, 1 < p < \lambda/(\lambda - \alpha)$, *and* $q = p/(p-1)$. *Then for any* $\phi \in C_0(\mathbb{R}^n)^+$,

$$\|M_0(\phi)\|^q_{L^1_{p,\lambda,\alpha}} \lesssim R_{\alpha,q,\lambda}(\phi). \tag{7.7}$$

Proof. By choosing a $f \in H^{q,\lambda}$ such that $I_\alpha * f \geq \phi$ on the support of $\phi \in C_0(\mathbb{R}^n)^+$, we have $M_0(\phi) \leq M_0(I_\alpha * f)$, and then

$$\|M_0(\phi)\|_{L^1_{p,\lambda,\alpha}} = \int_{\mathbb{R}^n} M_0(\phi) d\mathrm{Cap}_\alpha(\cdot; L^{p,\lambda}) \lesssim \|f\|_{Z^{q,\lambda}}, \tag{7.8}$$

by (7.4) and Theorem 3.3. Thus (7.7) follows from (7.8) and the definition of $\mathrm{R}_{\alpha,q,\lambda}(\phi)$.

Corollary 7.2, together (7.7) and (7.3), yields an interesting consequence as follows.

7.6 Corollary. *Let* $0 < \alpha < \lambda < n, 1 < p < \lambda/(\lambda - \alpha)$, *and* $q = p/(p-1)$. *Then:*

(i) *For any* $f \in C_0(\mathbb{R}^n)$,

$$\|M_0 f\|_{L^1_{p,\lambda,\alpha}} \sim R_{\alpha,q,\lambda}(f) \sim \|f\|_{L^1_{p,\lambda,\alpha}}.$$

(ii) *For any* $f \in C_0^\infty(\mathbb{R}^n)$ *and* $\alpha + \lambda/p \in \mathbb{N}$,

$$\|M_0 f\|_{L^1_{p,\lambda,\alpha}} \lesssim \|\nabla^{\alpha+\lambda/p} f\|_{L^1}.$$

This result corresponds nicely to Theorem A resp. Theorem B in [A3] on the Choquet integrals with respect to Hausdorff capacity.

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