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On Meinardus' Examples For the Conjugate Gradient Method[1]

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ABSTRACT

The Conjugate Gradient (CG) method is widely used to solve a positive definite linear system $Ax = b$ of order N. In 1963, Meinardus (*Numer. Math.*, 5 (1963), pp. 14–23.) proved that the relative residual of the kth approximate solution by CG (with the initial approximation $x_0 = 0$) is bounded above by

$$2\left[\Delta_\kappa^k + \Delta_\kappa^{-k}\right]^{-1} \quad \text{with} \quad \Delta_\kappa = \frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1},$$

where $\kappa \equiv \kappa(A) = \|A\|_2\|A^{-1}\|_2$ is A's spectral condition number. In the same paper he also gave an example to achieve this bound for $k = N-1$ but without saying anything about all other $1 \le k < N-1$. It is possible to construct examples to attain Meinardus' bound for any given k, with the help of his example, but such examples depend on k and, furthermore, it will be shown that if the kth residual achieve Meinardus' bound, then the $(k+1)$th residual must be exactly zero. Therefore it'd be interesting to know if there is any example on which the CG relative residuals are comparable to the bound for all $1 \le k \le N-1$. There are two contributions in this paper.

1. A closed formula for the CG residuals for all $1 \le k \le N-1$ on Meinardus' example is obtained, and in particular it implies that Meinardus' bound is always within a factor of $\sqrt{2}$ of the actual residuals;

2. A complete characterization of extreme positive linear systems for which the kth CG residual achieves Meinardus' bound is also presented. As a consequence, there is no positive linear system whose kth CG residual achieves Meinardus' bound for all $1 \le k < N$, unless $N = 2$.

[1]This report is available on the web at `http://www.ms.uky.edu/~math/MAreport/`.

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